Applied Mathematics E-Notes, 19(2019), 1-13 © ISSN 1607-2510
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More Results On 3-Step Hamiltonicity Of Graphs And Its Line Graphs*

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Received 22 January 2018

Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A (p, q)-graph $G = (V, E)$ is said to be $AL(k)$-traversal if there exists a sequence of vertices $(v_1, v_2, \ldots, v_p)$ such that for each $i = 1, 2, \ldots, p-1$, the distance between v_i and v_{i+1} is k. We call a graph G a k-step Hamiltonian graph (or say it admits a k-step Hamiltonian cycle) if it has an $AL(k)$-traversal in G and $d(v_p, v_1) = k$. In this paper, we give several construction of some families of graphs and its line graphs which admit a 3-step Hamiltonian cycle.

1 Introduction

Throughout this paper, we will consider only simple undirected graph $G = (V(G), E(G))$. The *distance* between two vertices u and v in G denoted by $d(u, v)$ is the length of a shortest u, v-path in G. The *line graph* $L(G)$ of a graph G has $E(G)$ as its vertex set and two vertices are adjacent in $L(G)$ if and only if they are adjacent as edges in G. A *matching* in a graph G is a set $M \in E(G)$ such that no edges in M have common endpoints. For a vertex $u \in V(G)$, we say u is *saturated* by a matching M if u is the endpoint of an edge of M, otherwise u is *unsaturated* by M. A *matching* M is called a *perfect matching* in a graph G if M saturates each vertex of G. For terminologies and notations which are not explained here, please refer West [8].

A graph G is said to be Hamiltonian if it contains a Hamiltonian cycle, i.e a spanning cycle that traverses each vertex of G exactly once. Determining whether such cycle exists in a given graph is one of the major classical problems in graph theory. There is no exact characterization to check the existence and non-existence of Hamiltonian cycle for a given graph. A good reference for recent development and open problems related to Hamiltonicity of graphs, please see [2]. This concept of Hamiltonicity is then

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extended by Lau et al. in [3] to k-step Hamiltonicity. They introduced the concept of $AL(k)$-traversal and k-step Hamiltonian graph as follows: For an integer $k \geq 1$, a (p, q)-graph G with p vertices and q edges is said to admit an $AL(k)$-traversal if the p vertices of G can be arranged as $v_1, v_2, \ldots, v_p$ such that $d(v_i, v_{i+1}) = k$ for each $i = 1, 2, \ldots, p-1$. A graph G is k-step Hamiltonian(or just k-SH) if G admits an $AL(k)$-traversal and $d(v_1, v_p) = k$. The sequence of vertices $v_1, v_2, \ldots, v_p, v_1$ is then called a k-SH cycle of G. Clearly, 1-SH graphs are Hamiltonian. The *distance-k graph*, $D_k(G)$ is a graph generated from a graph G such that $V(D_k(G)) = V(G)$ and $uv \in E(D_k(G))$ if and only if $d(u, v) = k$ in G. The following important results obtained by Lau et al. in [3] will be needed in our results.

LEMMA 1. A graph G is k-SH or admits an $AL(k)$-traversal if and only if $D_k(G)$ is Hamiltonian or has a Hamiltonian path, respectively.

LEMMA 2. A bipartite graph does not admit a k-SH cycle for even $k \geq 2$.

Lau et al. in [4] obtained the following necessary and sufficient condition for cycles C_n to be k-SH.

THEOREM 1. The cycle graph C_n, $n \geq 3$ admits a k-SH cycle for $k \geq 2$ if and only if $n \geq 2k + 1$ and $gcd(n, k) = 1$.

Several classes of k-SH graphs including trees, tripartite graphs, cycles, grid graphs, cubic graphs and subdivision of cycles, have been studied, see [3, 4, 5, 6, 7]. In [1], the authors investigated some families of graphs and its line graphs which admit a 3-SH cycle. In this paper, we extend the results in [1] and give new construction of some families of graphs and its line graphs which admit a 3-SH cycle.

2 Main Results

In [3], we know that the complete bipartite graph $K_{m,n}$ is not k-SH for all m, n and $k \geq 2$. Note that the line graph of complete bipartite graph $K_{m,n}$ is a graph obtained from a grid graph $P_m \times P_n$ such that vertices of the same horizontal (respectively vertical) path are also adjacent to each other. We denote (a, b) as the vertex on row a and column b of $P_m \times P_n$ for $1 \leq a \leq m$, $1 \leq b \leq n$. Two vertices (a, b) and (c, d) in $L(K_{m,n})$ are of distance 2 if $a \neq c$ and $b \neq d$. Otherwise, they are of distance 1. Therefore, we conclude that $L(K_{m,n})$ is not k-SH for all $k \geq 3$.

It is interesting to know about the k-step Hamiltonicity of the complete bipartite graph $K_{m,n}$ if some edges are deleted. But, from Lemma 2, we know that the graph, say G obtained from $K_{m,n}$ by deleting some edges is not k-SH for even $k \geq 2$ and the k-step Hamiltonicity of G for odd $k \geq 3$ is not studied yet.

We now check the 3-step Hamiltonicity of some graphs obtained from the complete bipartite graph $K_{m,n}$ by deleting two disjoint perfect matchings S and T. But here, we will consider only $K_{n,n}$, $n \geq 2$ since $K_{m,n}$ for $m \neq n$ does not have perfect matching. Let $V = \{a_1, a_2, \ldots, a_n\}$ and $W = \{a_1^*, a_2^*, \ldots, a_n^*\}$ be the partite sets of $K_{n,n}$ such that $E(K_{n,n}) = \{a_i a_j^* : 1 \leq j \leq n\}$. We then obtain the following results. Note that all subscripts are to be read modulo n.

LEMMA 3. For $S = \{a_i a_i^* : 1 \leq i \leq n\}$ and $T = \{a_i a_{i+1}^* : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S, T\}$ is 3-SH if and only if $n \geq 4$.

PROOF. It is obvious that G is disconnected when $n = 2$ and $n = 3$ so that G does not admit a 3-SH cycle. For $n \geq 4$, observe that $d(a_i^*, a_{i+1}) = d(a_i, a_{i+2}^*) = 1$ and $d(a_i, a_{i-1}) = d(a_i^*, a_{i-1}^*) = 2$ for $1 \leq i \leq n$. Since a_i^* is not adjacent to a_i and a_i is not adjacent to a_{i+1}^*, we have $d(a_i^*, a_i) = d(a_i, a_{i+1}^*) = 3$. Therefore, the sequence $a_1^*, a_1, a_2^*, a_2, \ldots, a_{n-1}^*, a_{n-1}, a_n^*, a_n, a_1^*$ is a possible 3- SH cycle of G.

LEMMA 4. For $S = \{a_i a_{i+1}^* : 1 \leq i \leq n\}$ and $T = \{a_i a_{i-1}^* : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S, T\}$ is 3-SH if and only if $n \geq 5$ is odd.

PROOF. We need $n \geq 3$ because when $n = 2$, we have $S = T$. For $n = 3$ and $n = 4$, graph G is disconnected and thus is not 3-SH. For $n \geq 6$ is even, $D_3(G)$ consists of 2 components each of size n so that $D_3(G)$ is not Hamiltonian. By Lemma 1, G is not 3-SH.

Now, consider odd $n \geq 5$. Note that for $1 \leq i \leq n$, $d(a_i, a_i^*) = 1$ and $d(a_i^*, a_{i+1}^*) = d(a_i, a_{i+1}) = 2$. Since a_i is not adjacent to a_{i+1}^* and a_i^* is not adjacent to a_{i+1}, we have $d(a_i, a_{i+1}^*) = d(a_i^*, a_{i+1}) = 3$. A 3-SH cycle is then given by a_1, a_2^*, a_3, $a_4^*, \ldots,$ a_{n-1}^*, a_n, a_1^*, a_2, $a_3^*, \ldots,$ a_{n-1}, a_n^*, a_1.

LEMMA 5. For $S = \{a_i a_i^* : 1 \leq i \leq n\}$ and $T = \{a_i a_{i+3}^* : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S, T\}$ is 3-SH if and only if $n \geq 4$, $n \not\equiv 0 \pmod 3$.

PROOF. We consider only $n = 2$ and $n \geq 4$ because when $n = 3$, we have $S = T$. It is obvious that G is disconnected when $n = 2$ and thus G is not 3-SH. Suppose $n \geq 6$, $n \equiv 0 \pmod 3$. We can observe that $D_3(G)$ consists of 3 components each of size $\frac{2n}{3}$ and so $D_3(G)$ is not Hamiltonian. By Lemma 1, G is not 3-SH. Suppose now $n \geq 4$, $n \not\equiv 0 \pmod 3$. Note that $d(a_i^*, a_{i+1}) = d(a_i, a_{i+1}^*) = 1$ and $d(a_i, a_{i-1}) = d(a_i^*, a_{i+2}^*) = 2$ for $1 \leq i \leq n$. Since a_i^* is not adjacent to a_i and a_i is not adjacent to a_{i+3}^*, we have $d(a_i^*, a_i) = d(a_i, a_{i+3}^*) = 3$. Then, G is 3-SH by choosing the sequence $a_1^*, a_1, a_4^*, a_4, \ldots, a_{n-3}^*, a_{n-3}, a_n^*, a_n, a_3^*, a_3, a_6^*, a_6, \ldots, a_{n-1}^*, a_{n-1}, a_2^*, a_2, a_5^*, a_5, \ldots, a_{n-2}^*,$ a_{n-2}, a_1^* for $n \equiv 1 \pmod 3$ and the sequence $a_1^*, a_1, a_4^*, a_4, \ldots, a_{n-1}^*, a_{n-1}, a_2^*, a_2, a_5^*, a_5$ $, \ldots, a_{n-3}^*, a_{n-3}, a_n^*, a_n, a_3^*, a_3, a_6^*, a_6, \ldots, a_{n-2}^*, a_{n-2}, a_1^*$ for $n \equiv 2 \pmod 3$ as the 3-SH cycle.

LEMMA 6. For $S = \{a_i a_i^* : 1 \leq i \leq n\}$ and $T = \{a_i a_{i+4}^* : 1 \leq i \leq n\}$, the graph $G = K_{n,n} - \{S, T\}$ is 3-SH if and only if $n \geq 5$ is odd.

PROOF. We consider only $n = 3$ and $n \geq 5$ because when $n = 2$ and $n = 4$, we have $S = T$. It is also obvious that G is disconnected when $n = 3$ so that G is not 3-SH. Suppose $n \geq 6$ is even. Observe that for $n \equiv 0 \pmod 4$, $D_3(G)$ consists of 4 components each of size $\frac{n}{2}$ and for $n \equiv 2 \pmod 4$, $D_3(G)$ consists of 2 components each

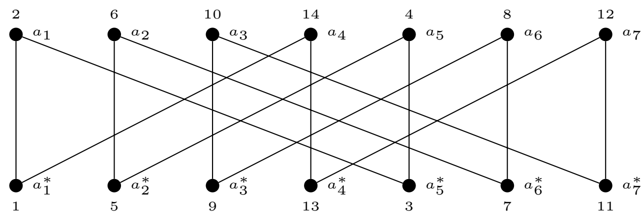


Figure 1: A Hamiltonian cycle of $D_3(G)$ when $n = 7$.

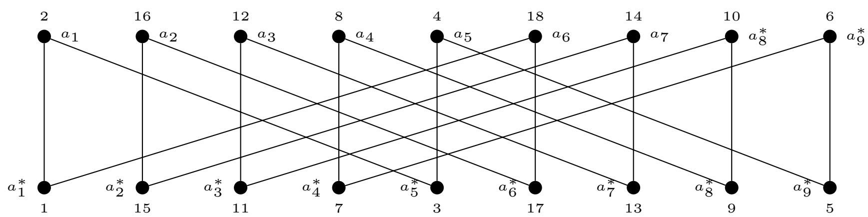


Figure 2: A Hamiltonian cycle of $D_3(G)$ when $n = 9$.

of size n. Therefore, for each case $D_3(G)$ is not Hamiltonian and thus by Lemma 1, G is not 3-SH. Suppose now $n \geq 5$ is odd. In Figure 1 and Figure 2, we give a labeling of Hamiltonian cycle for graph $D_3(G)$ when $n = 7$ and $n = 9$, respectively. Note that for all odd $n \geq 5$ such that $n \equiv 1 \pmod 4$, a Hamiltonian cycle of $D_3(G)$ can be obtained in a similar way to the labeling in Figure 2 and for all odd $n \geq 7$ such that $n \equiv 3 \pmod 4$, a labeling for Hamiltonian cycle follows those in Figure 1. By Lemma 1, we know that all these graphs G are 3-SH such that the Hamiltonian cycle in $D_3(G)$ is a 3-SH cycle of G.

As we can see from these 4 lemmas, we can get a 3-SH graph from the complete bipartite graph $K_{n,n}$ by deleting a set of edges. It is difficult to solve the 3-step Hamiltonicity of $G = K_{n,n}$-$\{S, T\}$ in general because there are $n!$ perfect matchings of $K_{n,n}$. There are a lot more cases that should be considered. We then propose the following problems.

PROBLEM 1. Solve the 3-step Hamiltonicity of $G = K_{n,n} - \{S, T\}$ for all cases of S and T.

PROBLEM 2. Study the 3-step Hamiltonicity of complete bipartite graph $K_{m,n}$ with more edges deleted.

Next, consider a graph G with n vertices. The corona product of G and any graph H, denoted by $G \odot H$, is a graph obtained by taking one copy of G and n copies $H_1, H_2, \ldots, H_n$ of H, and then joining the i-th vertex of G to every vertex in H_i.

Suppose G is a graph of order n that admits a Hamiltonian cycle given by the sequence $u_1, u_2, \ldots, u_n, u_1$ and 3-SH cycle given by $v_1, v_2, \ldots, v_n, v_1$ such that $v_1 = u_1$

and $v_n = u_{n-2}$.

THEOREM 2. The corona product of graph G described above and empty graph O_m of order m is 3-SH for all $m \geq 1$.

PROOF. We know that the graph $G \odot O_m$ is obtained from G by adding nm more vertices and nm more edges. Without loss of generality, we let the nm pendant vertices be $u_{i,1}, u_{i,2}, \ldots, u_{i,m}$ such that the added edges are $u_i u_{i,1}, u_i u_{i,2}, \ldots, u_i u_{i,m}$ for $i = 1, \ldots, n$. We can see that the sequence $v_1 = u_1, v_2, \ldots, v_n = u_{n-2}, u_{n,1}, u_{1,1}, u_{2,1}, \ldots, u_{n-1,1}, u_{n,2}, u_{1,2}, u_{2,2}, \ldots, u_{n-1,2}, u_{n,3}, \ldots, u_{n,m}, u_{1,m}, u_{2,m}, \ldots, u_{n-1,m}, u_1$ is a 3-SH cycle of $G \odot O_m$.

The corona product $C_n \odot K_1$, in particular, is the graph consisting of a cycle C_n, $n \geq 3$ (with edges $u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1$), n more pendant vertices $v_1, v_2, \ldots, v_n$ and n more edges u_iv_i for $i = 1, 2, \ldots, n$. We call this graph the sun graph S_n.

THEOREM 3. The sun graph S_n is 3-SH if and only if $n \geq 5$.

PROOF. Observe that all u_i are isolated in $D_3(S_n)$ if $n = 3$ and of degree 1 if $n = 4$ so that $D_3(S_n)$ cannot be Hamiltonian and thus S_3 and S_4 are not 3-SH. Suppose $n \geq 5$. We consider 2 cases.

Case 1. $n \equiv 0 \pmod 3$.
A 3-SH cycle is given by the sequence $v_1, u_3, u_6, \ldots, u_n, v_2, u_4, u_7, \ldots, u_{n-2}, u_1, v_3, u_5, u_8, \ldots, u_{n-1}, u_2, v_4, v_5, \ldots, v_n, v_1$. In Figure 3, we give a 3-SH cycle for S_9.

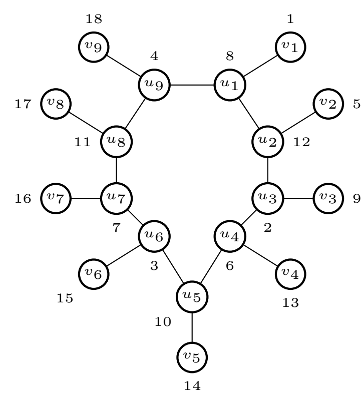


Figure 3: A 3-step Hamiltonian cycle for S_9.

Case 2. $n \not\equiv 0 \pmod 3$.
If $n = 5$, the sequence of vertices $v_1, u_3, v_5, u_2, v_4, u_1, v_3, u_5, v_2, u_4, v_1$ is a possible 3-SH cycle in S_5. For $n \geq 7$, since cycle C_n is 3-SH by Theorem 1, a possible 3-SH cycle in S_n is given in the proof of Theorem 2.
This completes the proof.

THEOREM 4. The line graph of S_n is 3-SH if and only if $n \geq 6$.

PROOF. We denote the vertices of $G = L(S_n)$ by $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$. Then, the edge set is $\{u_iu_{i+1}, u_nu_1 : i = 1, \ldots, n-1\} \cup \{u_iv_i : i = 1, \ldots, n\} \cup \{v_iu_{i+1}, v_nu_1 : i = 1, \ldots, n-1\}$. See Figure 4 for graph $L(S_5)$.

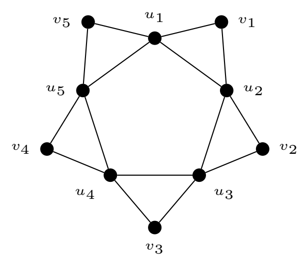


Figure 4: Graph $L(S_5)$.

Clearly, if $n = 3$, every vertex of G is a distance at most 2 from each other so that G is not 3-SH. Note that for $n = 4$ and $n = 5$, there exist isolated or pendant vertices in $D_3(G)$. Hence $D_3(G)$ is not Hamiltonian and thus G is not 3-SH. Next we assume $n \geq 6$. We consider 2 cases.

Case 1. n is odd. We consider 2 subcases.

(i) $n \equiv 0 \pmod 3$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-2}, u_1, u_4, \ldots, u_{n-2}, v_n, u_3, u_6, \ldots, u_n, v_2, u_5, u_8, \ldots, u_{n-1}, u_2, v_4, v_6, \ldots, v_{n-1}, v_1$.

(ii) $n \not\equiv 0 \pmod 3$.

A 3-SH cycle is given by $v_1, v_3, v_5, \ldots, v_n, v_2, v_4, \ldots, v_{n-1}$ followed by $u_2, u_5, \ldots, u_{n-1}$ such that $\{2, 5, 8, \ldots, n-1\} \pmod n$ is a set of distinct integers and it is clear that u_{n-1} is a distance 3 to v_1.

Case 2. n is even. We consider 3 subcases.

(i) $n \equiv 0 \pmod 3$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-3}, u_n, v_2, v_4, \ldots, v_{n-2}, u_1, u_4, \ldots, u_{n-2}, v_n, u_3, u_6, \ldots, u_{n-3}, v_{n-1}, u_2, u_5, \ldots, u_{n-1}, v_1$. Figure 5 shows the graph $L(S_6)$ with a 3-SH labeling in it.

(ii) $n \equiv 1 \pmod 3$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-1}, u_2, u_5, \ldots, u_{n-2}, u_1, u_4, \ldots, u_n, v_2, v_4, \ldots, v_n, u_3, u_6, \ldots, u_{n-1}, v_1$.

(iii) $n \equiv 2 \pmod 3$.

A 3-SH cycle is given by $v_1, v_3, \ldots, v_{n-1}, u_2, u_5, \ldots, u_n, v_2, v_4, \ldots, v_n, u_3, u_6, \ldots, u_{n-2}, u_1, u_4, \ldots, u_{n-1}, v_1$.

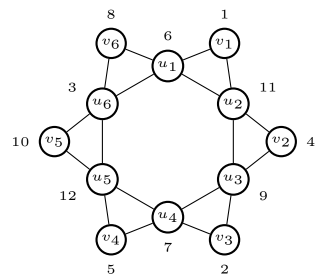
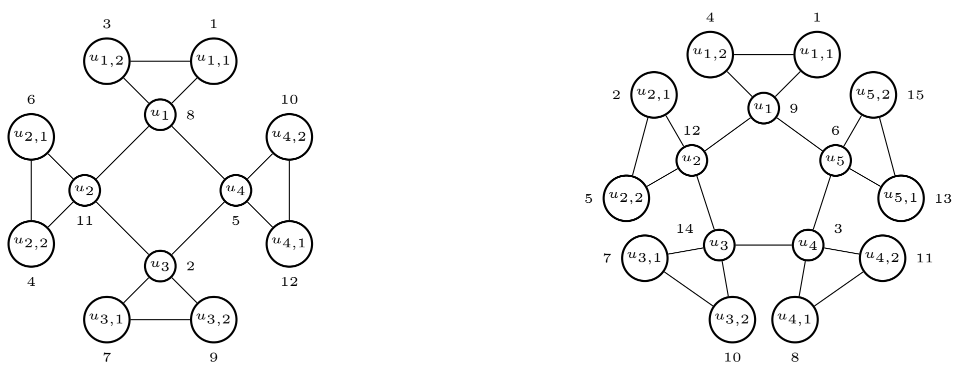


Figure 5: A 3-step Hamiltonian cycle for $L(S_6)$.

This completes the proof.

THEOREM 5. The corona product $C_n \odot P_2$ is 3-SH if and only if $n \geq 4$.

PROOF. Let the vertex set and edge set of $C_n \odot P_2$ be $\{u_i, u_{i,1}, u_{i,2} : 1 \leq i \leq n\}$ and $\{u_1u_n, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{i,1}u_{i,2}, u_iu_{i,1}, u_iu_{i,2} : 1 \leq i \leq n\}$, respectively. If $n = 3$, it is obvious that all u_i are a distance at most 2 from all other vertices of $C_n \odot P_2$ so that $C_n \odot P_2$ is not 3-SH. We now assume that $n \geq 4$. In Figure 6, we give a 3-SH labeling for graphs $C_4 \odot P_2$ and $C_5 \odot P_2$. For $n \geq 6$, we consider 2 cases:



(a) $C_4 \odot P_2$

(b) $C_5 \odot P_2$

Figure 6: 3-SH labeling for $C_4 \odot P_2$ and $C_5 \odot P_2$.

Case 1. $n \equiv 0 (\mathrm{mod}\, 3)$.
A sequence of vertices $u_{1,1}, u_{2,1}, \ldots, u_{n,1}, u_2, u_5, \ldots, u_{n-1}, u_{1,2}, u_3, u_6, \ldots, u_n, u_{2,2}, u_4,$ $u_7, \ldots, u_{n-2}, u_1, u_{3,2}, u_{4,2}, \ldots, u_{n,2}, u_{1,1}$ is a 3-SH cycle of graph $C_n \odot P_2$.

Case 2. $n \not\equiv 0 (\mathrm{mod}\, 3)$.
A possible 3-SH cycle is given by $u_{1,1}, u_{2,1}, \ldots, u_{n,1}, u_{1,2}, u_{2,2}, \ldots, u_{n,2}$ followed by $u_2, u_5, u_8, \ldots, u_{n-1}$ such that $\{2, 5, 8, \ldots, n-1\} (\mathrm{mod}\, n)$ is a set of distinct integers and we can see that $d(u_{1,1}, u_{n-1}) = 3$.
This completes the proof.

THEOREM 6. The line graph of the corona product $C_n \odot P_2$ is 3-SH if and only if $n \geq 5$.

PROOF. Let $G = L(C_n \odot P_2)$ with $V(G) = \{u_i, u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq 3\}$ and $E(G) = \{u_1u_n, u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_{i,j}u_{i,j+1}, u_{i,1}u_{i,3} : 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{u_iu_{i,1}, u_{i+1}u_{i,1}, u_iu_{i,3}, u_{i+1}u_{i,3} : 1 \leq i \leq n \,\text{and}\, i+1 \,\text{is taken modulo n}\}$. See Figure 7 for graph $L(C_3 \odot P_2)$. We consider 2 cases:

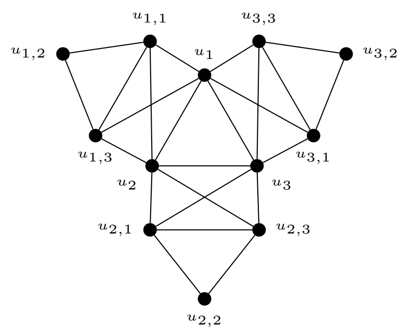


Figure 7: Graph $L(C_3 \odot P_2)$.

Case 1. n is odd.
For $n = 3$, note that all u_i are of degree 1 in $D_3(G)$ so that $D_3(G)$ is not Hamiltonian and thus G is not 3-SH. For $n = 5$, a 3-SH cycle is given by the sequence $u_{1,2}, u_{2,1}, u_5, u_{3,2}, u_{4,1}, u_2, u_{5,2}, u_{1,1}, u_4, u_{2,2}, u_{3,1}, u_1, u_{4,2}, u_{5,1}, u_3, u_{5,3}, u_{3,3}, u_{1,3}, u_{4,3}, u_{2,3}, u_{1,2}$. For $n \geq 7$, we consider 2 subcases:

Subcase 1.1. $n \equiv 0 (\bmod 3)$.

A 3-SH cycle is given by the sequence $u_{1,1}, u_4, u_7, \ldots, u_{n-2}, u_1, u_{2,2}, u_{3,1}, u_6, u_9, \ldots, u_n, u_3, u_{4,2}, u_{5,1}, u_8, u_{11}, \ldots, u_{n-1}, u_2, u_5, u_{6,2}, u_{7,1}, u_{8,2}, u_{9,1}, \ldots, u_{n-1,2}, u_{n,1}, u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,1}, u_{n,2}, u_{1,3}, u_{3,3}, u_{5,3}, \ldots, u_{n-2,3}, u_{n,3}, u_{2,3}, u_{4,3}, \ldots, u_{n-3,3}, u_{n-1,3}, u_{1,1}$.

Subcase 1.2. $n \not\equiv 0 (\bmod 3)$.

A possible 3-SH cycle is started with subsequence $u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,1}, u_{n,2}, u_{1,1}, u_{2,2}, u_{3,1}, u_{4,2}, \ldots, u_{n-1,2}, u_{n,1}, u_{2,3}, u_{4,3}, u_{6,3}, \ldots, u_{n-1,3}, u_{1,3}, u_{3,3}, u_{5,3}, \ldots, u_{n-2,3}, u_{n,3}$. We then completed the 3-SH cycle by traversing the vertices of cycle C_n in the sequence $u_3, u_6, u_9, \ldots, u_n$ such that $\{3, 6, 9, \ldots, n\} (\bmod n)$ is a set of distinct integers. Clearly the last vertex u_n is a distance 3 from $u_{1,2}$.

Case 2. n is even.
For $n = 4$, observe that all vertices in $\{u_i, u_{i,2} : 1 \leq i \leq 4\}$ are of degree 2 in $D_3(G)$, which by themselves forming a non-spanning cycle C_8, a contradiction. Hence, $D_3(G)$ is not Hamiltonian and thus G is not 3-SH. For $n \geq 6$, we consider 3 subcases:

Subcase 2.1. $n \equiv 0 (\bmod 3)$.

A 3-SH cycle is given by the sequence $u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,2}, u_{n,1}, u_3, u_6, \ldots, u_n, u_{2,3}, u_5, u_8, \ldots, u_{n-1}, u_{1,1}, u_{2,2}, u_{3,1}, u_{4,2}, \ldots, u_{n-1,1}, u_{n,2}, u_{1,3}, u_4, u_7, \ldots, u_{n-2}, u_1,$

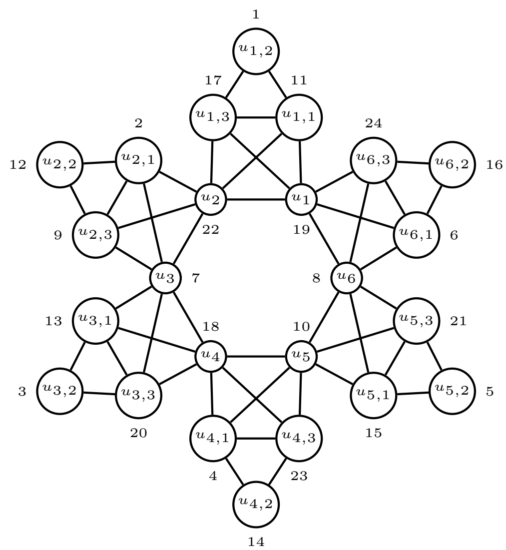


Figure 8: A 3-SH cycle for $L(C_6 \odot P_2)$.

$u_{3,3}, u_{5,3}, \ldots, u_{n-1,3}, u_2, u_{4,3}, u_{6,3}, \ldots, u_{n-2,3}, u_{n,3}, u_{1,2}$. In Figure 8, we give a 3-SH labeling for $L(C_6 \odot P_2)$.

Subcase 2.2. $n \equiv 1 (\mathrm{mod}\, 3)$.

A 3-SH cycle is given by the sequence $u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,2}, u_{n,1}, u_{2,3}, u_{4,3}, u_{6,3}, \ldots, u_{n,3}, u_3, u_6, \ldots, u_{n-1}, u_{n,2}, u_{1,1}, u_{2,2}, u_{3,1}, u_{4,2}, \ldots, u_{n-2,2}, u_{n-1,1}, u_2, u_5, \ldots, u_{n-2}, u_1, u_{3,3}, u_{5,3}, u_{7,3}, \ldots, u_{n-1,3}, u_{1,3}, u_4, u_7, \ldots, u_n, u_{1,2}$.

Subcase 2.3. $n \equiv 2 (\mathrm{mod}\, 3)$.

A 3-SH cycle is given by the sequence $u_{1,2}, u_{2,1}, u_{3,2}, u_{4,1}, \ldots, u_{n-1,2}, u_{n,1}, u_{2,3}, u_{4,3}, u_{6,3}, \ldots, u_{n,3}, u_3, u_6, \ldots, u_{n-2}, u_1, u_{2,2}, u_{3,1}, u_{4,2}, u_{5,1}, \ldots, u_{n-1,1}, u_{n,2}, u_{1,1}, u_4, u_7, \ldots, u_{n-1}, u_2, u_5, \ldots, u_{n-3}, u_{n-1,3}, u_{1,3}, u_{3,3}, u_{5,3}, \ldots, u_{n-3,3}, u_n, u_{1,2}$.
This completes the proof.

Let G be a graph and $G_1, G_2, \ldots, G_n, n \geq 2$ be n copies of graph G. Then, the graph obtained by adding an edge from G_i to G_{i+1}, $i = 1, 2, \ldots, n-1$ is called path union of G such that the added edges connecting the same pair of vertices from G_i to G_{i+1}. We denote path union of n copies of G by $P(G; n)$.

We now consider n copies of cycle C_m, $m \geq 3$ with $C_{i,m} = (u_{i,1}, u_{i,2}, \ldots, u_{i,m})$ be the i-th copy of C_m for $1 \leq i \leq n$. The path union of n copies of C_m denoted by $P(C_m; n)$, $n \geq 2$ is obtained by joining the first vertex of the i-th copy of C_m to the last vertex of the $(i+1)$-th copy of C_m for $i = 1, 2, \ldots, n-1$. See Figure 9 for graph $P(C_6; 2)$.

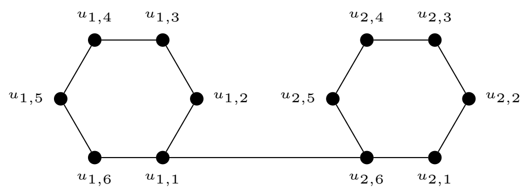


Figure 9: Graph $P(C_6; 2)$.

THEOREM 7. For any $m \geq 3$ and $n \geq 2$, $P(C_m; n)$ is not 3-SH.

PROOF. Obviously the vertex set of $P(C_m; n)$ is $\bigcup_{i=1}^{n} V(C_{i,m})$ and the edge set is $\bigcup_{i=1}^{n} E(C_{i,m}) \cup \{u_{i,1}u_{i+1,m} : 1 \leq i \leq n-1\}$.

Suppose $m = 3$. Note that for all $n \geq 2$, any possible 3-SH cycle in $P(C_m; n)$ must contain the sequence $u_{1,2}, u_{2,2}, u_{1,3}, u_{2,1}, u_{1,2}$, a contradiction. Thus, $P(C_m; n)$ is not 3-SH.

Suppose $4 \leq m \leq 6$. Observe that, in $D_3(P(C_m; n))$, there exist 2 or 4 pendant vertices so that it does not have any Hamiltonian cycle and thus $P(C_m; n)$ is not 3-SH.

Suppose $m \geq 7$. We consider 2 cases:

Case 1. $m \equiv 0 \pmod 3$.
Note that the vertices $u_{1,4}, u_{1,7}, \ldots, u_{1,n-2}$ and $u_{n,3}, u_{n,6}, \ldots, u_{n,m-3}$ are of degree 2 in $D_3(P(C_m; n))$ so that any possible Hamiltonian cycle in $D_3(P(C_m; n))$ necessarily contains the edges $u_{1,1}u_{1,4}, u_{1,4}u_{1,7}, \ldots, u_{1,n-5}u_{1,n-2}, u_{1,n-2}u_{1,1}$ and $u_{n,3}u_{n,6}, u_{n,6}u_{n,9}, \ldots, u_{n,m-3}u_{n,m}, u_{n,m}u_{n,3}$, forming 2 different cycles which is a contradiction. So we conclude that $D_3(P(C_m; n))$ is not Hamiltonian and thus $P(C_m; n)$ is not 3-SH.

Case 2. $m \not\equiv 0 \pmod 3$.
For all $n \geq 2$, the following observations hold:

(i) All the vertices in the sets $\{u_{1,4}, u_{1,5}, \ldots, u_{1,m-2}\}$, $\{u_{n,3}, u_{n,4}, \ldots, u_{n,m-3}\}$ and $\{u_{i,4}, u_{i,5}, \ldots, u_{i,m-3} : i \neq 1, n\}$ (when $n \geq 3$) are of degree 2 in $D_3(P(C_m; n))$.

(ii) The vertices $u_{i,3}$, $1 \leq i \leq n-1$ and $u_{1,m-1}$ are of degree 3 in $D_3(P(C_m; n))$ with $u_{1,3}$ and $u_{1,m-1}$ having a common neighbor $u_{2,m}$.

(iii) In any possible Hamiltonian cycle of $D_3(P(C_m; n))$, $u_{1,1}$ and $u_{n,m}$ have been traversed and no more visits available. Moreover, in $D_3(P(C_m; n))$, each $u_{i,3}$, $1 \leq i \leq n-1$, is adjacent to both $u_{i,m}$ (which has one more visit available in any Hamiltonian cycle of $D_3(P(C_m; n))$) and $u_{i+1,m}$.

From (i), (ii) and (iii), it is clear that $u_{n,m}$ is not available for $u_{n-1,3}$ so that the remaining 2 edges incident with $u_{n-1,3}$ are required to form Hamiltonian cycle in $D_3(P(C_m; n))$. The same result is then continuously applied to all other $u_{i,3}$, $i = n-2, n-3, \ldots, 1$. Finally, as vertex $u_{2,m}$ is no more available for $u_{1,m-1}$, any possible Hamiltonian cycle in $D_3(P(C_m; n))$ must necessarily contain a non-spanning cycle $u_{1,2}, u_{1,5}, u_{1,8}, \ldots, u_{1,m-2}, u_{1,1}, u_{1,4}, \ldots, u_{1,m}, u_{1,3}, u_{1,6}, \ldots, u_{1,m-1}, u_{1,2}$ for every $m \equiv 1 \pmod 3$, or a cycle $u_{1,2}, u_{1,5}, u_{1,8}, \ldots, u_{1,m}, u_{1,3}, u_{1,6}, \ldots, u_{1,m-2}, u_{1,1}, u_{1,4}, \ldots, u_{1,m-1}$, $u_{1,2}$ for every $m \equiv 2 \pmod 3$, a contradiction. Therefore, $D_3(P(C_m; n))$ is not Hamiltonian and thus $P(C_m;n)$ is not 3-SH.

This completes the proof.

From Theorem 1, we know that the cycle C_m when $m \not\equiv 0 \,(\mathrm{mod}\, 3)$ admits a 3-SH cycle. Therefore, Case 2 in the above theorem shows that the path union of any n $(n \geq 2)$ copies of 3-SH graph is not necessarily 3-SH. But, we can construct a 3-SH graph from two graphs as follows: Suppose H_1(respectively H_2) is a graph of order n(respectively m) with an $AL(3)$-traversal given by $u_1, u_2, \ldots, u_n$(respectively $v_1, v_2, \ldots, v_m$) such that $d(u_1, u_n) = d(v_1, v_m) = 2$. We join the vertex u_1 to v_1 to form a 3-SH graph with the vertex sequence u_1, $u_2, \ldots$, u_n, v_1, $v_2, \ldots$, v_m, u_1 as the 3-SH cycle.

THEOREM 8. Let G be a graph of order n with an $AL(3)$-traversal $u_1, u_2, \ldots, u_n$ such that $d(u_1, u_n) = 2$. Then, there exists a path union of two copies of G, $P(G;2)$ which admits a 3-SH cycle.

Suppose G is a graph of order p with a 3-SH cycle given by $u_1, u_2, \ldots, u_p, u_1$ and H is a graph of order q with an $AL(3)$-traversal $v_1, v_2, \ldots, v_q$ such that $d(v_1, v_q) = 1$. Since G is 3-SH, there exists a $u_p - u_1$ path of length 3, say u_p, a, b, u_1. Denote by G_{av_q} the graph obtained from G and H by joining the vertex a to v_q.

THEOREM 9. The graph G_{av_q} of order $p + q$ is 3-SH.

PROOF. Observe that $d(u_p, v_1) = d(v_q, u_1) = 3$ and thus the vertex sequence u_1, $u_2, \ldots$, u_p, v_1, $v_2, \ldots$, v_q, u_1 is a 3-SH cycle of G_{av_q}.

THEOREM 10. Let G be the line graph of $P(C_m;n)$, then

(i) G is not 3-SH for $3 \leq m \leq 5$ and all $n \geq 2$;

(ii) G is not 3-SH for $m \geq 6$, $m \equiv 0(\mathrm{mod}\, 3)$ and $n = 2$;

(iii) G is 3-SH for $m \geq 7$, $m \not\equiv 0(\mathrm{mod}\, 3)$ and $n \geq 3$.

PROOF. Let $V(G) = \{u_i, v_j : 1 \leq i \leq n, 1 \leq j \leq n-1\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ and $E(G) = \{u_i u_{i,1}, u_{i,j} u_{i,j+1}, u_i u_{i,m-1} : 1 \leq i \leq n, 1 \leq j \leq m-2\} \cup \{u_i v_i, v_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i u_{i,1}, v_i u_{i+1,m-1} : 1 \leq i \leq n-1\}$. Figure 10 shows the line graph $L(P(C_4;3))$.

(i) Suppose $m = 3$. Clearly for $n = 2$, vertex v_1 is a distance at most 2 to all other vertices of G so that G is not 3-SH. For all $n \geq 3$, any possible 3-SH cycle in G must consist of the subcycle $u_{1,1}, v_2, u_1, u_{2,1}, u_{1,1}$, a contradiction. Thus, G is not 3-SH. Suppose $m = 4$. For all $n \geq 2$, observe that the set of vertices $\{u_{1,3}, u_2, u_{1,2}, u_{2,3}\}$ induce a cycle in any possible 3-SH cycle of G so that G is not 3-SH. Suppose $m = 5$. For all $n \geq 2$, there exist exactly 2 pendant vertices in $D_3(G)$, from the first and last copy of C_m, respectively. Hence, $D_3(G)$ is not Hamiltonian and thus G is not 3-SH.

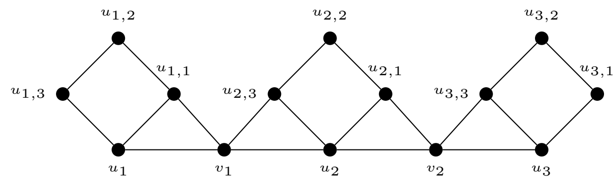


Figure 10: Graph $L(P(C_4;3))$.

(ii) Observe that v_1 is a cut-vertex in $D_3(G)$ so that it is not Hamiltonian. Hence, G is not 3-SH.

(iii) A 3-SH labeling for $L(P(C_8;5))$ and $L(P(C_7;6))$ are given in Figure 11 and in Figure 12, respectively. For $m \geq 7$ and odd $n \geq 3$, a 3-SH cycle can be constructed in a way similar to that in $L(P(C_8;5))$ whereas we can get a 3-SH labeling for $m \geq 7$ and even $n \geq 4$ by referring to the labeling pattern in $L(P(C_7;6))$.

This completes the proof.

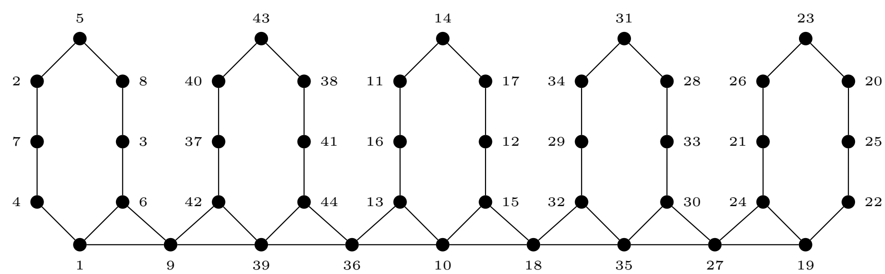


Figure 11: A 3-SH cycle for $L(P(C_8;5))$.

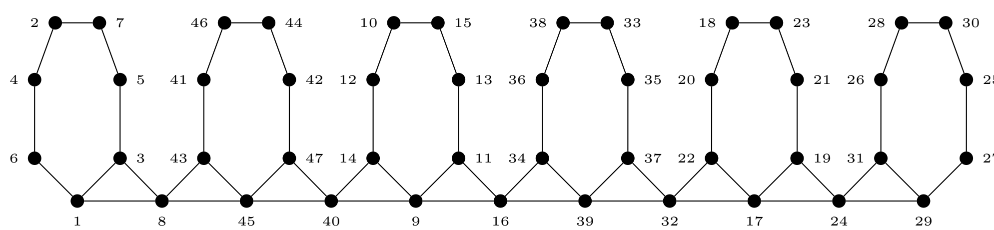


Figure 12: A 3-SH cycle for $L(P(C_7:6))$.

From Theorem 10, we pose the following open problem.

PROBLEM 3. Solve the 3-step Hamiltonicity of line graph of $P(C_m;n)$ for all $m \geq 3$ and $n \geq 2$.

Acknowledgment. The authors would like to thank the referee for his/her suggestions that improved the paper.

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