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Bifurcation Diagram Of The p-Laplacian Problem With Generalized Allen-Cahn Type Nonlinearities*†

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Abstract

We study the exact multiplicity of (classical) positive solutions and the bifurcation diagram of the p-Laplacian problem with generalized Allen-Cahn type nonlinearities. We give a complete classification of totally *six* qualitatively different bifurcation diagrams.

1 Introduction

In this paper we study the exact multiplicity of (classical) positive solutions and the bifurcation diagram of the p-Laplacian problem with generalized Allen-Cahn type nonlinearities

$$\begin{cases} \left(\varphi_p(u'(x))\right)' + \lambda u^{p-1}(a+u)^q(1-u)^r = 0, \ -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1}$$

where $p > 1$, $a, q, r > 0$, $\varphi_p(y) = |y|^{p-2}\, y$, $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian and bifurcation parameter $\lambda > 0$ is the *reciprocal* diffusion constant.

We first consider the p-Laplacian problem

$$\begin{cases} \left(\varphi_p(u'(x))\right)' + \lambda f(u) = 0, \ -1 < x < 1, \ \ u(-1) = u(1) = 0, \\ f(u) = u^{p-1} g(u). \end{cases} \tag{2}$$

We mainly consider $g(u)$ can take one of the following forms:

(a) **Logistic type:** $0 < g(0) < \infty$, $g'(u) < 0$ on $(0, D)$, $g(D) = 0$, and $g(u) < 0$ on (D, ∞) for some positive number D.

(b) **Weak Allee effect type:** $g(0) \geq 0$, $g'(u) > 0$ on $(0, A)$, $g'(u) < 0$ on (A, D), $g(D) = 0$, and $g(u) < 0$ on (D, ∞) for some positive numbers $A < D$.

(c) **Strong Allee effect type:** $g(0) < 0$, $g'(u) > 0$ on $(0, A)$, $g'(u) < 0$ on (A, D), $g(D) = 0$, and $g(u) < 0$ on (D, ∞) for some positive numbers $A < D$.

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Note that D is called the carrying capacity of the population. Sometimes, without loss of generality, we may assume $D = 1$.

The solutions of (2) are the steady state solutions of a p-Laplacian reaction-diffusion population model in one-dimensional case. A typical form of p-Laplacian reaction-diffusion population model equation is

$$\frac{\partial u}{\partial t} = \mathcal{D}\Delta_p u + u^{p-1}g(u), \tag{3}$$

where $p > 1$, $u(x,t)$ is the population density, $\mathcal{D} > 0$ is the diffusion constant, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u with respect to the variable x, and $f(u) = u^{p-1}g(u)$ is the growth rate. Note that, when $p = 2$, $g(u) = f(u)/u$ is the growth rate per capita. When $p = 2$, the p-Laplacian becomes the usual Laplacian and problem (3) has been studied intensively. We refer to the work of McCabe, Leach and Needham [13], Shi and Shivaji [17], Wang and Kot [18], and Xin [20] and the references therein. In the case when $p \neq 2$, Δ_p appears in numerous situations. For example, in the context of reaction-diffusions, Murray [14] suggested using diffusion of the form Δ_p in the study of diffusion-kinetic enzymes problems. See Díaz [5] for other reference. Problem (3) and the steady-state problem of it, associated with the p-Laplacian, have commanded growing interest, see, e.g., [1, 2, 3, 6, 7, 12, 15, 16].

If g is of logistic type, it is known that the bifurcation diagram of positive solutions of (2) consists of exactly one curve which is monotone on the $(\lambda, \|u\|_\infty)$-plane since f satisfies

$$(p-1)f(u) - uf'(u) = -u^p g'(u) > 0 \quad \text{on } (0, D).$$

See also [17, Theorem 2] for results of n-dimensional Dirichlet problem of (2) when $p = 2$.

If g is of weak Allee effect type, Hung and Wang [8, Theorem 2.1] proved that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram of positive solutions consists of exactly one curve with exactly one turning point where the curve turns to the right under some conditions. The next theorem is one of the results of [8, Theorem 2.1].

THEOREM 1. Let $p > 1$. Consider (2) where $g \in C[0, D] \cap C^2(0, D)$ and g is of weak Allee effect type for some positive numbers $A < D$. Assume that g is log-concave $((\log g(u))'' \leq 0)$ on (A, D), then the bifurcation diagram of positive solutions of (2) consists of exactly one curve with exactly one turning point where the curve turns to the right on the $(\lambda, \|u\|_\infty)$-plane.

Let $F(u) \equiv \int_0^u f(t)dt$, then the time map formula which we apply to study the p-Laplacian problem (2) takes the form as follows:

$$\lambda^{1/p} = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [F(\alpha) - F(u)]^{-1/p}\, du \equiv T(\alpha) \quad \text{for } 0 < \alpha < D, \tag{4}$$

see, e.g., [4, Lemmas 2.1 and 2.2] and [10, Lemma 2.4] for the derivation of the time map formula $T(\alpha)$ for (2). The next proposition contains some basic results on the time map formula $T(\alpha)$, in which we determine the limits of $T(\alpha)$ at 0 and D. Let, for $p > 1$,

$$
\begin{aligned}
C_p &\equiv (p-1)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \\
&= (p-1)\left(\int_0^1 (1-v)^{-1/p} dv\right)^p \\
&= \frac{p-1}{p}\left(\int_0^1 \left[\int_v^1 s^{p-1} ds\right]^{-1/p} dv\right)^p > 0.
\end{aligned} \tag{5}
$$

PROPOSITION 2. Let $p > 1$. Consider (2), where $f(u) = u^{p-1}g(u) \in C[0,D] \cap C^2(0,D)$ for some finite positive number D, and $g(D) = 0$, $g(u) > 0$ on $(0,D)$. Then the following assertions (i)–(iv) hold:

(i)
$$
\lim_{\alpha\to 0^+} T(\alpha) = \begin{cases} 0 & \text{if } \lim_{u\to 0^+} g(u) = \infty, \\ (C_p/g(0))^{1/p} & \text{if } 0 < g(0) < \infty, \\ \infty & \text{if } g(0) = 0. \end{cases}
$$

(ii) If $0 \le \lim_{u\to D^-} g(u)/(D-u)^{p-1} < \infty$, then $\lim_{\alpha\to D^-} T(\alpha) = \infty$.

(iii) If
$$
0 < \lim_{u\to D^-} \frac{g(u)}{(D-u)^{p-1}\left(\log\frac{1}{D-u}\right)^\eta} \le \infty \quad \text{for some } \eta > p, \tag{6}
$$
then $0 < \lim_{\alpha\to D^-} T(\alpha) < \infty$.

(iv) If g is decreasing on (D_0, D) for some finite positive number $D_0 < D$ and $0 < \lim_{\alpha\to D^-} T(\alpha) < \infty$, then
$$
\begin{aligned}
\lim_{\alpha\to D^-} T(\alpha) &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 s^{p-1} g(Ds) ds\right]^{-1/p} dv \\
&= \left(\frac{p-1}{p}\right)^{1/p} \int_0^D [F(D) - F(u)]^{-1/p} du \equiv T(D).
\end{aligned} \tag{7}
$$

Proposition 2(i)–(ii) are slight generalization of [9, Theorems 2.6, 2.9 and 2.10] from $p = 2$ to $p > 1$, and consequently, we omit the proofs. Proposition 2(iii) follows from [11, Theorem 5.2] after slight modification. The proof of Proposition 2(iv) is easy but tedious; we omit it. Note that condition (6) implies that f does not satisfy a Lipschitz condition of order $p-1$ at D^-.

In the time map formulas (4) and (7), any (classical) positive solutions u of (2) correspond to

$$
\|u\|_\infty = \alpha \begin{cases} \in (0,D) & \text{if } \lim_{\alpha\to D^-} T(\alpha) = \infty, \\ \in (0,D] & \text{if } 0 < \lim_{\alpha\to D^-} T(\alpha) < \infty \end{cases} \tag{8}
$$

and $T(\alpha) = \lambda^{1/p}$. Thus studying the exact number of positive solutions u of (2) is equivalent to studying the shape of the time map $T(\alpha)$ on $(0, D)$ if $\lim_{\alpha\to D^-} T(\alpha) = \infty$ and that on $(0, D]$ if $\lim_{\alpha\to D^-} T(\alpha) < \infty$.

We define

$$\hat{\lambda} = \begin{cases} (\lim_{\alpha\to 0^+} T(\alpha))^p = C_p/g(0) & \text{if } 0 < g(0) < \infty, \\ \infty & \text{if } g(0) = 0 \end{cases} \tag{9}$$

and

$$\bar{\lambda} = \begin{cases} (\lim_{\alpha\to D^-} T(\alpha))^p & \text{if } 0 < \lim_{\alpha\to D^-} T(\alpha) < \infty, \\ \infty & \text{if } \lim_{\alpha\to D^-} T(\alpha) = \infty. \end{cases} \tag{10}$$

2 Main Result

In next Theorem 3, we apply Theorem 1 to p-Laplacian problem (1) with generalized Allen-Cahn type nonlinearities, we give a complete classification of bifurcation diagrams of positive solutions $u \le 1$ of (1) for constants $a > 0$, $p > 1$ and parameters $q, r > 0$ (see Fig. 1.)

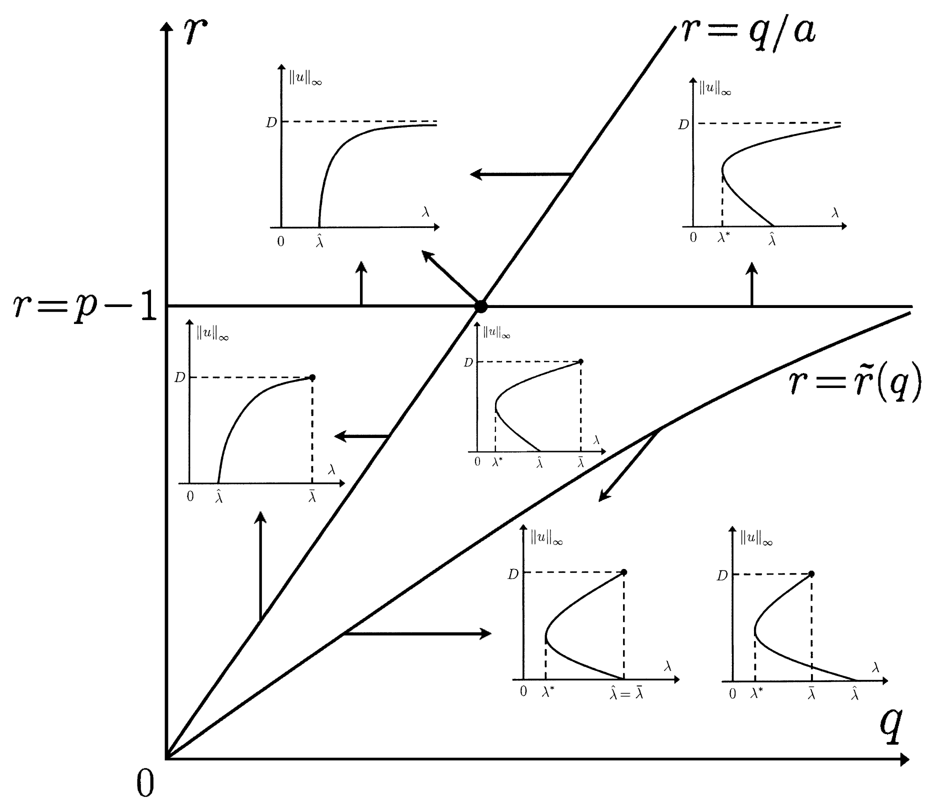

Fig. 1. Classified six bifurcation diagrams of (1), drawn on the first quadrant of the (q, r)-parameter plane. $p > 1$ and $D = 1$.

THEOREM 3 (See Fig. 1). Let $p > 1$. Consider (1). Then, for any fixed $q > 0$, $\hat{\lambda} = C_p/a^q$ and there exists a positive $\tilde{r}(q) < \min\{q/a, p-1\}$ such that the following assertions (i)–(vii) hold:

(i) If $r \geq \max\{q/a, p-1\}$, then (1) has exactly one positive solution u_λ for $\lambda > \hat{\lambda}$, and no positive solution for $0 < \lambda \leq \hat{\lambda}$. Moreover, $\lim_{\lambda \to \hat{\lambda}^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = 1$.

(ii) If $q/a \leq r < p-1$, then $0 < \hat{\lambda} < \bar{\lambda} < \infty$, and (1) has exactly one positive solution u_λ for $\hat{\lambda} < \lambda \leq \bar{\lambda}$, and no positive solution for $0 < \lambda \leq \hat{\lambda}$ and $\lambda > \bar{\lambda}$. Moreover, $\lim_{\lambda \to \hat{\lambda}^+} \|u_\lambda\|_\infty = 0$ and $\|u_{\bar{\lambda}}\|_\infty = 1$.

(iii) If $p - 1 \leq r < q/a$, then there exists $\lambda^* \in (0, \hat{\lambda})$ such that (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $\lambda^* < \lambda < \hat{\lambda}$, exactly one positive solution v_λ for $\lambda = \lambda^*$ and $\lambda \geq \hat{\lambda}$, and no positive solution for $0 < \lambda < \lambda^*$. Moreover, $\lim_{\lambda \to \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \to \infty} \|v_\lambda\|_\infty = 1$.

(iv) If $\tilde{r}(q) < r < \min\{q/a, p-1\}$, then $1 < \bar{\lambda}/\hat{\lambda} < c(r)$ where

$$c(r) = \left(\frac{\int_0^1 \left[\int_v^1 s^{p-1}(1+s/a)^{ar}(1-s)^r ds \right]^{-1/p} dv}{\int_0^1 \left[\int_v^1 s^{p-1} ds \right]^{-1/p} dv} \right)^p \in (1, \infty), \tag{11}$$

and there exists $\lambda^* \in (0, \hat{\lambda})$ such that (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $\lambda^* < \lambda < \hat{\lambda}$, exactly one positive solution v_λ for $\lambda = \lambda^*$ and $\hat{\lambda} \leq \lambda \leq \bar{\lambda}$, and no positive solution for $0 < \lambda < \lambda^*$ and $\lambda > \bar{\lambda}$. Moreover, $\lim_{\lambda \to \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ and $\|v_{\bar{\lambda}}\|_\infty = 1$.

(v) If $r = \tilde{r}(q)$, then $\bar{\lambda}/\hat{\lambda} = 1$ and there exists $\lambda^* \in (0, \hat{\lambda})$ such that (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $\lambda^* < \lambda < \hat{\lambda}$, exactly one positive solution v_λ for $\lambda = \lambda^*$ and $\lambda = \hat{\lambda}$, and no positive solution for $0 < \lambda < \lambda^*$ and $\lambda > \hat{\lambda}$. Moreover, $\lim_{\lambda \to \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ and $\|v_{\bar{\lambda}}\|_\infty = 1$.

(vi) If $0 < r < \tilde{r}(q)$, then

$$(\frac{a}{a+1})^q < \frac{\bar{\lambda}}{\hat{\lambda}} < 1$$

and there exists $\lambda^* \in (0, \bar{\lambda})$ such that (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $\lambda^* < \lambda \leq \bar{\lambda}$, exactly one positive solution u_λ for $\lambda = \lambda^*$ and $\bar{\lambda} < \lambda < \hat{\lambda}$, and no positive solution for $0 < \lambda < \lambda^*$ and $\lambda \geq \hat{\lambda}$. Moreover, $\lim_{\lambda \to \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ and $\|v_{\bar{\lambda}}\|_\infty = 1$.

Furthermore,

(vii) $\tilde{r}(q)$ is a continuous, strictly increasing function of q on $(0, \infty)$. Moreover, $\lim_{q \to 0^+} \tilde{r}(q) = 0$ and $\lim_{q \to \infty} \tilde{r}(q) = p - 1$.

3 Proof of Main Result

For $a, q, r > 0$, let $f(u) = u^{p-1}(a+u)^q(1-u)^r$, then problem (1) is a p-Laplacian problem (2) with generalized Allen-Cahn type nonlinearities.

First, for any $q > 0$ and $r \geq p-1$, the time-map function $T(\alpha)$ defined in (4) satisfies $\lim_{\alpha\to 1^-} T(\alpha) = \infty$ easily by Proposition 2(ii). Secondly, for $q > 0$ and $0 < r < p-1$, letting $\eta = p+1$ in (6), we have

$$\lim_{u\to 1^-} \frac{(a+u)^q(1-u)^r}{(1-u)^{p-1}\left(\log\frac{1}{1-u}\right)^{p+1}} = \lim_{u\to 1^-} \frac{(a+1)^q}{(1-u)^{p-1-r}\left(\log\frac{1}{1-u}\right)^{p+1}} = \infty.$$

So $\lim_{\alpha\to 1^-} T(\alpha) \in (0,\infty)$ by Proposition 2(iii).

Thirdly, for any $q, r > 0$ in (1), we adopt the results of Proposition 2 to obtain that

$$T_{q,r}(0) \equiv \lim_{\alpha\to 0^+} T(\alpha) = (C_p/g(0))^{1/p} = (C_p/a^q)^{1/p} \in (0,\infty) \tag{12}$$

and

$$T_{q,r}(1) \equiv \lim_{\alpha\to 1^-} T(\alpha) \begin{cases} = \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 s^{p-1}(a+s)^q(1-s)^r ds\right]^{-1/p} dv \\ \quad \in (0,\infty) \ \text{ if } r \in (0, p-1), \\ = \infty \ \text{ if } r \geq p-1. \end{cases} \tag{13}$$

Finally, we have

LEMMA 4. Let $p > 1$. Consider (1). Then

(i) For any fixed $q > 0$, $\frac{T_{q,r}(1)}{T_{q,r}(0)}$ is a continuous, strictly increasing function of r on $(0, p-1)$. Moreover,

$$\left(\frac{a}{a+1}\right)^{q/p} < \lim_{r\to 0^+} \frac{T_{q,r}(1)}{T_{q,r}(0)} < 1 \quad \text{and} \quad \lim_{r\to (p-1)^-} \frac{T_{q,r}(1)}{T_{q,r}(0)} = \infty.$$

(ii) For any fixed $r \in (0, p-1)$, $\frac{T_{q,r}(1)}{T_{q,r}(0)}$ is a continuous, strictly decreasing function of q on (ar, ∞). Moreover,

$$1 < \lim_{q\to (ar)^+} \frac{T_{q,r}(1)}{T_{q,r}(0)} < c(r)^{1/p} \quad \text{and} \quad \lim_{q\to\infty} \frac{T_{q,r}(1)}{T_{q,r}(0)} = 0,$$

where $c(r)$ is defined in (11).

PROOF. For any $q > 0$, $r \in (0, p-1)$, (12) and (13) implies

$$\begin{aligned} \frac{T_{q,r}(1)}{T_{q,r}(0)} &= \left(\frac{a^q}{C_p}\right)^{1/p} \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 s^{p-1}(a+s)^q(1-s)^r ds\right]^{-1/p} dv \\ &= \left(\frac{p-1}{pC_p}\right)^{1/p} \int_0^1 \left[\int_v^1 s^{p-1}(1+s/a)^q(1-s)^r ds\right]^{-1/p} dv. \end{aligned} \tag{14}$$

Now, for any fixed $q > 0$, $(1-s)^r$ is a strictly decreasing function of r on $(0, p-1]$ for any $s \in (0,1)$, hence $\frac{T_{q,r}(1)}{T_{q,r}(0)}$ is a strictly increasing function of r on $(0, p-1)$ by (12) and (13). For fixed $r_0 \in (0, p-1)$, we have

$$\begin{aligned}\lim_{r\to r_0}\frac{T_{q,r}(1)}{T_{q,r}(0)} &= \lim_{r\to r_0}\left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^q(1-s)^r ds\right]^{-1/p} dv\\ &= \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^q(1-s)^{r_0} ds\right]^{-1/p} dv = \frac{T_{q,r_0}(1)}{T_{q,r_0}(0)}\end{aligned}$$

by the Monotone convergence theorem [19, p. 75]. So $\frac{T_{q,r}(1)}{T_{q,r}(0)}$ is a continuous function of r on $(0, p-1)$. Moreover, (5), (12)–(14) and the Monotone convergence theorem [19, p. 75] imply

$$\lim_{r\to(p-1)^-}\frac{T_{q,r}(1)}{T_{q,r}(0)} = \frac{T_{q,p-1}(1)}{T_{q,p-1}(0)} = \infty,$$

$$\begin{aligned}\lim_{r\to 0^+}\frac{T_{q,r}(1)}{T_{q,r}(0)} &= \lim_{r\to 0^+}\left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^q(1-s)^r ds\right]^{-1/p} dv\\ &= \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^q ds\right]^{-1/p} dv\\ &< \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1} ds\right]^{-1/p} dv = 1,\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to 0^+}\frac{T_{q,r}(1)}{T_{q,r}(0)} &= \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^q ds\right]^{-1/p} dv\\ &> \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+1/a)^q ds\right]^{-1/p} dv = (\frac{a}{a+1})^{q/p}.\end{aligned}$$

On the other hand, we assume $r \in (0, p-1)$ be fixed. Since $(1+s/a)^q$ is a strictly increasing function of q on (ar, ∞) for any $s \in (0,1)$, $\frac{T_{q,r}(1)}{T_{q,r}(0)}$ is a strictly decreasing function of q on (ar, ∞). It is easy to check $\frac{T_{q,r}(1)}{T_{q,r}(0)}$ is a continuous function of q on $[ar, \infty)$ by similar argument as the above analysis. Moreover, (5), (12)–(14) and the Monotone convergence theorem [19, p. 75] imply

$$\lim_{q\to\infty}\int_v^1 s^{p-1}(1+s/a)^q(1-s)^r ds = \infty \quad \text{for any } v \in (0,1),$$

$$\lim_{q\to\infty}\frac{T_{q,r}(1)}{T_{q,r}(0)} = \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1 \lim_{q\to\infty}\left[\int_v^1 s^{p-1}(1+s/a)^q(1-s)^r ds\right]^{-1/p} dv = 0,$$

and

$$\begin{aligned}\lim_{q\to(ar)^+}\frac{T_{q,r}(1)}{T_{q,r}(0)} &= \left(\frac{p-1}{pC_p}\right)^{1/p}\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^{ar}(1-s)^r ds\right]^{-1/p}dv\\ &= \frac{\int_0^1\left[\int_v^1 s^{p-1}(1+s/a)^{ar}(1-s)^r ds\right]^{-1/p}dv}{\int_0^1\left[\int_v^1 s^{p-1}ds\right]^{-1/p}dv} = c(r)^{1/p}.\end{aligned}$$

It is easy to check $(1+s/a)^{ar}(1-s)^r<1$ for any $0<s<1$, and hence

$$\lim_{q\to(ar)^+}\frac{T_{q,r}(1)}{T_{q,r}(0)} = c(r)^{1/p}\in(1,\infty).$$

The proof of Lemma 4 is now complete.

We are now in a position to prove Theorem 3.

PROOF. For $T(\alpha)$ in (4), we compute that

$$T'(\alpha)=\left(\frac{p-1}{p^{p+1}}\right)^{1/p}\frac{1}{\alpha}\int_0^\alpha\frac{\theta(\alpha)-\theta(u)}{[F(\alpha)-F(u)]^{(p+1)/p}}du, \tag{15}$$

where

$$\theta(u)=pF(u)-uf(u). \tag{16}$$

For $a,q,r>0$, $f(u)=u^{p-1}(a+u)^q(1-u)^r\in C[0,1]\cap C^2(0,1)$. By (16), we compute that

$$\theta'(u)=(p-1)f(u)-uf'(u)=-u^p g'(u)=(q+r)\,u^p(a+u)^{q-1}(1-u)^{r-1}(u-A), \tag{17}$$

where $A\equiv\frac{q-ar}{q+r}$.

Case (i) $r\geq\max\{q/a,p-1\}$. It is easy to see that $g(u)=\frac{f(u)}{u^{p-1}}=(a+u)^q(1-u)^r$ satisfies $g(0)=a^q$, $g(1)=0$ and $g(u)>0$ on $(0,1)$. By (17), we have $\theta'(u)>0$ on $(0,1)$, and hence $T(\alpha)$ is strictly increasing on $(0,1)$ by (15). In addition,

$$T_{q,r}(0)=(C_p/a^q)^{1/p}\quad\text{and}\quad T_{q,r}(1)=\infty$$

by (12) and (13), and hence

$$\hat\lambda=(T_{q,r}(0))^p=C_p/a^q$$

by (9). So by (4) and (8), we obtain the exact multiplicity result in this Case (i).

Case (ii) $q/a\leq r<p-1$**.** The proof of Case (ii) are almost the same as that of Case (i), the only difference is that $T_{q,r}(1)\in(0,\infty)$. To show it, letting $\eta=p+1$ in (6), we have

$$\lim_{u\to1^-}\frac{(a+u)^q(1-u)^r}{(1-u)^{p-1}\left(\log\frac{1}{1-u}\right)^{p+1}}=\lim_{u\to1^-}\frac{(a+u)^q}{(1-u)^{p-1-r}\left(\log\frac{1}{1-u}\right)^{p+1}}=\infty$$

for $0 < r < p-1$. So $T_{q,r}(1) \in (0,\infty)$ by Proposition 2, and

$$\bar{\lambda} = (T_{q,r}(1))^p \in (0,\infty)$$

by (10). So by (4), (7) and (8), we obtain the exact multiplicity result in this Case (ii).

Case (iii) $p-1 \le r < q/a$.

It is easy to see that $g(u) = \frac{f(u)}{u^{p-1}} = (a+u)^q(1-u)^r \in C[0,1] \cap C^2(0,1)$ satisfies $g(0) = a^q$, $g(1) = 0$, and

$$\begin{aligned} \theta'(u) &= (p-1)f(u) - uf'(u) = -u^p g'(u) \\ &= (q+r)\, u^p (a+u)^{q-1}(1-u)^{r-1}(u-A) \begin{cases} < 0 & \text{on } (0,A), \\ = 0 & \text{when } u = A, \\ > 0 & \text{on } (A,1), \end{cases} \end{aligned}$$

where $A \equiv \frac{q-ar}{q+r} \in (0,1)$. So g is of weak Allee effect type. In addition, we compute that

$$(\log g(u))'' = -\frac{q}{(a+u)^2} - \frac{r}{(1-u)^2} < 0 \quad \text{on } (A,1).$$

So g is log-concave on $(A,1)$. By Theorem 1, $T(\alpha)$ has exactly one critical point, a minimum, on $(0,1)$. In addition,

$$T_{q,r}(0) = (C_p/a^q)^{1/p} \quad \text{and} \quad T_{q,r}(1) = \infty$$

by (12) and (13), and hence

$$\hat{\lambda} = (T_{q,r}(0))^p = C_p/a^q$$

by (9). So by (4) and (8), we obtain the exact multiplicity result in this Case (iii). Note that, for $\lambda^* < \lambda < \hat{\lambda}$, the ordering of u_λ, v_λ can be proved easily; we omit it.

Cases (iv)–(vi) $0 < r < \min\{q/a, p-1\}$. By similar argument in the proof of Case (ii) and Case (iii), we can prove that $T(\alpha)$ has exactly one critical point, a minimum, on $(0,1)$. Moreover,

$$T_{q,r}(0) = (C_p/a^q)^{1/p} \quad \text{and} \quad T_{q,r}(1) \in (0,\infty).$$

So

$$\hat{\lambda} = (T_{q,r}(0))^p = C_p/a^q \quad \text{and} \quad \bar{\lambda} = (T_{q,r}(1))^p \in (0,\infty)$$

by (9) and (10).

For any fixed $q > 0$, if $q/a < p-1$,

$$\lim_{r \to (q/a)^-} \frac{T_{q,r}(1)}{T_{q,r}(0)} = \lim_{q \to (ar)^+} \frac{T_{q,r}(1)}{T_{q,r}(0)} = c(r)^{1/p} \in (1,\infty)$$

by Lemma 4(ii). So applying Lemma 4(i), there exists a positive $\tilde{r}(q) < \min\{ar, p-1\}$ such that

$$\frac{T_{q,r}(1)}{T_{q,r}(0)} \begin{cases} < 1 & \text{if } r \in (0, \tilde{r}(q)), \\ = 1 & \text{if } r = \tilde{r}(q), \\ > 1 & \text{if } r \in (\tilde{r}(q), \min\{q/a, p-1\}). \end{cases} \tag{18}$$

If $0 < r < \tilde{r}(q)$,

$$\frac{\bar{\lambda}}{\hat{\lambda}} = \left(\frac{T_{q,r}(1)}{T_{q,r}(0)}\right)^p > \lim_{r\to 0^+} \left(\frac{T_{q,r}(1)}{T_{q,r}(0)}\right)^p > (\frac{a}{a+1})^q \tag{19}$$

by Lemma 4(i). If $\tilde{r}(q) < r < \min\{q/a, p-1\}$,

$$\frac{\bar{\lambda}}{\hat{\lambda}} = \left(\frac{T_{q,r}(1)}{T_{q,r}(0)}\right)^p < \lim_{q\to (ar)^+} \left(\frac{T_{q,r}(1)}{T_{q,r}(0)}\right)^p = c(r) \tag{20}$$

by Lemma 4(ii).

So by (4), (7), (8), (18)–(20), we obtain the exact multiplicity result and all properties of Cases (iv)–(vi). Note that the ordering of u_λ, v_λ can be proved easily; we omit it.

Case (vii) For any fixed $r \in (0, p-1)$, $\lim_{q\to(ar)^+} \frac{T_{q,r}(1)}{T_{q,r}(0)} = c(r)^{1/p} \in (1,\infty)$ by Lemma 4(ii). So there exists $\tilde{q}(r) \in (ar, \infty)$ such that

$$\frac{T_{q,r}(1)}{T_{q,r}(0)} \begin{cases} > 1 & \text{if } q \in (ar, \tilde{q}(r)), \\ = 1 & \text{if } q = \tilde{q}(r), \\ < 1 & \text{if } q \in (\tilde{q}(r), \infty). \end{cases} \tag{21}$$

By (18) and (21), we obtain that $\tilde{r}(q)$ is a continuous, strictly increasing function of q on $(0,\infty)$, $\lim_{q\to 0^+} \tilde{r}(q) = 0$ and $\lim_{q\to\infty} \tilde{r}(q) = p-1$. Indeed, $\tilde{q}(r)$ is the inverse function of $\tilde{r}(q)$ on $(0, p-1)$. More precisely, $\tilde{q}(r)$ is a continuous, strictly increasing function of r on $(0, p-1)$, $\lim_{r\to 0^+} \tilde{q}(r) = 0$ and $\lim_{r\to(p-1)^-} \tilde{q}(r) = \infty$.

The proof of Theorem 3 is now complete.

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