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Formulas For Sequential Pareto Subdifferentials Of The Sums Of Vector Mappings And Applications To Optimality Conditions*

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Abstract

In this paper, we establish three new sequential weak, proper and strong Pareto subdifferential sums rule formulas for m cone-convex vector valued mappings with $m \geq 2$. As an application, we derive necessary and sufficient sequential weak, proper and strong efficiency optimality conditions for general vector optimization problem with geometric and cone constraints. We also give sequential characterizations of weak and strong efficient solutions of general multi-objective fractional programming problem with geometric and cone constraints. The sequential results are stated without any qualification condition.

1 Introduction

The sequential subdifferential calculus and the sequential efficiency are important and active topics of Mathematical Optimization. Recently, Laghdir et al. [6] have shown a sequential formula for the weak and proper Pareto subdifferential of the sum of two proper convex lower semicontinuous (lsc) vector valued mappings. As a corollary, they derived sequential efficiency optimality conditions for vector optimization problems with geometric constraint. The contribution [6] motivates the present work.

In this paper, by applying interesting results of Boţ and Wanka [1] and Jeyakumar [5], we obtain three new sequential formulas without conditions of qualification for the weak, proper and strong Pareto subdifferential of the sums of $m \geq 2$ proper, cone-convex and Penot-Théra lower semicontinuous vector valued mappings. The first formula is expressed in terms of the epigraphs of the conjugate of the data vector valued mappings. The second involves the approximate subdifferential. The third one is by means of the scalar subdifferential and extends to m vector mappings the sum rule formula of [6]. It is worth noting that in the latter situation the induction principle is useless. As an application, we provide sequential without a constraint qualification

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necessary and sufficient optimality conditions for weak, proper and strong efficient solution of general vector optimization problem with geometric and cone constraints generalizing the corresponding result of [6]. We also give sequential weak and strong efficiency for general multi-objective fractional programming problem with geometric and cone constraints. To the best of our knowledge it is the first time sequential Pareto subdifferential calculus is used in the context of multi-objective fractional optimization to derive sequential efficiency optimality conditions.

The outline of this work is as follows. The next section presents preliminary facts. In section 3, we establish the sequential Pareto subdifferentials sums rule formulas for vector mappings. In sections 4 and 5 we derive the sequential efficiency optimality conditions for vector and multi-objective fractional optimization respectively.

2 Preliminaries

Throughout this paper, let X and Y be two real reflexive Banach spaces paired in duality by $\langle .,. \rangle$ with their topological duals X^* and Y^*. For simplicity, the norms and the dual norms as well as the associated topologies are denoted by $\|.\|$ and $\|.\|_*$ respectively. We will use the symbol w^* for the weak-star topology on the dual spaces and τ_R for the Euclidean topology on the real line R. The product space $X \times Y$ will be endowed with the norm $\|(x,y)\| := \sqrt{\|x\|^2 + \|y\|^2}$ and similarly the norm on $X^* \times Y^*$ will be choosen.

Let Y_+ be a nontrivial convex cone of Y with nonempty topological interior int Y_+. The associated dual and strict polar cones are :

$$Y_+^* := \{y^* \in Y^* \ : \ \langle y^*, y \rangle \geq 0, \ \forall y \ \in Y_+\},$$

$$(Y_+^*)^\circ := \{y^* \in Y^* \ : \ \langle y^*, y \rangle > 0, \ \forall y \ \in Y_+ \setminus l(Y_+)\}.$$

When the lineality $l(Y_+) := Y_+ \cap -Y_+$ reduces to $\{0\}$, Y_+ is said to be pointed. The cone Y_+ induces the following binary relations :

$$\begin{aligned} y_1 \quad &\leq_{Y_+} \quad y_2 :\Longleftrightarrow \ y_2 - y_1 \in Y_+, \\ y_1 \quad &<_{Y_+} \quad y_2 :\Longleftrightarrow \ y_2 - y_1 \in \text{int } Y_+, \\ y_1 \quad &\lneq_{Y_+} \quad y_2 :\Longleftrightarrow \ y_2 - y_1 \in Y_+ \backslash l(Y_+), \end{aligned}$$

for $y_1, y_2 \in Y$. With respect to " $\leq_{Y_+}$ " the augmented set $Y \cup \{+\infty_Y\}$ is considered where $+\infty_Y$ is an abstract element verifying natural relations :

$$y \leq_{Y_+} +\infty_Y,$$

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y,$$

$$\alpha \cdot (+\infty_Y) = +\infty_Y,$$

for every $y \in Y \cup \{+\infty_Y\}$ and $\alpha \geq 0$.

DEFINITION 2.1. Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be a vector valued mapping.

- f is said to be proper if its effective domain

$$\text{dom}f := \{x \in X \ : \ f(x) \in Y\} \neq \emptyset.$$

- f is said to be Y_+-convex if for every $\lambda \in [0,1]$ and $x_1, x_2 \in X$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{Y_+} \lambda f(x_1) + (1-\lambda) f(x_2).$$

- f is said to be Y_+-epi-closed if its epigraph

$$\text{epi}f := \{(x,y) \in X \times Y \ : \ f(x) \leq_{Y_+} y\} \text{ is closed.}$$

- f is said to be lower semicontinuous [7, 8] at $\bar{x} \in \text{dom}f$ if for every neighborhood V of $f(\bar{x})$ in Y, there exists a neighborhood U of $\bar{x}$ such that

$$f(U) \subseteq (V + Y_+) \cup \{+\infty_Y\}.$$

  When $f(\bar{x}) = +\infty_Y$, f is said to be lower semicontinuous at $\bar{x}$ if for any $y \in Y$, any neighborhood V of y, there exists a neighborhood U of $\bar{x}$ such that the above inclusion is satisfied.

  f is said to be lower semicontinuous if it is lower semicontinuous at every point of X.

- The weak subdifferential of f at $\bar{x} \in \text{dom}f$ is

$$\partial^w f(\bar{x}) := \{A \in \text{L}(X,Y) \ : \ \nexists x \in X,\ f(x) - f(\bar{x}) <_{Y_+} A(x - \bar{x})\}.$$

- The proper subdifferential of f at $\bar{x} \in \text{dom}f$ is

$$\begin{aligned}\partial^p f(\bar{x}) :=\{&A \in \text{L}(X,Y) : \exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that} \\ &Y_+ \setminus l(Y_+) \subseteq \text{int}\hat{Y}_+,\ \nexists x \in X,\ f(x) - f(\bar{x}) \lneq_{\hat{Y}_+} A(x - \bar{x})\}.\end{aligned}$$

- The strong subdifferential of f at $\bar{x} \in \text{dom}f$ is

$$\partial^s f(\bar{x}) := \{A \in \text{L}(X,Y) \ : \ \forall x \in X,\ A(x - \bar{x}) \leq_{Y_+} f(x) - f(\bar{x})\}.$$

Here $\text{L}(X,Y)$ is the space of linear continuous operators from X to Y.

REMARK 2.2. By applying Proposition 1.1 in [7], we obtain that if $f : X \longrightarrow Y \cup \{+\infty_Y\}$ is lower semicontinuous in the sense of Penot-Théra then the scalar function $y^* \circ f$ for any $y^* \in Y_+^* \setminus \{0\}$ is lower semicontinuous. This fact is needed.

DEFINITION 2.3. Let S be a nonempty subset of X.

- The vector indicator mapping of S is

$$\begin{aligned}\delta_S^v : \quad & X \longrightarrow Y \cup \{+\infty_Y\} \\ & x \longrightarrow \begin{cases} 0 & \text{if } x \in S, \\ +\infty_Y & \text{otherwise.} \end{cases}\end{aligned}$$

- The vector normal cone $N_S^v(\bar{x})$ of S at $\bar{x} \in S$ is the strong subdifferential of δ_S^v at $\bar{x}$.

REMARK 2.4. If $Y = R$, $Y_+ = [0, +\infty[$, δ_S^v reduces to the scalar indicator function denoted by δ_S and $N_S^v(\bar{x})$ becomes, for S convex, the classical normal cone defined by

$$N_S(\bar{x}) := \{x^* \in X^* \ : \ \forall x \in S, \ \langle x^*, x - \bar{x}\rangle \leq 0\}.$$

Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be a vector valued mapping and S a nonempty subset of X. The vector minimization problem

$$(P) \qquad \inf_{x \in S} f(x)$$

is considered in the following senses

DEFINITION 2.5. $\bar{x} \in dom f \cap S$ is called :

- a weak efficient solution of (P) if $\nexists x \in S, \ f(x) <_{Y_+} f(\bar{x})$.
- a proper efficient solution of (P) if

  $$\exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that } Y_+ \setminus l(Y_+) \subseteq \text{int}\hat{Y}_+, \ \nexists x \in S, \ f(x) \lneq_{\hat{Y}_+} f(\bar{x}).$$

- a strong efficient solution of (P) if $\forall x \in S, \ f(\bar{x}) \leq_{Y_+} f(x)$.

The set of weak, proper and strong efficient solutions of (P) are respectively denoted by $E_w(f, S)$, $E_p(f, S)$ and $E_s(f, S)$.

We deduce two useful relations ($\sigma \in \{w, p, s\}$) (see also [4]).

$$\bar{x} \in E_\sigma(f, X) \Longleftrightarrow 0 \in \partial^\sigma f(\bar{x}). \tag{1}$$

$$\bar{x} \in E_\sigma(f, S) \Longleftrightarrow \bar{x} \in E_\sigma(f + \delta_S^v, X). \tag{2}$$

REMARK 2.6. In the particular case where $Y = R^q$ is the q dimensional Euclidian space, $Y_+ = R_+^q$ is the nonnegative orthant and (P) is of the form

$$\inf_{x \in S} \{r_1(x), \cdots, r_q(x)\},$$

with $r_1, ..., r_q : X \longrightarrow R$, the definition of weak efficient solution becomes : $\bar{x} \in S$ is a weak efficient solution of (P) if there does not exist $x \in S$ such that

$$r_i(x) < r_i(\bar{x})$$

for all $i \in \{1, ..., q\}$.

The next convention is adopted. Let $g : Y \longrightarrow R \cup \{+\infty\}$ be a scalar function and $h : X \longrightarrow Y \cup \{+\infty_Y\}$ a vector valued mapping then the composed function $g \circ h : X \longrightarrow R \cup \{+\infty\}$ is given by

$$(g \circ h)(x) := \begin{cases} g(h(x)) & \text{if } x \in \text{dom} h, \\ +\infty & \text{otherwise.} \end{cases}$$

Analogously the composed mapping is defined when g is vector valued.

For convenience we also recall well known concepts from scalar convex analysis.

DEFINITION 2.7. Let $f : X \to R \cup \{+\infty\}$ be a convex scalar function.

- The ϵ-approximate subdifferential of f at $\bar{x} \in dom f$ with $\epsilon \geq 0$ is

$$\partial_\epsilon f(\bar{x}) := \{x^* \in X^* \ : \ \forall x \in X,\ f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - \epsilon\}.$$

- The subdifferential of f at $\bar{x} \in dom f$ is :

$$\partial f(\bar{x}) := \{x^* \in X^* \ : \ \forall x \in X,\ f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle\}.$$

- The conjugate of f is the function given by :

$$\begin{array}{cccc} f^* & : & X^* & \longrightarrow & R \cup \{-\infty, +\infty\} \\ & & x^* & \longmapsto & \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} \end{array}.$$

We point out the relation between the subdifferential and the conjugate :

$$\partial f(\bar{x}) := \{x^* \in X^* \ : \ f(\bar{x}) + f^*(x^*) = \langle x^*, \bar{x}\rangle\}.$$

The following results will be useful.

LEMMA 2.8 (Boţ and Wanka [1]). Let $f_1, f_2 : X \longrightarrow R \cup \{+\infty\}$ be two proper, convex and lower semicontinuous scalar functions verifying $dom\, f_1 \cap dom\, f_2 \neq \emptyset$. Then,

$$\text{epi}(f_1 + f_2)^* = \text{cl}_{w^* \times \tau_R}(\text{epi} f_1^* + \text{epi} f_2^*),$$

where cl denotes the topological closure.

Using induction, the fact that the conjugate is convex lower semicontinuous and the relation $cl(clA + clB) = cl(A + B)$ for subsets in a topological vector space, the precedent lemma is easily reformulated for m scalar functions with $m \geq 2$ and for the dual norm.

LEMMA 2.9. Let $f_1, ..., f_m : X \longrightarrow R \cup \{+\infty\}$ be m proper, convex and lower semicontinuous scalar functions satisfying $\cap_{i=1}^m dom\, f_i \neq \emptyset$. Then,

$$\text{epi}(\sum_{i=1}^m f_i)^* = \text{cl}_{\|.\|_* \times \tau_R}(\sum_{i=1}^m \text{epi}\, f_i^*).$$

LEMMA 2.10 (Jeyakumar [5]). Let $f : X \longrightarrow R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous scalar function. Let $a \in dom\, f$. Then

$$\text{epi}\, f^* = \bigcup_{\epsilon \geq 0} \{(s, \epsilon + \langle s, a\rangle - f(a)) : s \in \partial_\epsilon f(a)\}.$$

LEMMA 2.11 (Thibault [9], [2]). Let $f : X \longrightarrow R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous scalar function. Let $\bar{x} \in dom\ f$, then for any real number $\epsilon > 0$ and any $x^* \in \partial_\epsilon f(\bar{x})$, there exists $(x_\epsilon, x_\epsilon^*) \in X \times X^*$ such that

$$x_\epsilon^* \in \partial f(x_\epsilon),$$

$$\|x_\epsilon - \bar{x}\| \leq \sqrt{\epsilon},$$

$$\|x_\epsilon^* - x^*\|_* \leq \sqrt{\epsilon},$$

$$|f(x_\epsilon) - \langle x_\epsilon^*, x_\epsilon - \bar{x}\rangle - f(\bar{x})| \leq 2\epsilon.$$

LEMMA 2.12 (El Maghri and Laghdir [4]). Let $f : X \longrightarrow Y \cup \{+\infty_Y\}$ be Y_+-convex vector valued mapping and $\bar{x} \in X$ and

$$Y_+^\sigma := \begin{cases} Y_+^* \setminus \{0\} & \text{if } \sigma \in \{w, s\}, \\ (Y_+^*)^\circ & \text{if } \sigma = p. \end{cases}$$

Case $\sigma \in \{w, p\}$ with Y_+ pointed as $\sigma = p$:

$$\partial^\sigma f(\bar{x}) = \bigcup_{y^* \in Y_+^\sigma} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ f)(\bar{x})\}.$$

Case $\sigma = s$ and Y_+ is closed :

$$\partial^s f(\bar{x}) = \bigcap_{y^* \in Y_+^s} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ f)(\bar{x})\}.$$

REMARK 2.13. In the sequel Z is a normed space. All the above notations, concepts and results stated with Y remain true for Z. Sometimes the departure space will be taken equal to $X \times Y$.

3 Sequential Weak, Proper and Strong Pareto Subdifferential Sums Rules

Using Lemma 2.9 and Lemma 2.10, that describe the epigraph of the conjugate of the sums of functions and the relationship of the epigraph of the conjugate with the approximate subdifferential respectively, and a refined version of the well known Brøndsted-Rockafellar theorem (Lemma 2.11), we show three new sequential without constraint qualification weak, proper and strong Pareto subgradient characterization formulas for the sums of m proper, cone-convex and lower semicontinuous vector valued mappings with $m \geq 2$.

The first sequential formula is by means of the epigraphs of the conjugate of data vector valued mappings.

THEOREM 3.1. Let $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ be m proper, Z_+-convex and lower semicontinuous vector valued mappings. Let $\bar{x} \in \cap_{i=1}^m dom\, f_i$ and suppose that Z_+ is pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$) and there exist sequences $\{(u_{i,n}^*, r_{i,n})\}_n \subseteq epi\,(z^* \circ f_i)^*$ with $i \in \{1, \cdots, m\}$ and $\{t_n\}_n \subseteq R_+$ satisfying

$$\sum_{i=1}^m u_{i,n}^* \xrightarrow{\|.\|_*} z^* \circ A,$$

$$t_n + \sum_{i=1}^m r_{i,n} \xrightarrow[n\to\infty]{} \langle z^* \circ A, \bar{x}\rangle - \sum_{i=1}^m (z^* \circ f_i)(\bar{x}).$$

PROOF. Let $\sigma \in \{w, p\}$ and $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$. By applying Lemma 2.12, there exists some $z^* \in Z_+^\sigma$ such that $z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x})$. For each $x^* \in X^*$, we introduce the function φ_{x^*} by

$$\begin{array}{cccc} \varphi_{x^*} & : & X & \longrightarrow & R \cup \{+\infty\} \\ & & x & \longmapsto & [\sum_{i=1}^m (z^* \circ f_i)(x)] - \langle x^*, x - \bar{x}\rangle \end{array}.$$

Then, one can check that

$$z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x}) \Longleftrightarrow (0, -\varphi_{z^* \circ A}(\bar{x})) \in epi(\varphi_{z^*\circ A})^* \tag{3}$$

and also

$$epi(-\langle z^* \circ A, . - \bar{x}\rangle)^* = \{-z^* \circ A\} \times [-\langle z^* \circ A, \bar{x}\rangle, +\infty[. \tag{4}$$

Using successively (3), Lemma 2.9 and (4), we obtain that

$$A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$$

if and only if

$$(0, -\varphi_{z^*\circ A}(\bar{x})) \in \mathrm{cl}_{\|.\|_* \times \tau_R}((\sum_{i=1}^m \mathrm{epi}\,(z^* \circ f_i)^*) + \{-z^* \circ A\} \times [-\langle z^* \circ A, \bar{x}\rangle, +\infty[)$$

or equivalently there exist $\{(u_{i,n}^*, r_{i,n})\}_n \subseteq epi\,(z^* \circ f_i)^*$ and $\{s_n\}_n \subseteq [-\langle z^* \circ A, \bar{x}\rangle, +\infty[$ $(i \in \{1 \cdots m\})$ such that

$$(\sum_{i=1}^m u_{i,n}^*) - z^* \circ A \xrightarrow{\|.\|_*} 0,$$

$$(\sum_{i=1}^{m} r_{i,n}) + s_n \xrightarrow[n\to\infty]{} -\varphi_{z^*\circ A}(\bar{x}).$$

By putting $t_n := s_n + \langle z^* \circ A, \bar{x}\rangle$ for $n \in N$, the announced result follows. The case $\sigma = s$ is analogous.

The second formula is in terms of the approximate subdifferentials.

THEOREM 3.2. Let $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ be m proper, Z_+-convex and lower semicontinuous vector valued mappings. Let $\bar{x} \in \cap_{i=1}^m \mathit{dom}\, f_i$ and suppose that Z_+ is pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$), there exist sequences $\{\epsilon_n\}_n \subseteq R_+$ and $\{u_{i,n}^*\}_n \subseteq X^*$ with $i \in \{1, \cdots, m\}$ satisfying

$$\epsilon_n \xrightarrow[n\to\infty]{} 0,$$

$$\sum_{i=1}^{m} u_{i,n}^* \overset{\|\cdot\|_*}{\longrightarrow} z^* \circ A,$$

$$u_{i,n}^* \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$ and $n \in N$.

PROOF. The focus is on the case $\sigma \in \{w, p\}$.
($\Longrightarrow$) Let $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$. By applying Theorem 3.1, there exist $z^* \in Z_+^\sigma$ and sequences $\{(u_{i,n}^*, r_{i,n})\}_n \subseteq \mathit{epi}\,(z^* \circ f_i)^*$ with $i \in \{1, \cdots, m\}$ and $\{t_n\}_n \subseteq R_+$ satisfying

$$\sum_{i=1}^{m} u_{i,n}^* \overset{\|\cdot\|_*}{\longrightarrow} z^* \circ A \tag{5}$$

$$t_n + \sum_{i=1}^{m} r_{i,n} \xrightarrow[n\to\infty]{} \langle z^* \circ A, \bar{x}\rangle - \sum_{i=1}^{m} (z^* \circ f_i)(\bar{x}) \tag{6}$$

and according to Lemma 2.10, there exist another sequences $\{\epsilon_{i,n}\}_n \subseteq R_+$ that satisfy

$$u_{i,n}^* \in \partial_{\epsilon_{i,n}}(z^* \circ f_i)(\bar{x}) \tag{7}$$

$$r_{i,n} = \epsilon_{i,n} + \langle u_{i,n}^*, \bar{x}\rangle - (z^* \circ f_i)(\bar{x}) \tag{8}$$

with $i \in \{1, \cdots, m\}$ and $n \in N$. From (7) and by setting $\epsilon_n := \max_{i\in\{1,\cdots,m\}}(\epsilon_{i,n})$ for each $n \in N$, we have

$$u_{i,n}^* \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x})$$

with $i \in \{1, \cdots, m\}$ and $n \in N$. Now in view of (8), it is easy to see that

$$0 \leq \epsilon_n \leq t_n + \textstyle\sum_{i=1}^m r_{i,n} + [\sum_{i=1}^m (z^* \circ f_i)(\bar{x})] - \langle \sum_{i=1}^m u_{i,n}^*, \bar{x}\rangle.$$

Then, by (5) and (6), we obtain that $\epsilon_n \xrightarrow[n\to\infty]{} 0$.

To prove the converse ($\Longleftarrow$), it suffices to sum over i and let $n \to +\infty$ in accordance to the variational inequalities associated to $u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x})$, that is, $(z^* \circ f_i)(x) - (z^* \circ f_i)(\bar{x}) \geq \langle u^*_{i,n}, x - \bar{x}\rangle - \epsilon_n$ for all $x \in X$, with $i \in \{1, \cdots, m\}$ and $n \in N$. Then $z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x})$. We conclude by Lemma 2.12.

The third formula involves the scalar subdifferentials.

THEOREM 3.3. Let $f_1, ..., f_m : X \longrightarrow Z \cup \{+\infty_Z\}$ be m proper, Z_+-convex and lower semicontinuous vector valued mappings. Let $\bar{x} \in \cap_{i=1}^m dom\, f_i$ and suppose that Z_+ is pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exists $z^* \in Z^\sigma_+$ (resp. for every $z^* \in Z^s_+$ as $\sigma = s$), there exist sequences $\{x_{i,n}\}_n \subseteq dom\, f_i$ and $\{x^*_{i,n}\}_n \subseteq X^*$ with $i \in \{1, \cdots, m\}$ satisfying

$$x^*_{i,n} \in \partial(z^* \circ f_i)(x_{i,n}),$$

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},$$

$$\sum_{i=1}^m x^*_{i,n} \xrightarrow{\|\cdot\|_*} z^* \circ A,$$

$$(z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} (z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$ and $n \in N$.

PROOF. We treat the situation with $\sigma \in \{w, p\}$, the case $\sigma = s$ is similar.
($\Longrightarrow$) By Theorem 3.2, $A \in \partial^\sigma(\sum_{i=1}^m f_i)(\bar{x})$ if and only if there exist $z^* \in Z^\sigma_+$, sequences $\{\epsilon_n\}_n \subseteq R_+$ and $\{u^*_{i,n}\}_n \subseteq X^*$ with $i \in \{1, \cdots, m\}$ satisfying

$$\epsilon_n \xrightarrow[n\to\infty]{} 0, \tag{9}$$

$$\sum_{i=1}^m u^*_{i,n} \xrightarrow{\|\cdot\|_*} z^* \circ A, \tag{10}$$

$$u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$ and $n \in N$.

Therefore from Lemma 2.11 with $u^*_{i,n} \in \partial_{\epsilon_n}(z^* \circ f_i)(\bar{x})$, we obtain sequences $\{x_{i,n}\}_n \subseteq dom\, f_i$ and $\{x^*_{i,n}\}_n \subseteq X^*$ with $i \in \{1, \cdots, m\}$ such that

$$x^*_{i,n} \in \partial(z^* \circ f_i)(x_{i,n}),$$

$$\|x_{i,n} - \bar{x}\| \leq \sqrt{\epsilon_n}, \tag{11}$$

$$\|x^*_{i,n} - u^*_{i,n}\|_* \leq \sqrt{\epsilon_n}, \tag{12}$$

$$|(z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle - (z^* \circ f_i)(\bar{x})| \leq 2\epsilon_n, \tag{13}$$

with $i \in \{1, \cdots, m\}$ and $n \in N$.

Hence by letting $n \to +\infty$ in (11) and (13), we get

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},$$

$$(z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} (z^* \circ f_i)(\bar{x}),$$

with $i \in \{1, \cdots, m\}$.

It remains to prove that $\sum_{i=1}^m x^*_{i,n} \xrightarrow{\|\cdot\|_*} z^* \circ A$ and this easily follows by using (10), (12) and (9).

($\Longleftarrow$) The variational inequalities associated to $x^*_{i,n} \in \partial\,(z^* \circ f_i)(x_{i,n})$ lead us to

$$(z^* \circ f_i)(x) - ((z^* \circ f_i)(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle) \geq \langle x^*_{i,n}, x - \bar{x}\rangle, \quad \forall x \in X$$

with $i \in \{1, \cdots, m\}$ and $n \in N$. Thus by summing over i and letting $n \to +\infty$, we obtain $z^* \circ A \in \partial(\sum_{i=1}^m (z^* \circ f_i))(\bar{x})$.

REMARK 3.4. The above formulas are also valid when all data vector valued mappings f_i satisfy that for any $z^* \in Z^*_+ \setminus \{0\}$, the scalar function $z^* \circ f_i$ is lower semicontinuous.

4 Application to Sequential Weak, Proper and Strong Efficiency of General Vector Optimization Problem

In this section we use the sequential Pareto subdifferential calculus to obtain sequential without any constraint qualification necessary and sufficient weak, proper and strong efficient optimality conditions of the following general vector optimization problem with geometric and cone constraints

$$(VOP): \quad \inf_{\substack{x \in C \\ h(x) \in -Y_+}} f(x)$$

where

- $f : X \longrightarrow Z \cup \{+\infty_Z\}$ is proper, Z_+-convex and lower semicontinuous vector valued mapping.
- $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is proper, Y_+-convex and Y_+-epi-closed vector valued mapping.
- C is a nonempty closed convex subset of X.
- Y_+ is a nonempty closed convex cone of Y.

For convenience efficient solutions are shortened as σ-efficient solution with $\sigma \in \{w,p,s\}$.

THEOREM 4.1. Let $\bar{x} \in dom f \cap C \cap h^{-1}(-Y_+)$, $\sigma \in \{w,p,s\}$ and assume Z_+ pointed as $\sigma = p$ (resp. closed as $\sigma = s$). Then, $\bar{x}$ is a σ-efficient solution of (VOP) if and only if there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$), there exist sequences $\{x_n\}_n \subseteq dom\, f$, $\{c_n\}_n \subseteq C$, $\{y_n\}_n \subseteq -Y_+$, $\{(u_n,v_n)\}_n \subseteq epih$ and $\{x_n^*\}_n, \{c_n^*\}_n, \{u_n^*\}_n \subseteq X^*$, $\{y_n^*\}_n, \{v_n^*\}_n \subseteq Y^*$ satisfying

$$x_n^* \in \partial\, (z^* \circ f)(x_n),\; c_n^* \in N_C(c_n),$$

$$y_n^* \in Y_+^*,\; \langle y_n^*, y_n\rangle = 0,$$

$$u_n^* \in \partial\, ((-v_n^*) \circ h)(u_n),\; v_n^* \in -Y_+^*,\; \langle v_n^*, v_n - h(u_n)\rangle = 0,$$

$$x_n \xrightarrow{\|\cdot\|} \bar{x}, c_n \xrightarrow{\|\cdot\|} \bar{x},\; u_n \xrightarrow{\|\cdot\|} \bar{x},\; y_n \xrightarrow{\|\cdot\|} h(\bar{x}),\; v_n \xrightarrow{\|\cdot\|} h(\bar{x}),$$

$$x_n^* + c_n^* + u_n^* \xrightarrow{\|\cdot\|_*} 0, \quad y_n^* + v_n^* \xrightarrow{\|\cdot\|_*} 0,$$

$$(z^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle \xrightarrow[n\to\infty]{} (z^* \circ f)(\bar{x}),$$

$$\langle c_n^*, c_n - \bar{x}\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle y_n^*, y_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle u_n^*, u_n - \bar{x}\rangle + \langle v_n^*, v_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

with $n \in N$.

PROOF. By (2) and (1), $\bar{x}$ is a σ-efficient solution of (VOP) if and only if

$$0 \in \partial^\sigma(f + \delta_C^v + \delta_{-Y_+}^v \circ h)(\bar{x}). \tag{14}$$

Introduce the vector valued mappings with arrival set $Z \cup \{+\infty_Z\}$ defined by : $f_1(x,y) := f(x)$, $f_2(x,y) := \delta_C^v(x)$, $f_3(x,y) := \delta_{-Y_+}^v(y)$, $f_4(x,y) := \delta_{\mathrm{epi}h}^v(x,y)$, where $(x,y) \in X \times Y$. By using the definition of Pareto subdifferentials it is not difficult to see that (14) is equivalent to $(0,0) \in \partial^\sigma(f_1+f_2+f_3+f_4)(\bar{x}, h(\bar{x}))$. Taking into account Remark 3.4 and by Theorem 3.3, there exists $z^* \in Z_+^\sigma$ (resp. for every $z^* \in Z_+^s$ as $\sigma = s$), there exist sequences $\{(x_n,\bar{x}_n)\}_n \subseteq dom\, f \times Y$, $\{(c_n,\bar{c}_n)\}_n \subseteq C \times Y$, $\{(\bar{y}_n, y_n)\}_n \subseteq X \times -Y_+$, $\{(u_n,v_n)\}_n \subseteq epih$ and $\{x_n^*\}_n, \{c_n^*\}_n, \{\bar{y}_n^*\}_n, \{u_n^*\}_n \subseteq X^*$, $\{\bar{x}_n^*\}_n, \{\bar{c}_n^*\}_n, \{y_n^*\}_n, \{v_n^*\}_n \subseteq Y^*$ satisfying for every $n \in N$:

$$\begin{aligned}(x_n^*, \bar{x}_n^*) &\in \partial\,(z^* \circ f_1)(x_n,\bar{x}_n) = \partial\,(z^* \circ f)(x_n) \times \{0\},\\ (c_n^*, \bar{c}_n^*) &\in \partial\,(z^* \circ f_2)(c_n,\bar{c}_n) = N_C(c_n) \times \{0\},\\ (\bar{y}_n^*, y_n^*) &\in \partial\,(z^* \circ f_3)(\bar{y}_n, y_n) = \{0\} \times N_{-Y_+}(y_n),\\ (u_n^*, v_n^*) &\in \partial\,\delta_{\mathrm{epi}h}(u_n,v_n),\\ x_n &\xrightarrow{\|\cdot\|} \bar{x}, c_n \xrightarrow{\|\cdot\|} \bar{x}, \bar{y}_n \xrightarrow{\|\cdot\|} \bar{x}, u_n \xrightarrow{\|\cdot\|} \bar{x},\end{aligned} \tag{15}$$

$$\bar{x}_n \xrightarrow{\|\cdot\|} h(\bar{x}), \bar{c}_n \xrightarrow{\|\cdot\|} h(\bar{x}), y_n \xrightarrow{\|\cdot\|} h(\bar{x}), v_n \xrightarrow{\|\cdot\|} h(\bar{x}),$$
$$x_n^* + c_n^* + \bar{y}_n^* + u_n^* \xrightarrow{\|\cdot\|_*} 0,$$
$$\bar{x}_n^* + \bar{c}_n^* + y_n^* + v_n^* \xrightarrow{\|\cdot\|_*} 0,$$
$$(z^* \circ f)(x_n) - \langle x_n^*, x_n - \bar{x}\rangle - \langle \bar{x}_n^*, \bar{x}_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} (z^* \circ f)(\bar{x}),$$
$$\langle c_n^*, c_n - \bar{x}\rangle + \langle \bar{c}_n^*, \bar{c}_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$
$$\langle \bar{y}_n^*, \bar{y}_n - \bar{x}\rangle + \langle y_n^*, y_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$
$$\langle u_n^*, u_n - \bar{x}\rangle + \langle v_n^*, v_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0.$$

Now, for $(x^*, y^*) \in X^* \times Y^*$, simple computations show that

$$\delta^*_{\mathrm{epi}h}(x^*, y^*) = ((-y^*) \circ h)^*(x^*) + \delta^*_{Y_+}(y^*).$$

Then, for $n \in N$, (15) is equivalent to

$$((-v_n^*) \circ h)^*(u_n^*) + \delta^*_{Y_+}(v_n^*) - \langle u_n^*, u_n\rangle - \langle v_n^*, v_n\rangle = 0. \tag{16}$$

Since $(u_n, v_n) \in epi\, h$, we define $z_n := v_n - h(u_n) \in Y_+$ for $n \in N$. Consequently (16) is reformulated as follows

$$[((-v_n^*) \circ h)^*(u_n^*) + ((-v_n^*) \circ h)(u_n) - \langle u_n^*, u_n\rangle] + [\delta^*_{Y_+}(v_n^*) + \delta_{Y_+}(z_n) - \langle v_n^*, z_n\rangle] = 0.$$

According to Fenchel-Young inequality, this is equivalent to

$$u_n^* \in \partial\,((-v_n^*) \circ h)(u_n),\ v_n^* \in N_{Y_+}(v_n - h(u_n))$$

and from the fact that Y_+ is a convex cone, we obtain

$$v_n^* \in N_{Y_+}(v_n - h(u_n)) \Longleftrightarrow \begin{cases} v_n^* \in -Y_+^* \\ \langle v_n^*, v_n - h(u_n)\rangle = 0 \end{cases}.$$

Therefore the result follows after observing that $\{\bar{x}_n^*\}_n, \{\bar{c}_n^*\}_n,\ \{\bar{y}_n^*\}_n$ are null and $\{\bar{x}_n\}_n, \{\bar{c}_n\}_n, \{\bar{y}_n\}_n$ are superfluous.

5 Application to Sequential Weak and Strong Efficiency in Multi-objective Fractional Programming

Multi-objective fractional optimization problems appear in many practical areas such as economics and management science. In this section we are interested in establishing sequential without any constraint qualification necessary and sufficient weak and strong efficient optimality conditions for the following multi-objective fractional optimization problem.

$$(MFP): \quad \inf_{\substack{x\in C\\ h(x)\in -Y_+}} \left\{\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_q(x)}{g_q(x)}\right\}$$

where :

- $f_1, ..., f_q : X \longrightarrow [0, +\infty[$ are convex and lower semicontinuous scalar functions.
- $g_1, ..., g_q : X \longrightarrow]0, +\infty[$ are concave and upper semicontinuous scalar functions.
- $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is a proper, Y_+-convex and Y_+-epi-closed vector valued mapping.
- C is a nonempty closed convex subset of X.
- Y_+ is a nonempty closed convex cone of Y.

The approach is based on sequential Pareto subdifferential calculus.

THEOREM 5.1. Let $\bar{x} \in C \cap h^{-1}(-Y_+)$ $\sigma \in \{w, s\}$ and

$$\Omega := \{k \in \{1, \cdots, q\} : f_k(\bar{x}) > 0\}.$$

Then $\bar{x}$ is a σ-efficient solution of (MFP) if and only if there exist index set $\Delta \subseteq \{1, \cdots, q\}$ nonempty, $\{\lambda_i\}_{i\in\Delta} \subseteq]0, +\infty[$ (resp. for every Δ and every $\{\lambda_i\}_{i\in\Delta}$ as $\sigma = s$) and there exist sequences $\{x_{i,n}\}_n$, $\{w_{j,n}\}_n \subseteq X$, $\{c_n\}_n \subseteq C, \{y_n\}_n \subseteq -Y_+$, $\{(u_n, v_n)\}_n \subseteq epi\, h$, $\{x^*_{i,n}\}_n$, $\{w^*_{j,n}\}_n$, $\{c^*_n\}_n$, $\{u^*_n\}_n \subseteq X^*$, $\{y^*_n\}_n$, $\{v^*_n\}_n \subseteq Y^*$ with $i \in \Delta$, $j \in \Delta \cap \Omega$ such that :

$$x^*_{i,n} \in \partial f_i(x_{i,n}),\ w^*_{j,n} \in \partial(-g_j)(w_{j,n}),\ c^*_n \in N_C(c_n),$$

$$y^*_n \in Y^*_+,\ \langle y^*_n, y_n\rangle = 0,$$

$$u^*_n \in \partial((-v^*_n) \circ h)(u_n),\ v^*_n \in -Y^*_+,\ \langle v^*_n, v_n - h(u_n)\rangle = 0,$$

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},\ w_{j,n} \xrightarrow{\|\cdot\|} \bar{x},\ c_n \xrightarrow{\|\cdot\|} \bar{x},\ u_n \xrightarrow{\|\cdot\|} \bar{x},\ y_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|} h(\bar{x}),$$

$$[\sum_{i\in\Delta} \lambda_i x^*_{i,n}] + [\sum_{j\in\Delta\cap\Omega} (\lambda_j \frac{f_j(\bar{x})}{g_j(\bar{x})}) w^*_{j,n}] + c^*_n + u^*_n \xrightarrow{\|\cdot\|_*} 0,\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_*} 0,$$

$$f_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} f_i(\bar{x}),$$

$$g_j(w_{j,n}) + \langle w^*_{j,n}, w_{j,n} - \bar{x}\rangle \xrightarrow[n\to\infty]{} g_j(\bar{x}),$$

$$\langle c^*_n, c_n - \bar{x}\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle y^*_n, y_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle u^*_n, u_n - \bar{x}\rangle + \langle v^*_n, v_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

with $i \in \Delta$, $j \in \Delta \cap \Omega$ and $n \in N$.

PROOF. We study the situation $\sigma = w$.
First we proceed using the parametric approach [3] by considering

$$(MFP_{\bar{x}}) : \qquad \inf_{\substack{x\in C \\ h(x)\in -Y_+}} \left\{ f_1(x) - \frac{f_1(\bar{x})}{g_1(\bar{x})} g_1(x), \cdots, f_q(x) - \frac{f_q(\bar{x})}{g_q(\bar{x})} g_q(x) \right\}.$$

Directly from Remark 2.6, we deduce that $\bar{x}$ is a weak efficient solution of (MFP) if and only if $\bar{x}$ is a weak efficient solution of $(MFP_{\bar{x}})$. By (2) and (1), this is equivalent to

$$0 \in \partial^w((\sum_{i=1}^{q} F^i) + (\sum_{i=1}^{q} G^i_{\bar{x}}) + \delta^v_C + \delta^v_{-Y_+} \circ h)(\bar{x}). \tag{17}$$

where

$$F^i: \begin{array}{ccc} X & \longrightarrow & R^q \\ x & \longrightarrow & \big(0, ..., f_i(x), ..., 0\big) \end{array}, \quad G^i_{\bar{x}}: \begin{array}{ccc} X & \longrightarrow & R^q \\ x & \longrightarrow & \big(0, ..., -\frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x), ..., 0\big) \end{array}.$$

Now consider the following auxiliary vector valued mappings with values in R^q :

$$\phi_i(x,y) := F^i(x),$$
$$\varphi_i(x,y) := G^i_{\bar{x}}(x),$$
$$\gamma(x,y) := \delta^v_C(x),$$
$$\psi(x,y) := \delta^v_{-Y_+}(y),$$
$$H(x,y) := \delta^v_{\text{epi}h}(x,y),$$

where $(x,y) \in X \times Y$ and $i \in \{1, \cdots, q\}$. Similar to the preceding section, we have that (17) is equivalent to :

$$(0,0) \in \partial^w((\sum_{i=1}^{q} \phi_i) + (\sum_{i=1}^{q} \varphi_i) + \gamma + \psi + H)(\bar{x}, h(\bar{x})).$$

According to Theorem 3.3, there exist $\lambda^* := (\lambda_1, \cdots, \lambda_q) \in R^q_+ \setminus \{0\}$ and sequences $\{(x_{i,n}, \bar{x}_{i,n})\}_n, \{(w_{i,n}, \bar{w}_{i,n})\}_n \subseteq X \times Y$, $\{(c_n, \bar{c}_n)\}_n \subseteq C \times Y$, $\{(\bar{y}_n, y_n)\}_n \subseteq X \times -Y_+$, $\{(u_n, v_n)\}_n \subseteq epi\, h$, $\{(\hat{x}^*_{i,n}, \bar{x}^*_{i,n})\}_n$, $\{(\hat{w}^*_{i,n}, \bar{w}^*_{i,n})\}_n$, $\{(c^*_n, \bar{c}^*_n)\}_n$, $\{(\bar{y}^*_n, y^*_n)\}_n$, $\{(u^*_n, v^*_n)\}_n \subseteq X^* \times Y^*$ with $i \in \{1, \cdots, q\}$ satisfying

$$(\hat{x}^*_{i,n}, \bar{x}^*_{i,n}) \in \partial\,(\lambda^* \circ \phi_i)(x_{i,n}, \bar{x}_{i,n}) = \partial\,(\lambda_i f_i)(x_{i,n}) \times \{0\},$$

$$(\hat{w}^*_{i,n}, \bar{w}^*_{i,n}) \in \partial\,(\lambda^* \circ \varphi_i)(w_{i,n}, \bar{w}_{i,n}) = \partial\,(\lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})}(-g_i))(w_{i,n}) \times \{0\},$$

$$(c^*_n, \bar{c}^*_n) \in \partial\,(\lambda^* \circ \gamma)(c_n, \bar{c}_n) = N_C(c_n) \times \{0\},$$

$$(\bar{y}^*_n, y^*_n) \in \partial\,(\lambda^* \circ \psi)(\bar{y}_n, y_n) = \{0\} \times N_{-Y_+}(y_n),$$

$$(u^*_n, v^*_n) \in \partial\,(\lambda^* \circ H)(u_n, v_n) = \partial\,\delta_{\text{epi}h}(u_n, v_n), \tag{18}$$

$$x_{i,n} \xrightarrow{\|\cdot\|} \bar{x},\ w_{i,n} \xrightarrow{\|\cdot\|} \bar{x},\ c_n \xrightarrow{\|\cdot\|} \bar{x},\ \bar{y}_n \xrightarrow{\|\cdot\|} \bar{x},\ u_n \xrightarrow{\|\cdot\|} \bar{x},$$

$$\bar{x}_{i,n} \xrightarrow{\|\cdot\|} h(\bar{x}),\ \bar{w}_{i,n} \xrightarrow{\|\cdot\|} h(\bar{x}),\ \bar{c}_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ y_n \xrightarrow{\|\cdot\|} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|} h(\bar{x}),$$

$$(\sum_{i=1}^{q} \hat{x}^*_{i,n}) + (\sum_{i=1}^{q} \hat{w}^*_{i,n}) + c^*_n + \bar{y}^*_n + u^*_n \xrightarrow{\|\cdot\|_*} 0,\ (\sum_{i=1}^{q} \bar{x}^*_{i,n}) + (\sum_{i=1}^{q} \bar{w}^*_{i,n}) + \bar{c}^*_n + y^*_n + v^*_n \xrightarrow{\|\cdot\|_*} 0,$$

$$\lambda_i f_i(x_{i,n}) - \langle \hat{x}^*_{i,n}, x_{i,n} - \bar{x}\rangle - \langle \bar{x}^*_{i,n}, \bar{x}_{i,n} - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} \lambda_i f_i(\bar{x}),$$

$$\lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})}(-g_i)(w_{i,n}) - \langle \hat{w}^*_{i,n}, w_{i,n} - \bar{x}\rangle - \langle \bar{w}^*_{i,n}, \bar{w}_{i,n} - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} \lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})}(-g_i)(\bar{x}),$$

$$\langle c^*_n, c_n - \bar{x}\rangle + \langle \bar{c}^*_n, \bar{c}_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle \bar{y}^*_n, \bar{y}_n - \bar{x}\rangle + \langle y^*_n, y_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

$$\langle u^*_n, u_n - \bar{x}\rangle + \langle v^*_n, v_n - h(\bar{x})\rangle \xrightarrow[n\to\infty]{} 0,$$

with $i \in \{1, \cdots, q\}$ and $n \in N$.

As in the section before, (18) is equivalent to :

$$u^*_n \in \partial\left((-v^*_n) \circ h\right)(u_n),\ v^*_n \in -Y^*_+,\ \langle v^*_n, v_n - h(u_n)\rangle = 0.$$

For $i \in \{1, \cdots, q\}$, $\{\bar{x}^*_{i,n}\}_n$, $\{\bar{w}^*_{i,n}\}_n$, $\{\bar{c}^*_n\}_n$, $\{\bar{y}^*_n\}_n$ are null and $\{\bar{x}_{i,n}\}_n$, $\{\bar{w}_{i,n}\}_n$, $\{\bar{c}_n\}_n$, $\{\bar{y}_n\}_n$ are superfluous. Thus the announced result follows by setting $\Delta := \{k \in \{1, \cdots, q\} : \lambda_k > 0\}$. Similarly we have the result of strong efficient solutions.

REMARK 5.2. Using the approach of sequential Pareto subdifferential calculus, the case of sequential proper efficiency of (MFP) is more delicate. Also, it is interessant to provide the nonsmooth counterpart of Theorem 5.1 or even develop second order approach. These may be the objects of future research.

References

[1] R. I. Boţ and G. Wanka, A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces, Nonlinear Anal., 64(2006), 2787–2804.

[2] A. Brøndsted and R. T. Rockafellar, On the subdifferential of convex functions, Proc. Amer. Math. Soc., 16(1965), 605–611.

[3] W. Dinkelbach, On nonlinear fractional programming, Manag. Sci., 13(1967), 492–498.

[4] M. El Maghri and M. Laghdir, Pareto subdifferential calculus for convex vector mappings and applications to vector optimization, SIAM J. Optim., 19(2009), 1970–1994.

[5] V. Jeyakumar, Asymptotic dual conditions characterizing optimality for infinite convex programs, J. Optim. Theory Appl., 93(1997), 153–165.

[6] M. Laghdir, A. Rikouane, Y. A. Fajri and E. Tazi, Sequential Pareto subdifferential sum rule and sequential efficiency, Appl. Math. E-Notes, 16(2016), 133–143.

[7] M. Laghdir and M. Volle, A general formula for the horizon function of a convex composite function, Arch. Math., 73(1999), 291–302.

[8] J. -P. Penot and M. Théra, Semi-continuous mappings in general topology, Arch. Math., 38(1982), 158–166.

[9] L. Thibault, Sequential convex subdifferential calculus and sequential Lagrange multipliers, SIAM J. Control Optim., 35(1997), 1434–1444.