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A Note On The Exponential Of A Matrix Whose Elements Are All 1*

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Abstract

Let J_n denote an $n \times n$ matrix whose elements are all 1. It is well-known that

$$e^{J_n} = \exp(J_n) = I_n + \frac{e^n - 1}{n} J_n,$$

where I_n is the identity matrix of order n. The aim of our work is to establish a generalization of this equality. To prove the main results, we will use the well-known Helmert matrix.

1 Introduction

In linear algebra and matrix theory there are many special and important matrices. For example, the *exponential of a matrix* is one of them. The exponential of a complex square matrix of order n such as $A_n = [a_{ij}]_{n \times n}$ is defined as:

$$e^{A_n} = \exp(A_n) = \sum_{z=0}^{\infty} \frac{A_n^z}{z!} = I_n + A_n + \frac{A_n^2}{2!} + \frac{A_n^3}{3!} + \frac{A_n^4}{4!} + ..., \tag{1}$$

where I_n is the identity matrix of order n. Let J_n denote an $n \times n$ matrix whose elements are all 1. It is well-known that

$$e^{J_n} = I_n + \frac{e^n - 1}{n} J_n. \tag{2}$$

Since $J_n^z = n^{z-1} J_n$ for $z = 1, 2, ...$, if one puts J_n in equation (1), the proof of (2) is then immediate.

Now, in this paper it is shown that

$$\exp^{(k)}(J_n) = \underbrace{\exp\left(\exp\left(... \exp\left(J_n\right)\right)\right)}_{\text{apply the exponential of } J_n \text{ in } k \text{ times}} .$$

Clearly

$$\exp^{(k+1)}(J_n) = \exp(\exp^{(k)}(J_n)). \tag{3}$$

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We will prove the following equation:

$$\exp^{(k+1)}(J_n) = {}^{k}eI_n + \frac{\exp_e^{k+1}(n) - {}^{k}e}{n} J_n, \tag{4}$$

where ${}^{k}e$ denotes the *power tower of order* k, namely

$${}^{k}e = \underbrace{e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}}_{k}. \tag{5}$$

and $\exp_e^k(n)$ denotes the *iterated exponential* that is defined as:

$$\exp_e^k(n) = e^{e^{\cdot^{\cdot^{\cdot^{e^n}}}}}. \tag{6}$$

Obviously, the special case of iterated exponential is the power tower of order k, i.e., ${}^{k}e = \exp_e^k(1)$.

In subsequent sections, the following notation will be used:

a) $\mathbb{R}$ denotes the set of all real numbers.

b) $Diag[d_1 \quad d_2 \quad ... \quad d_n]$ denotes an $n \times n$ diagonal matrix with diagonal entries $d_1, d_2, ..., d_n$.

c) A_n^{-1} denotes the invers of a matrix A_n.

d) A_n^T denotes the transpose of a matrix A_n.

2 Preliminaries

2.1 Similarity and Diagonalization

In linear algebra two square matrices of order n such as A_n and B_n are said to be *similar*, denoted $A_n \sim B_n$, if there exists an invertible matrix T_n, such that

$$T_n^{-1} A_n T_n = B_n. \tag{7}$$

The matrix T_n is called the *similarity transformation matrix*. Similar matrices have the same set of eigenvalues [1]. Hence, it is important if a matrix is similar to a diagonal matrix, since the eigenvalues of a diagonal matrix are its diagonal elements. In particular, we have the following definition:

DEFINITION 1 ([7, page 3]). A matrix is said to be diagonalizable if it is similar to a diagonal matrix.

Hence, if a real square matrix A_n is diagonalizable, then A_n is similar to a diagonal matrix $D_n = Diag[d_1 \quad d_2 \quad ... \quad d_n]$, and there exists an invertible matrix T_n such that $T_n^{-1} A_n T_n = D_n$ or equivalently $A_n = T_n D_n T_n^{-1}$ which is the *diagonalized form* of A_n.

The diagonalized form of a matrix is useful for some matrix calculations. Consider the following proposition which gives us a simple method to compute the exponential of a diagonalizable matrix:

PROPOSITION 1 ([4, Proposition 2.3]). Let A_n is a diagonalizable matrix with $A_n = T_n D_n T_n^{-1}$, where $D_n = Diag[d_1 \quad d_2 \quad ... \quad d_n]$. Then $e^{A_n} = T_n e^{D_n} T_n^{-1}$, where $e^{D_n} = Diag[e^{d_1} \quad e^{d_2} \quad ... \quad e^{d_n}]$.

Notice that not all matrices can be diagonalizable [1]. However, we know that any symmetric matrix is diagonalizable (see [1, page 255]). Since the matrix J_n that is the subject of the paper is symmetric, therefore the materials of this part, though brief, are very useful to prove the main results in the next section.

2.2 The Helmert Matrix

A *Helmert matrix* of order n is a square matrix that was introduced by H. O. Lancaster in 1965 [5]. The Helmert matrix of order n is defined as:

$$H_n = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{-3}{\sqrt{12}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n}. \tag{8}$$

Moreover, the first row of the Helmert matrix of order n, has the following form

$$[\underbrace{\frac{1}{\sqrt{n}} \quad \frac{1}{\sqrt{n}} \quad \frac{1}{\sqrt{n}} \quad \cdots \quad \frac{1}{\sqrt{n}}}_{n \quad items}]. \tag{9}$$

And the other i-th rows $2 \leq i \leq n$ are formed by

$$[\underbrace{\frac{1}{\sqrt{i(i-1)}} \quad \frac{1}{\sqrt{i(i-1)}} \quad \cdots \quad \frac{1}{\sqrt{i(i-1)}}}_{i-1 \quad items} \quad \frac{-(i-1)}{\sqrt{i(i-1)}} \quad \underbrace{0 \quad \cdots \quad 0}_{n-i \quad items}]. \tag{10}$$

Furthermore, we know that the Helmert matrix is orthogonal [2]:

$$H_n H_n^T = H_n^T H_n = I_n. \tag{11}$$

In other words, $H_n^{-1} = H_n^T$.

The Helmert matrix is usually used in statistics for the analysis of variance (ANOVA), see [2, 6]. Recently, in 2017, R. Farhadian and N. Asadian showed that the Helmert matrix can be used in stochastic processes [3].

3 Main results

First, let us consider the following lemma:

LEMMA 1. Let H_n be the Helmert matrix of order n and $\beta \in \mathbb{R}$. Then

$$\beta J_n H_n^T = \begin{bmatrix} \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}.$$

PROOF. First recall (8)–(10). Hence

$$J_n H_n^T = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}_{n\times n} \times \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & 0 & \frac{-3}{\sqrt{12}} & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}} & 0 & 0 & 0 & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} \frac{n}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}.$$

By multiplying the above equation by β the proof is then complete.

THEOREM 1. Let H_n be the Helmert matrix of order n and

$$D_n = Diag[\gamma + n\beta \quad \gamma \quad \gamma \quad ... \quad \gamma],$$

where $\gamma, \beta \in \mathbb{R}$. Then

$$H_n^T D_n H_n = \gamma I_n + \beta J_n. \tag{12}$$

PROOF. Starting from the left side of (12), we advance the proof:

$H_n^T D_n$

$$= \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \dots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \dots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & \dots & \frac{1}{\sqrt{n(n-1)}} \\ \frac{1}{\sqrt{n}} & 0 & 0 & \frac{-3}{\sqrt{12}} & \dots & \frac{1}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}} & 0 & 0 & 0 & \dots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n} \times \begin{bmatrix} \gamma+n\beta & 0 & 0 & 0 & \dots & 0 \\ 0 & \gamma & 0 & 0 & \dots & 0 \\ 0 & 0 & \gamma & 0 & \dots & 0 \\ 0 & 0 & 0 & \gamma & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \gamma \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} \frac{\gamma+n\beta}{\sqrt{n}} & \frac{\gamma}{\sqrt{2}} & \frac{\gamma}{\sqrt{6}} & \frac{\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n+1)}} \\ \frac{\gamma+n\beta}{\sqrt{n}} & \frac{-\gamma}{\sqrt{2}} & \frac{\gamma}{\sqrt{6}} & \frac{\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n+1)}} \\ \frac{\gamma+n\beta}{\sqrt{n}} & 0 & \frac{-2\gamma}{\sqrt{6}} & \frac{\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n+1)}} \\ \frac{\gamma+n\beta}{\sqrt{n}} & 0 & 0 & \frac{-3\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n+1)}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma+n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & \frac{-(n-1)\gamma}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n}$$

$$= \begin{bmatrix} \frac{\gamma}{\sqrt{n}} & \frac{\gamma}{\sqrt{2}} & \frac{\gamma}{\sqrt{6}} & \frac{\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n-1)}} \\ \frac{\gamma}{\sqrt{n}} & \frac{-\gamma}{\sqrt{2}} & \frac{\gamma}{\sqrt{6}} & \frac{\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n-1)}} \\ \frac{\gamma}{\sqrt{n}} & 0 & \frac{-2\gamma}{\sqrt{6}} & \frac{\gamma}{\sqrt{12}} & \dots & \frac{\gamma}{\sqrt{n(n-1)}} \\ \frac{\gamma}{\sqrt{n}} & 0 & 0 & \frac{-3\gamma}{12} & \dots & \frac{\gamma}{\sqrt{n(n-1)}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma}{\sqrt{n}} & 0 & 0 & 0 & \dots & \frac{-(n-1)\gamma}{\sqrt{n(n-1)}} \end{bmatrix}_{n\times n} + \begin{bmatrix} \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}$$

$$= \gamma H_n^T + \begin{bmatrix} \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n\beta}{\sqrt{n}} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}. \tag{13}$$

Using Lemma 1 in (13), we obtain

$$H_n^T D_n = \gamma H_n^T + \beta J_n H_n^T. \tag{14}$$

Since the Helmert matrix is orthogonal (see (11)), by multiplying both sides of equation (14) by the Helmert matrix from the right, we have

$$H_n^T D_n H_n = (\gamma H_n^T + \beta J_n H_n^T) H_n = \gamma \overbrace{H_n^T H_n}^{I_n} + \beta J_n \overbrace{H_n^T H_n}^{I_n} = \gamma I_n + \beta J_n.$$

The theorem is proved.

COROLLARY 1. Let H_n be the Helmert matrix of order n. Then

$$J_n \sim Diag[n \quad 0 \quad ... \quad 0],$$

with similarity transformation matrix H_n.

PROOF. Using Theorem 1 with $\gamma = 0$ and $\beta = 1$ and equation (7), the proof is immediate.

COROLLARY 2. The matrix e^{J_n} is a diagonalizable matrix as

$$e^{J_n} = H_n^T D_n H_n,$$

where H_n denotes the Helmert matrix of order n and $D_n = Diag[e^n \quad 1 \quad ... \quad 1]$.

PROOF. Recall equation (2) of e^{J_n}. Then the corollary is an immediate consequence of Theorem 1 (with $\gamma = 1$ and $\beta = \frac{e^n-1}{n}$), equation (7) and Definition 1.

Now, in the next theorem we shall prove (4).

THEOREM 2.

$$\exp^{(k+1)}(J_n) = {}^k e I_n + \frac{\exp_e^{k+1}(n) - {}^k e}{n} J_n.$$

PROOF. By Corollary 2, we know that

$$\exp(J_n) = H_n^T \times \begin{bmatrix} e^n & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n} \times H_n.$$

(3), (7), Definition 1, Proposition 1, Corollary 2 and Theorem 1 give

$$\begin{aligned} \exp^{(2)}(J_n) &= \exp(\exp(J_n)) = H_n^T \times \begin{bmatrix} e^{e^n} & 0 & \dots & 0 \\ 0 & e & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e \end{bmatrix}_{n \times n} \times H_n \\ &= eI_n + \frac{e^{e^n} - e}{n} J_n. \end{aligned} \tag{15}$$

(3), (7), Definition 1, Proposition 1, (15) and Theorem 1 give

$$\exp^{(3)}(J_n) = \exp(\exp^{(2)}(J_n)) = H_n^T \times \begin{bmatrix} e^{e^{e^n}} & 0 & \dots & 0 \\ 0 & e^e & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^e \end{bmatrix}_{n \times n} \times H_n$$

$$= \quad e^{e} I_n + \frac{e^{e^{e^{n}}} - e^{e}}{n} J_n. \tag{16}$$

(3), (7), Definition 1, Proposition 1, (16) and Theorem 1 give

$$\begin{aligned}\exp^{(4)}(J_n) &= \exp(\exp^{(3)}(J_n)) = H_n^T \times \begin{bmatrix} e^{e^{e^{e^{n}}}} & 0 & \dots & 0 \\ 0 & e^{e^{e}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{e^{e}} \end{bmatrix}_{n\times n} \times H_n \\ &= e^{e^{e}} I_n + \frac{e^{e^{e^{e^{n}}}} - e^{e^{e}}}{n} J_n.\end{aligned}$$

In general, since (3) is a recurrence relation, repeating the above actions (use the equation (7), Definition 1, Proposition 1 and Theorem 1 for $\exp^{(r)}(J_n)$ where $r = 5, 6, ..., k, k+1$), we have

$$\exp^{(k+1)}(J_n) = e^{e^{\cdot^{\cdot^{\cdot^{e}}}}} I_n + \frac{e^{e^{\cdot^{\cdot^{\cdot^{e^{e^{n}}}}}}} - e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}}{n} J_n. \tag{17}$$

Using (5) and (6) (with $\exp_e^{k+1}(n) = e^{e^{\cdot^{\cdot^{\cdot^{e^{e^{n}}}}}}}$) in (17) we obtain

$$\exp^{(k+1)}(J_n) = {}^{k}e I_n + \frac{\exp_e^{k+1}(n) - {}^{k}e}{n} J_n.$$

The theorem is proved.

We can also establish a similar generalization for the matrix $n^{-1}J_n$. In fact since $n^{-1}J_n$ is an orthogonal projection matrix onto the space spanned by the vector of $\mathbf{1}_n$ it has a single eigenvalue of 1 which is the dimension of the space on which it projects and an eigenvalue of zero with multiplicity $n-1$. Hence, $n^{-1}J_n$ is an idempotent matrix. Following [1, page 248], if A_n is an idempotent matrix, then $\exp(A_n) = I_n + (e-1)A_n$. Therefore $\exp\left(n^{-1}J_n\right) = I_n + \frac{e-1}{n} J_n$. Now, let us consider the following theorem to generalize the equality $\exp\left(n^{-1}J_n\right) = I_n + \frac{e-1}{n} J_n$ in order to get $\exp^{(k+1)}\left(n^{-1}J_n\right)$.

THEOREM 3.

$$\exp^{(k+1)}\left(n^{-1}J_n\right) = {}^{k}e I_n + \frac{{}^{k+1}e - {}^{k}e}{n} J_n.$$

PROOF. Since $\exp\left(n^{-1}J_n\right) = I_n + \frac{e-1}{n} J_n$, so by Theorem 1, we have $\exp\left(n^{-1}J_n\right) = H_n^T D_n H_n$, where $D_n = Diag[e \quad 1 \quad ... \quad 1]$ and H_n is the Helmert matrix of order n. Then the proof is similar to the proof of the former theorem. The theorem is proved.

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