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Existence Of Positive Radial Solutions For $(p(x), q(x))$-Laplacian Systems*

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Abstract

In this paper, we prove the existence of weak positive radial solutions for a system of differential equations with some given conditions via sub-super solutions concept.

1 Introduction

We consider the following system of differential equations

$$\begin{cases} -\triangle_{p(x)}u = \lambda^{p(x)}\left[\lambda_1 f(v) + \mu_1 h(u)\right] \text{ in } \Omega, \\ -\triangle_{q(x)}v = \theta^{q(x)}\left[\lambda_2 g(u) + \mu_2 \gamma(v)\right] \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where

$$\Omega = B(0, r) \subset \mathbb{R}^N, \tag{2}$$

$\lambda, \theta, \lambda_1, \lambda_2, \mu_1, \mu_2$ are positive parameters, and

$$1 < p(x), q(x) \in C^1\left(\overline{\Omega}\right)$$

are radial symmetric positive functions, that is to say

$$p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|). \tag{3}$$

Operator $\Delta_{p(x)}$ is a $p(x)$-Laplacian defined as:

$$\Delta_{p(x)}u = \text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right),$$

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with $f,\ g,\ h,\ \gamma$ are monotone functions in $[0,+\infty[$ and satisfy:

$$\lim_{u\to+\infty} f(u)=+\infty,\quad \lim_{u\to+\infty} g(u)=+\infty,\quad \lim_{u\to+\infty} h(u)=+\infty,\quad \lim_{u\to+\infty} \gamma(u)=+\infty.$$

The differential equations and variational problems with nonstandard $p(x)$-growth conditions has been extensively studied in the last two decades and it is a new and interesting topic. It modelized from nonlinear elasticity theory, electro-rheological fluids, etc. (For more information see [10, 22]). Many results regarding the existence of solution of this kind of problems are given by many authors, see for example [1, 2, 3, 8, 9, 10, 12, 13, 15, 16, 18]. Moreover, in [7, 11, 15, 20], the regularity and existence of solutions for some class of this problem has been studied, considering that $p(x)=q(x)=p$ (a constant). Then, in [11], the author considered the existence and nonexistence of positive weak solutions to the following class of quasilinear elliptic system

$$\begin{cases} -\triangle_p u=\lambda u^{\alpha}v^{\gamma} \text{ in } \Omega, \\ -\triangle_q v=\lambda u^{\delta}v^{\beta} \text{ in } \Omega, \\ \quad u=v=0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

where, he used the first eigenfunction to construct the subsolution of problem (4) and he got the following results:

(i) If $\alpha,\beta\geq 0,\ \ \gamma,\delta>0$ and $\theta=(p-1-\alpha)(q-1-\beta)-\gamma\delta>0$, then the problem (4) has a positive weak solution for each $\lambda>0$.

(ii) If $\theta=0$ and $p\gamma=q(p-1-\alpha)$, then there exists $\lambda_0>0$ such that for $0<\lambda<\lambda_0$. Thus, problem (4) has no nontrivial nonnegative weak solution. We refer to [9, 19] for further system generalizations of (4).

In this current paper, motivated by the previous results given for some classes of the differential equations and variational problems with nonstandard $p(x)$-growth conditions in the previous mentioned references and our obtained results in [23, 24], we prove the existence of weak positive radial solutions of a new class of the system of differential equations with respect to (2) and (3), while maintaining the symmetry conditions in [23, 24].

The outline of the paper is as follows: In section 2, we introduce some necessary technical assumptions and auxiliary results. Then in section 3 we give our main result which is the existence of weak positive radial solutions of a new class of the system of differential equations (1) via sub-super solutions concept.

2 Preliminary Results

In this section, we need to introduce $W_0^{1,p(x)}(\Omega)$. First, we give some basic spaces properties of $W_0^{1,p(x)}(\Omega)$. We define

$L^{p(x)}(\Omega)$

$$= \left\{ u : u \text{ is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}$$

with the norm

$$|u(x)|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

and

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. Assume that:

(H1) $\Omega = B(0, r) \subset \mathbb{R}^N$ is an open ball with center 0 and radius $r > 0$;

(H2) $p(x), q(x) \in C^1(\overline{\Omega})$ are radial symmetric functions, $1 < p^- \leq p^+$ and $1 < q^- \leq q^+$;

(H3) $f, g, h, \gamma : [0, +\infty[\to \mathbb{R}$ are C^1, monotone functions such that

$$\lim_{s\to+\infty} f(s) = +\infty, \quad \lim_{s\to+\infty} g(s) = +\infty, \quad \lim_{s\to+\infty} h(s) = +\infty, \quad \lim_{s\to+\infty} \gamma(s) = +\infty;$$

(H4) $\lim_{s\to+\infty} \frac{f\left(M(g(s))^{\frac{1}{q^- -1}}\right)}{s^{p^- -1}} = 0$ for all $M > 0$;

(H5) $\lim_{s\to+\infty} \frac{h(s)}{s^{p^- -1}} = 0$ and $\lim_{s\to+\infty} \frac{\gamma(s)}{s^{q^- -1}} = 0$.

We define

$$\langle L(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \forall u, v \in W_0^{1,p(x)}(\Omega).$$

Thus

$$L : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega) \right)^*$$

is a continuous, bounded and strictly monotone operator, and it is a homeomorphism, see [17, Theorem 3.1].

Define $A : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega) \right)^*$ as

$$\langle A(u), \varphi \rangle = \int_{\Omega} \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + h(x, u) \varphi \right) dx, \text{ for all } u, \varphi \in W_0^{1,p(x)}(\Omega),$$

where $h(x,u)$ is continuous on $\overline{\Omega}\times\mathbb{R}$.and $h(x,.)$ is increasing. It can be checked that A is a continuous bounded mapping according to the result of Lemma 1 in [23].

DEFINITION 1. Let $(u,v)\in\left(W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)\right)$, (u,v) is said a weak solution of (1) if it satisfies

$$\begin{cases}\int_\Omega |\nabla u|^{p(x)-2}\nabla u.\nabla\varphi dx=\int_\Omega \lambda^{p(x)}\left[\lambda_1 f(v)+\mu_1 h(u)\right]\varphi dx,\\ \int_\Omega |\nabla v|^{q(x)-2}\nabla v.\nabla\psi dx=\int_\Omega \theta^{q(x)}\left[\lambda_2 g(u)+\mu_2\gamma(v)\right]\psi dx,\end{cases}$$

for all $(\varphi,\psi)\in\left(W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)\right)$ with $(\varphi,\psi)\geq 0$.

3 Main Result

In the present paper, we use $(\lambda,\theta)>(\lambda^*,\theta^*)$ to denote $\lambda>\lambda^*,\theta>\theta^*$ and the same meaning for other cases, and denote by $\rho(x)=|x|$. Then we have the following result:

THEOREM 1. If (H1)–(H5) hold, then there exists $(\lambda^*,\theta^*)>(0,0)$ such that for any $(\lambda,\theta)>(\lambda^*,\theta^*)$, problem (1) has at least one positive solution.

PROOF. Construct a positive subsolution (ϕ_1,ϕ_2) and supersolution (z_1,z_2) of problem (1), where $\phi_1\leq z_1$ and $\phi_2\leq z_2$, i.e., (ϕ_1,ϕ_2) and (z_1,z_2) satisfy

$$\begin{cases}\int_\Omega |\nabla \phi_1|^{p(x)-2}\nabla \phi_1.\nabla\varphi dx\leq\int_\Omega \lambda^{p(x)}\left[\lambda_1 f(\phi_2)+\mu_1 h(\phi_1)\right]\varphi dx,\\ \int_\Omega |\nabla \phi_2|^{q(x)-2}\nabla \phi_2.\nabla\psi dx\leq\int_\Omega \theta^{q(x)}\left[\lambda_2 g(\phi_1)+\mu_2\gamma(\phi_2)\right]\psi dx,\end{cases}$$

and

$$\begin{cases}\int_\Omega |\nabla z_1|^{p(x)-2}\nabla z_1.\nabla\varphi dx\geq\int_\Omega \lambda^{p(x)}\left[\lambda_1 f(z_2)+\mu_1 h(z_1)\right]\varphi dx,\\ \int_\Omega |\nabla z_2|^{q(x)-2}\nabla z_2.\nabla\psi dx\geq\int_\Omega \theta^{q(x)}\left[\lambda_2 g(z_1)+\mu_2\gamma(z_2)\right]\psi dx,\end{cases}$$

for all $(\varphi,\psi)\in\left(W_0^{1,p(x)}(\Omega)\times W_0^{1,q(x)}(\Omega)\right)$ with $(\varphi,\psi)\geq 0$. By using to the sub-super solution concept for $p(x)$-Laplacian equations, see [16], problem (1) has a positive solution such that $(\lambda,\theta)>(\lambda^*,\theta^*)$.

Step 1. Construct a subsolution of problem (1).

By (H3)–(H5), we see that there exists $M>2$ such that

$$\lambda_1 f(0)+\mu_1 g(0)\geq 1\quad\text{and}\quad\lambda_2 h(0)+\mu_2\gamma(0)\geq 1.$$

Let

$$\sigma=\frac{\ln M}{k}\quad\text{and}\quad\tau=\frac{\ln M}{l}.$$

Then there exists $k_1 = l_1 > 1$ such that for any $k > k_1, l > l_1$, we have $\sigma, \tau \in (0, r)$. We denote

$$\phi_1(x) = \phi_1(\rho) = \begin{cases} e^{k(r-\rho)} - 1, & r - \sigma < \rho \leq r, \\ e^{k\sigma} - 1 + \int\limits_{\rho}^{r-\sigma} ke^{k\sigma} \left(\frac{t}{r-\sigma}\right)^{\frac{1}{p(t)-1}} dt, & 0 \leq \rho \leq r - \sigma, \end{cases}$$

and

$$\phi_2(x) = \phi_1(\rho) = \begin{cases} e^{l(r-\rho)} - 1, & r - \tau < \rho \leq r, \\ e^{l\tau} - 1 + \int\limits_{\rho}^{r-\tau} le^{l\tau} \left(\frac{t}{r-\tau}\right)^{\frac{1}{q(t)-1}} dt, & 0 \leq \rho \leq r - \tau. \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\overline{\Omega})$. It can be easily got by some simple calculations

$$\begin{aligned} -\triangle_{p(x)}\phi_1 &= -k\left(ke^{k(r-\rho)}\right)^{p(\rho)-1}\Big[k(p(\rho)-1) - p'(\rho)\ln k \\ &\quad -kp'(\rho)(r-\rho) - \frac{N-1}{\rho}\Big] \text{ for } r-\sigma < \rho < r, \end{aligned} \tag{5}$$

$$\begin{aligned} -\triangle_{p(x)}\phi_1 &= -\left(le^{l\sigma}\right)^{p(\rho)-1}\Big[p'(\rho)(\ln k + k\sigma)\frac{\rho}{r-\sigma} \\ &\quad -\frac{1}{r-\sigma} + \frac{N-1}{\rho}\frac{\rho}{r-\sigma}\Big] \text{ for } 0 < \rho < r-\sigma, \end{aligned} \tag{6}$$

$$\begin{aligned} -\triangle_{q(x)}\phi_2 &= -l\left(le^{l(r-\rho)}\right)^{q(\rho)-1}\Big[l(q(\rho)-1) - q'(\rho)\ln l \\ &\quad -lq'(\rho)(r-\rho) - \frac{N-1}{\rho}\Big] \text{ for } r-\tau < \rho < r, \end{aligned}$$

and

$$\begin{aligned} -\triangle_{q(x)}\phi_2 &= -\left(le^{l\tau}\right)^{q(\rho)-1}\Big[q'(\rho)(\ln l + l\tau)\frac{\rho}{r-\tau} - \frac{1}{r-\tau} \\ &\quad +\frac{N-1}{\rho}\frac{\rho}{r-\tau}\Big] \text{ for } 0 < \rho < r-\tau. \end{aligned}$$

Denote

$$\alpha_1 = \min\left\{\frac{\inf p(x) - 1}{4(\sup|\nabla p(x)| + 1)}, 1\right\}, \quad \alpha_2 = \min\left\{\frac{\inf q(x) - 1}{4(\sup|\nabla q(x)| + 1)}, 1\right\},$$

$$\zeta_1 = \lambda_1 f(0) + \mu_1 h(0) \quad \text{and} \quad \zeta_2 = \lambda_2 g(0) + \mu_2 \gamma(0).$$

From (5) and (6), there exists $k_2 > 0$ such that when $k > k_2$, we have

$$-\triangle_{p(x)}\phi_1 \leq -k^{p(x)}\alpha_1, \quad r - \sigma < \rho < r. \tag{7}$$

Let $\lambda = \frac{k\alpha_1}{\zeta_1}$. We have $k^{p(x)}\alpha_1 \geq \lambda^{p(x)}\zeta_1$. Then

$$\begin{aligned} -\triangle_{p(x)}\phi_1 &\leq -\lambda^{p(x)}\zeta_1 \leq \lambda^{p(x)}\left(\lambda_1 f(0) + \mu_1 h(0)\right) \\ &\leq \lambda^{p(x)}\left(\lambda_1 f(\phi_2) + \mu_1 h(\phi_1)\right) \quad \text{for } r - \sigma < \rho < r. \end{aligned} \tag{8}$$

When $0 < \rho < r - \sigma$, there exists $C_1 > 0$ such that

$$-\triangle_{p(x)}\phi_1 \leq C_1 \left(ke^{k\sigma}\right)^{p(\rho)-1} \ln k. \tag{9}$$

Then there exists $k_3 > 0$ such that when $k > k_3, \lambda = \frac{k\alpha_1}{\zeta_1}$, we have

$$C_1 \left(ke^{k\sigma}\right)^{p(\rho)-1} \ln k \leq \lambda^{p(x)}\left(\lambda_1 + \mu_1\right). \tag{10}$$

From (9) and (10), we have

$$-\triangle_{p(x)}\phi_1 \leq \lambda^{p(x)}\left(\lambda_1 f(\phi_2) + \mu_1 h(\phi_1)\right), \quad 0 < \rho < r - \sigma. \tag{11}$$

Let $k^* = \{k_1, k_2, k_3\}$. Similarly, we obtain l_2 and l_3. Denote

$$\lambda^* = \frac{\alpha_1}{\zeta_1} k^* \quad \text{and} \quad \theta^* = \frac{\alpha_2}{\zeta_2} l^* \quad \text{where } l^* = \{l_1, l_2, l_3\}.$$

Then for any $(\lambda, \theta) > (\lambda^*, \theta^*)$, we let

$$\sigma = \frac{\alpha_1 \ln M}{\zeta_1 \lambda} \quad \text{and} \quad \tau = \frac{\alpha_2 \ln M}{\zeta_2 \lambda}$$

and (7) (11) still hold, that is

$$-\triangle_{p(x)}\phi_1 \leq \lambda^{p(x)}\left(\lambda_1 f(\phi_2) + \mu_1 h(\phi_1)\right) \quad a.e \text{ on } \Omega. \tag{12}$$

Similarly, we have

$$-\triangle_{q(x)}\phi_2 \leq \lambda^{q(x)}\left(\lambda_2 g(\phi_1) + \mu_2 \gamma(\phi_2)\right) \quad a.e \text{ on } \Omega. \tag{13}$$

From (12) and (13), it can be seen that (ϕ_1, ϕ_2) is a sub-solution of (1) for all $(\lambda, \theta) > (\lambda^*, \theta^*)$.

Step 2. Construct a supersolution of (1):

$$\begin{cases} -\triangle_{p(x)} z_1 = \lambda^{p+}\left(\lambda_1 + \mu_1\right)\mu \text{ in } \Omega, \\ -\triangle_{q(x)} z_2 = \theta^{q+}\left(\lambda_2 + \mu_2\right) g\left(\beta\left(\lambda^{p+}\left(\lambda_1 + \mu_1\right)\mu\right)\right) \text{ in } \Omega, \\ z_1 = z_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{14}$$

where $\omega_1 = \omega_1\left(\lambda^{p+}\left(\lambda_1 + \mu_1\right)\mu\right) = \max_{x\in\overline{\Omega}} z_1(x)$. We shall prove that (z_1, z_2) is a supersolution of problem (1). It can be seen

$$z_1 = \int_\rho^r \left(\frac{\lambda^{p+}\left(\lambda_1 + \mu_1\right)\mu}{N} t\right)^{\frac{1}{p(t)-1}} dt,$$

$$z_2 = \int_\rho^r \left(\frac{\theta^{q+} (\lambda_2 + \mu_2) g (\beta (\lambda^{p+} (\lambda_1 + \mu_1) \mu))}{N} t \right)^{\frac{1}{q(t)-1}} dt,$$

are the positive solutions of problem (14). Certainly, there exists a $\eta \in [0, r]$ such that

$$\begin{aligned} \omega_1 &= \max_{x\in\overline{\Omega}} z_1(x) = \int_\rho^r \left(\frac{\lambda^{p+} (\lambda_1 + \mu_1) \mu}{N} t \right)^{\frac{1}{p(t)-1}} dt \\ &= \left[\lambda^{p+} (\lambda_1 + \mu_1) \mu\right]^{\frac{1}{p(\eta)-1}} \int_\rho^r \left(\frac{t}{N} \right)^{\frac{1}{p(t)-1}} dt, \end{aligned}$$

when μ is large. Then we obtain

$$C_2 \left[\lambda^{p+} (\lambda_1 + \mu_1) \mu\right]^{\frac{1}{p^+-1}} \le \omega_1 \le C_2 \left[\lambda^{p+} (\lambda_1 + \mu_1) \mu\right]^{\frac{1}{p^--1}} \tag{15}$$

where

$$C_2 = \int_\rho^r \left(\frac{t}{N} \right)^{\frac{1}{p(t)-1}} dt$$

is a positive constant. Similarly, we have

$$\begin{aligned} & C_3 \left[\theta^{q+} (\lambda_2 + \mu_2) g (\beta (\lambda^{p+} (\lambda_1 + \mu_1) \mu))\right]^{\frac{1}{q^+-1}} \\ \le\ & \omega_2 \le C_3 \left[\theta^{q+} (\lambda_2 + \mu_2) g (\beta (\lambda^{p+} (\lambda_1 + \mu_1) \mu))\right]^{\frac{1}{q^--1}}. \end{aligned}$$

For any $\psi \in W_0^{1,q(x)}(\Omega)$ with $\psi \ge 0$, it is easy to see that

$$\begin{aligned} & \int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 . \nabla \psi dx \\ =\ & \int_\Omega \theta^{q+} (\lambda_2 + \mu_2) g (\beta (\lambda^{p+} (\lambda_1 + \mu_1) \mu)) \psi dx \\ \ge\ & \int_\Omega \theta^{q+} \lambda_2 g (z_1) \psi dx + \int_\Omega \theta^{q+} \mu_2 g \left(\beta \left(\lambda^{p+} (\lambda_1 + \mu_1) \mu\right)\right) \psi dx. \end{aligned}$$

By (H4) and (H5), for μ large enough, we have

$$\begin{aligned} g \left(\beta \left(\lambda^{p+} (\lambda_1 + \mu_1) \mu\right)\right) &\ge \gamma \left(C_2 \left[\theta^{q+} (\lambda_2 + \mu_2) g \left(\beta \left(\lambda^{p+} (\lambda_1 + \mu_1) \mu\right)\right)\right]^{\frac{1}{q^--1}} \right) \\ &\ge \gamma (z_2). \end{aligned} \tag{16}$$

Hence

$$\int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 . \nabla \psi dx \ge \int_\Omega \theta^{q+} \lambda_2 g (z_1) \psi dx + \int_\Omega \theta^{q+} \mu_2 \gamma (z_2) \psi dx. \tag{17}$$

Also, for $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, it is easy to see that

$$\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1.\nabla \varphi dx = \int_\Omega \lambda^{p+} (\lambda_1 + \mu_1) \mu \varphi dx.$$

By (H3) and (H4), when μ is sufficiently large, we have

$$\begin{aligned}(\lambda_1 + \mu_1) \mu &\geq \frac{1}{\lambda^{p+}} \left[\frac{1}{C_2} \beta \left(\lambda^{p+} (\lambda_1 + \mu_1) \mu\right)\right]^{p^- -1} \\ &\geq \mu_1 h \left(\beta \left(\lambda^{p+} (\lambda_1 + \mu_1) \mu\right)\right) \\ &\quad + \lambda_1 f \left(C_2 \left[\theta^{q+} (\lambda_2 + \mu_2) g \left(\beta \left(\lambda^{p+} (\lambda_1 + \mu_1) \mu\right)\right)\right]^{\frac{1}{q^- -1}}\right).\end{aligned}$$

Then

$$\int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1.\nabla \varphi dx \geq \int_\Omega \lambda^{p+} \lambda_1 f(z_2) \varphi dx + \int_\Omega \lambda^{p+} \mu_1 h(z_1) \varphi dx. \qquad (18)$$

According to (17) and (18), we can conclude that (z_1, z_2) is a supersolution of problem (1).

Now, we only need to show that $(\phi_1, \phi_2) \leq (z_1, z_2)$ in Ω. When μ is large enough, we have

$$\lim_{\rho \to r^-} \frac{\phi_1(\rho)}{z_1(\rho)} = \frac{k}{\left(\frac{\lambda^{p+}(\lambda_1+\mu_1)\mu}{N} r\right)^{\frac{1}{p(r)-1}}} < 1.$$

By the continuity of $\phi_1(x)$ and $z_1(x)$, there exists $\varepsilon > 0$ such that

$$\phi_1(x) \leq z_1(x), \qquad r - \varepsilon < \rho \leq r.$$

When $0 \leq \rho \leq r - \varepsilon$, we can see that $\phi_1(x)$ is bounded and

$$z_1 = \int_\rho^r \left(\frac{\lambda^{p+} (\lambda_1 + \mu_1) \mu}{N} t\right)^{\frac{1}{p(t)-1}} dt \geq \int_{r-\varepsilon}^r \left(\frac{\lambda^{p+} (\lambda_1 + \mu_1) \mu}{N} t\right)^{\frac{1}{p(t)-1}} dt \to \infty \text{ as } \mu \to \infty.$$

Then $\phi_1(x) \leq z_1(x),\ x \in \Omega$ when μ is large enough. Similarly, when μ is large enough, we obtain

$$\phi_2(x) \leq z_2(x), \quad x \in \Omega.$$

We complete the proof of Theorem 1.

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