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Semilocal Convergence Of Sixth Order Method By Using Recurrence Relations In Banach Spaces*

Kalyanasundaram Madhu†

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Abstract

In this work, we study the semilocal convergence of the sixth order iterative method by using the recurrence relations for solving nonlinear equation in Banach spaces. This scheme is finally used to estimate the solutions of systems of nonlinear equations and so, the theoretical results are numerically checked. We use this example to show the better efficiency of the current method compared with other existing ones, including Newton's scheme.

1 Introduction

This paper is concerned with the problem of approximating a solution x^* of nonlinear equation $F(x) = 0$, where $\mathbb{X}$ and $\mathbb{Y}$ are Banach space, $F : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ a nonlinear twice Frechet differentiable operator in an open convex domain Ω, and this equation can represent differential equations, integral equations, a system of nonlinear equations, etc. One of the most well known method for solving nonlinear equation is Newton's method (NM) [9] which has convergence order two. Many papers have been written in a Banach space setting for the Newton-Kantorovich method as well as Newton-type methods and their applications. The semi-local convergence of Newton's method in Banach spaces was established by Kantorovich in [8]. Recently, the convergence of iterative methods for solving nonlinear operator equation in Banach spaces is established from the convergence of majorizing sequences. This technique has been used in order to establish the order of convergence of the variants of Newton's methods in the literature [3, 4, 14]. In [11], Rall has suggested a different approach for the convergence of these methods, based on recurrence relations. Recently, numerous variants of Newton's methods are developed by using this idea to prove the semilocal convergence for several methods of different orders (see [1, 2, 5, 10, 12]).

In the present paper, we study the semilocal convergence of the sixth order method ($M6$) proposed by K. Madhu [7]. The extension of this method in Banach spaces is given by

$$y_n = x_n - \Gamma_n F(x_n), \quad z_n = y_n - \tau\ \Gamma_n F(y_n), \quad x_{n+1} = z_n - \tau\ \Gamma_n F(z_n), \tag{1}$$

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†Department of Mathematics, Saveetha Engineering College, Chennai - 602105, India

where $\tau = 2I - \Gamma_n F'(y_n)$ and $\Gamma_n = [F'(x_n)]^{-1}$ for $n \in \mathbb{N}$. Recurrence relations, consists of generating a sequence of positive real numbers that guarantees the convergence of the iterative scheme in Banach spaces, providing a suitable convergence domain. This technique allows us to establish weak semilocal convergence conditions for new iterative method with sixth-order convergence. Even more, we get a result of semilocal convergence under the same conditions of Kantorovich Theorem for Newton's method, which has quadratic convergence. This allows us to apply the sixth-order convergence method for solving nonlinear equations $F(x) = 0$ under the same conditions that assures us the convergence of Newton's method.

In section 2 we describe the recurrence relations. In section 3, we describe the properties needed to prove the semilocal convergence of proposed method. In Section 4, numerical examples are given to illustrate the efficiency of the iterative methods.

2 Recurrence Relations

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach space and $F : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ be a nonlinear twice Frechet differentiable operator in an open convex domain Ω. Let us assume that the inverse of F' at x_0, $[F'(x_0)]^{-1} = \Gamma_0 \in \mathbb{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathbb{L}(Y, X)$ is the set of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$.

In the following conditions we will assume that $y_0, z_0 \in \Omega$ and

(i) $||\Gamma_0|| \leq \beta$,

(ii) $||\Gamma_0 F(x_0)|| \leq \eta$,

(iii) $||F'(x) - F'(y)|| \leq K||x - y||$, $x, y \in \Omega$,

in order to obtain the recurrence relations that satisfy the steps that appear in the iterative process (1). Note that these are the classical Kantorovich's conditions [8] for the semilocal convergence of Newton's method.

Let us also denote by $a_0 = K\beta\eta$ and define the sequence $a_{n+1} = a_n f(a_n)^2 g(a_n)$ where

$$f(x) = \frac{1}{1 - x(h(x) + 1)}, \tag{2}$$

$$g(x) = \frac{1}{2}x + (x+1)h(x) + \frac{1}{2}xh(x)^2, \tag{3}$$

and

$$h(x) = \frac{3}{2}x + \frac{5}{2}x^2 + \frac{13}{8}x^3 + \frac{7}{8}x^4 + \frac{3}{8}x^5 + \frac{1}{8}x^6. \tag{4}$$

To study the convergence of $\{x_n\}$ defined by (1) to a solution of $F(x) = 0$ in a Banach space, we have to prove that $\{x_n\}$ is a Cauchy sequence. To do this, we need to analyze some properties of sequence $\{a_n\}$ and real functions described in (2)–(4) respectively.

LEMMA 1. Let $f(x)$, $g(x)$ and $h(x)$ be the real functions described in (2)–(4). Then

(i) f is increasing and $f(x) > 1$ for $x \in (0, 0.4)$,

(ii) h and g are increasing for $x \in (0, 0.4)$.

LEMMA 2. Let $f(x)$ and $g(x)$ as before and $a_0 \in (0, 0.1799...)$. Then

(i) $f(a_0)^2 g(a_0) < 1$,

(ii) $f(a_0) g(a_0) < 1$,

(iii) the sequence $\{a_n\}$ is decreasing and $a_n < 0.1799...$, for $n \geq 0$.

PROOF. (i) follows trivially. From (i) and $f(a_0) > 1$, we obtain (ii). We are going to prove (iii) by induction on $n \geq 0$. Firstly, from (i) and the definition of a_1, we have that $a_1 < a_0$. Now, it is supposed that $a_k < a_{k-1}$, for $k \leq n$. Then

$$a_{n+1} = a_n f(a_n)^2 g(a_n) < a_{n-1} f(a_n)^2 g(a_n) < a_{n-1} f(a_{n-1})^2 g(a_{n-1}) = a_n,$$

as f and g are increasing and $f(x) > 1$. Finally, for all $n \geq 0$, $a_n < 0.1799...$, since $\{a_n\}$ is a decreasing sequence and $a_0 < 0.1799....$ Let us also note that $a_0 = 0.1799...$ is the value of the solution of equation $f(a_0)^2 g(a_0) - 1 = 0$. Using Taylor's expansion of $F(y_0)$ around x_0,

$$z_0 = y_0 - (2 - \Gamma_0 F'(y_0))\Gamma_0 F(y_0),$$

$$\begin{aligned} z_0 - x_0 &= y_0 - x_0 - (2 - \Gamma_0 F'(y_0))\Gamma_0 F''(x_0 + t(y_0 - x_0))(y_0 - x_0)^2 \\ &= y_0 - x_0 - (2 - \Gamma_0 F'(y_0))\Gamma_0 \int_0^1 F'(x_0 + t(y_0 - x_0))(y_0 - x_0)dt, \end{aligned}$$

then

$$||z_0 - x_0|| \leq ||y_0 - x_0|| + \frac{\Gamma_0 K}{2}||y_0 - x_0||^2 + \frac{\Gamma_0^2 K^2}{2}||y_0 - x_0||^3.$$

We have

$$\begin{aligned} ||z_0 - y_0|| &\leq \frac{\Gamma_0 K \eta}{2}||y_0 - x_0|| + \frac{\Gamma_0^2 K^2 \eta^2}{2}||y_0 - x_0||, \\ &\leq \frac{1}{2}(a_0 + a_0^2)||y_0 - x_0||. \end{aligned}$$

By using Taylor's expansion of $F(z_0)$ and (1), then we have

$$x_1 = x_0 - \Gamma_0 F(x_0) - \Big(2I - \Gamma_0 F'(y_0)\Big)\, \Gamma_0 F(y_0) - \Big(2I - \Gamma_0 F'(y_0)\Big)\, \Gamma_0 F(z_0),$$

$$
\begin{aligned}
& \|x_1 - x_0\| \\
= \ & \left\| -\Gamma_0\left(F(x_0) + \left(2I - \Gamma_0 F'(y_0)\right)F(y_0) + \left(2I - \Gamma_0 F'(y_0)\right)F(z_0)\right)\right\| \\
\leq \ & \left\| -\Gamma_0\left(F(x_0) + \left(I + \Gamma_0 K||y_0 - x_0||\right)F(y_0) + \left(I + \Gamma_0 K||y_0 - x_0||\right)F(z_0)\right)\right\| \\
\leq \ & \left\| -\Gamma_0\left(F(x_0) + \left(I + \Gamma_0 K||y_0 - x_0||\right)\int_{x_0}^{y_0}(F'(x) - F'(x_0))dx \right.\right. \\
& -\left(I + \Gamma_0 K||y_0 - x_0||\right)^2 \int_{x_0}^{y_0}(F'(x) - F'(x_0))dx \\
& \left.\left. +\left(I + \Gamma_0 K||y_0 - x_0||\right)\int_{x_0}^{z_0}(F'(x) - F'(x_0))dx\right)\right\| \\
\leq \ & \left\| -\Gamma_0\left(F(x_0) + \left(I + \Gamma_0 K||y_0 - x_0||\right)\frac{1}{2}K||y_0 - x_0||^2 \right.\right. \\
& \left.\left. -\left(I + \Gamma_0 K||y_0 - x_0||\right)^2 \frac{1}{2}K||y_0 - x_0||^2 + \left(I + \Gamma_0 K||y_0 - x_0||\right)\frac{1}{2}K||z_0 - x_0||^2\right)\right\| \\
\leq \ & ||y_0 - x_0|| + \left(I + \beta K||y_0 - x_0||\right)\frac{1}{2}K\beta||y_0 - x_0||^2 \\
& +\left(I + \beta K||y_0 - x_0||\right)^2 \frac{1}{2}K\beta||y_0 - x_0||^2 + \left(I + \beta K||y_0 - x_0||\right)\frac{1}{2}K\beta||z_0 - x_0||^2 \\
\leq \ & \left(1 + \frac{3}{2}a_0 + \frac{5}{2}a_0^2 + \frac{13}{8}a_0^3 + \frac{7}{8}a_0^4 + \frac{3}{8}a_0^5 + \frac{1}{8}a_0^6\right)\eta \leq (1 + h(a_0))\eta. \qquad (5)
\end{aligned}
$$

Now assuming that $a_0 < 0.4$ and applying assumptions (i)–(iii), we have

$$
\begin{aligned}
||I - \Gamma_0 F'(x_1)|| \ & \leq \ ||\Gamma_0|| \, ||F'(x_1) - F'(x_0)|| \leq \beta K||x_1 - x_0|| \leq \beta K\eta(1 + h(a_0)) \\
& \leq \ a_0(1 + h(a_0)) < 1,
\end{aligned}
$$

by Banach Lemma, Γ_1 exists and

$$
\Gamma_1 \leq \frac{1}{1 - a_0(1 + h(a_0))}\Gamma_0 = f(a_0)||\Gamma_0||.
$$

Let us remark that we required $a_0 < 0.4$ in order to guaranty $a_0(1 + h(a_0)) < 1$. Also note that $K||\Gamma_0|| \, ||y_0 - x_0|| \leq a_0$, so it can be deduced that x_1 is well defined and

$$
\begin{aligned}
\|x_1 - x_0\| \ & \leq \ ||\Gamma_0||\left\|F(x_0) + \left(2I - \Gamma_0 F'(y_0)\right)F(y_0) + \left(2I - \Gamma_0 F'(y_0)\right)F(z_0)\right\| \\
& \leq \ (1 + h(a_0))||\Gamma_0 F(x_0)||. \qquad (6)
\end{aligned}
$$

Then assuming that $x_n, y_n, z_n \in \Omega$ and $a_n < 0.4$ for all $n \geq 1$, the following inequalities can be proved by induction on $n \geq 1$:

(I_n) $||\Gamma_n|| \leq f(a_{n-1})\ ||\Gamma_{n-1}||,$

(II_n) $||y_n - x_n|| \leq ||\Gamma_n F(x_n)|| \leq f(a_{n-1})g(a_{n-1})\ ||y_{n-1} - x_{n-1}||,$

(III_n) $||z_n - y_n|| \leq \frac{\beta K}{2} f(a_0)^n\ (1 + \beta K f(a_0)^n ||y_n - x_n||)\ ||y_n - x_n||^2,$

(IV_n) $K||\Gamma_n||\ ||y_n - x_n|| \leq a_n,$

(V_n) $||x_n - x_{n-1}|| \leq (1 + h(a_{n-1}))\ ||y_{n-1} - x_{n-1}||.$

Let us consider $n = 1$, so (I_1) has been proved before.

(II_1): Using Taylor's formula

$$\begin{aligned}
& F(x_1) \\
= \;& F(y_0) + (x_1 - y_0)F'(y_0) + F(x_1) - F(y_0) - (x_1 - y_0)F'(y_0) \\
= \;& F(y_0) + (x_1 - y_0)F'(y_0) + \int_{y_0}^{x_1} (F'(x) - F'(y_0))dx \\
= \;& \int_0^1 \Big(F'(x_0) + t(y_0 - x_0) - F'(x_0)\Big)(y_0 - x_0)dt \\
& -\Big(F'(y_0) - F'(x_0) + F'(x_0)\Big)\Gamma_0(2I - \Gamma_0 F'(y_0))(F(y_0) + F(z_0)) \\
& -\Gamma_0(2I - \Gamma_0 F'(y_0))(F(y_0) + F(z_0)) \int_0^1 \Big(F'(x_0) + t(y_0 - x_0) - F'(x_0)\Big)dt. \quad (7)
\end{aligned}$$

On the other hand, from equ. (5) we have

$$||2I - \Gamma_0 F'(y_0)||\Big(||F(y_0) + F(z_0)||\Big) \leq \eta \frac{h(a_0)}{\beta}.$$

Then from equ. (7)

$$||F(x_1)|| \leq \frac{1}{2}K\eta^2 + K\eta^2 h(a_0) + \eta\frac{h(a_0)}{\beta} + \frac{K}{2}\eta^2 h(a_0)^2.$$

Therefore,

$$\begin{aligned}
\|y_1 - x_1\| &\leq \Gamma_1 \|F(x_1)\| \leq f(a_0)\Gamma_0||F(x_1)|| \\
&\leq f(a_0)\Big(\frac{1}{2}a_0 + (1 + a_0)h(a_0) + \frac{1}{2}a_0 h(a_0)^2\Big)\eta \\
&\leq f(a_0)g(a_0)||y_0 - x_0||.
\end{aligned}$$

(III_1): It is clear that

$$\begin{aligned}
\|z_1 - y_1\| &\leq \Gamma_1||2 - \Gamma_1 F'(y_1)||\ \ \|F(y_1)\| \\
&\leq \Gamma_1\Big(||1 + \Gamma_1||F'(y_1) - F'(x_1)||\Big)\ ||F(y_1)|| \\
&\leq \Big(1 + f(a_0)\Gamma_0 K||y_1 - x_1||\Big)f(a_0)\Gamma_0\ \frac{K}{2}||y_1 - x_1||^2 \\
&\leq \frac{\beta K}{2} f(a_0)\Big(1 + \beta K f(a_0)||y_1 - x_1||\Big)\ ||y_1 - x_1||^2.
\end{aligned}$$

(IV_1): Using (I_1) and (II_1),

$$K\Gamma_1||y_1 - x_1|| \leq K\beta\eta f(a_0)^2 g(a_0) \leq a_0 f(a_0)^2 g(a_0) \leq a_1.$$

(V_1): Has been shown in (6) that

$$||x_1 - x_0|| \leq (1 + h(a_0))||\Gamma_0 F(x_0)|| \leq (1 + h(a_0))||y_0 - x_0||.$$

By considering that the induction hypothesis of items (I_n) to (V_n) are true for a fixed $n \geq 1$, it can be proved (I_{n+1}) to (V_{n+1}) in a similar way and the induction is complete. Let us note that condition $a_n < 0.4$, for $n \geq 1$, is necessary for the existence of operators Γ_n, $n \geq 1$. The above recurrence relations for proposed given in (1) allow us to establish a new semilocal convergence result for this method $M6$ under mild conditions.

3 Semilocal Convergence Analysis

We are able to prove the semilocal convergence of method (1) under mild conditions by using technical Lemmas 1 and 2 and recurrence relations.

THEOREM 1. Let X and Y be Banach space and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Frechet differentiable operator in an open convex domain Ω. Let us assume that $\Gamma_0 = [F'(x_0)]^{-1} \in \mathbb{L}(Y, X)$ exists at some $x_0 \in \Omega$ and assumptions

(i) $\|\Gamma_0\| \leq \beta$,

(ii) $\|\Gamma_0 F(x_0)\| \leq \eta$,

(iii) $\|F'(x) - F'(y)\| \leq K\,\|x - y\|\,, x, y \in \Omega$,

are satisfied. Let us denote $a_0 = K\beta\eta$ and suppose that $a_0 < 0.1799....$ Then, if $B(x_0, R\eta) = \{x \in X : ||x - x_0|| < R\eta\} \subset \Omega$, where $R = \frac{1}{2}(a_0 + a_0^2) + \frac{1+h(a_0)}{1-f(a_0)g(a_0)}$ the sequence $\{x_n\}$ defined in (1) and starting at x_0 converges to a solution x^* of the equation $F(x) = 0$. In that case, the solution x^* and the iterates x_n, y_n and z_n belong to $\overline{B(x_0, R\eta)}$, and x^* is the only solution of $F(x) = 0$ in $B(x_0, \frac{2}{K\beta} - R\eta) \cap \Omega$.

PROOF. Let us recall that Γ_n exists for $n \geq 1$, since $a_0 < 0.1799....$ Moreover, we are going to prove that y_n and z_n belong to $B(x_0, R\eta) \subset \Omega$. By recurrence relation (V_n), it is easy to observe that

$$\begin{aligned}
\|x_1 - x_0\| &\leq (1 + h(a_0))\,\|y_0 - x_0\|\,,\\
\|x_2 - x_1\| &\leq (1 + h(a_0))\,\|y_1 - x_1\| \leq (1 + h(a_0))f(a_0)g(a_0)||y_0 - x_0||,\\
&\vdots\\
\|x_n - x_{n-1}\| &\leq (1 + h(a_0))(f(a_0)g(a_0))^{n-1}||y_0 - x_0||,
\end{aligned}$$

adding above relations then we obtained

$$||x_n - x_0|| \leq (1 + h(a_0))||y_0 - x_0|| \sum_{k=0}^{n-1}(f(a_0)g(a_0))^k.$$

So

$$\begin{aligned}\|y_n - x_0\| &\leq ||y_n - x_n|| + ||x_n - x_0|| \\ &\leq (1+h(a_0))(f(a_0)g(a_0))^n \ ||y_0 - x_0|| \\ &\quad +(1+h(a_0))||y_0 - x_0|| \ \sum_{k=0}^{n-1}(f(a_0)g(a_0))^k \\ &< (1+h(a_0))\eta \sum_{k=0}^{n} f(a_0)g(a_0) \\ &< (1+h(a_0))\frac{1-(f(a_0)g(a_0))^{n+1}}{1-f(a_0)g(a_0)}\eta < R\eta, \ \text{using geometric sequence.}\end{aligned}$$

By applying recurrence relations (I_n) and (II_n), we have

$$\begin{aligned}\|z_n - y_n\| &\leq ||2I - \Gamma_n F'(y_n)|| \ ||\Gamma_n|| \ ||F(y_n)|| \\ &\leq \frac{\beta K}{2} f(a_0)^n \Big(1 + \beta K f(a_0)^n \ ||y_n - x_n||\Big) \ ||y_n - x_n||^2 \\ &\leq \frac{a_0}{2}\Big(1 + a_0(f(a_0)^2 g(a_0))^n\Big)(f(a_0)^3 g(a_0)^2)^n ||y_0 - x_0||.\end{aligned}$$

Therefore,

$$\begin{aligned}\|z_n - x_0\| &\leq ||z_n - y_n|| + ||y_n - x_0|| \\ &\leq \frac{a_0}{2}\Big(1 + a_0(f(a_0)^2 g(a_0))^n\Big)(f(a_0)^3 g(a_0)^2)^n ||y_0 - x_0|| \\ &\quad +(1+h(a_0))\frac{1-(f(a_0)g(a_0))^{n+1}}{1-f(a_0)g(a_0)}||y_0 - x_0|| \\ &< \Big(\frac{1}{2}(a_0 + a_0^2) + (1+h(a_0))\frac{1-(f(a_0)g(a_0))^{n+1}}{1-f(a_0)g(a_0)}\Big)\eta < R\eta.\end{aligned}$$

In order to prove the convergence of the sequence $\{x_n\}$, let us state that

$$\begin{aligned}|x_{n+1} - x_n|| &\leq (1+h(a_n)) \ ||y_n - x_n|| \\ &\leq (1+h(a_n)) f(a_{n-1}) g(a_{n-1}) \ ||y_{n-1} - x_{n-1}|| \\ &\leq \cdots \leq (1+h(a_n)) \prod_{j=0}^{n-1} f(a_j) g(a_j) \ ||y_0 - x_0|| \end{aligned} \tag{8}$$

by (V_n) and (II_n). From (8) we have

$$\begin{aligned}&\|x_{n+m} - x_n\| \\ \leq\ & ||x_{n+m} - x_{n+m-1}|| + ||x_{n+m-1} - x_{n+m-2}|| + \cdots + ||x_{n+1} - x_n|| \\ \leq\ & (1+h(a_{n+m-1}))\eta \prod_{j=0}^{n+m-2} f(a_j)g(a_j) \\ & +(1+h(a_{n+m-2}))\eta \prod_{j=0}^{n+m-3} f(a_j)g(a_j) + \cdots + (1+h(a_n))\eta \prod_{j=0}^{n-1} f(a_j)g(a_j),\end{aligned}$$

as h is increasing and $\{a_n\}$ is decreasing by Lemmas 1 and 2,

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq (1+h(a_0))\eta\sum_{l=0}^{m-1}\left(\prod_{j=0}^{n+l-1} f(a_j)g(a_j)\right)\\ &\leq (1+h(a_0))\eta\sum_{l=0}^{m-1}\Big(f(a_0)g(a_0)\Big)^{l+n},\end{aligned}$$

since functions f and g are also increasing. So, by applying the partial sum of a geometric sequence,

$$\begin{aligned}\|x_{n+m}-x_n\| &\leq (1+h(a_0))\Big(f(a_0)g(a_0)\Big)^n\eta\sum_{l=0}^{m-1}\Big(f(a_0)g(a_0)\Big)^l\\ &\leq (1+h(a_0))\Big(f(a_0)g(a_0)\Big)^n\frac{1-(f(a_0)g(a_0))^m}{1-f(a_0)g(a_0)}\eta.\end{aligned}$$

Then, we conclude that $\{x_n\}$ is a Cauchy sequence if $f(a_0)g(a_0)<1$. In order to prove that x^* is a solution of $F(x)=0$, we will start with the bound of $||F'(x_n)||$,

$$\begin{aligned}||F'(x_n)|| &\leq ||F'(x_0)||+||F'(x_n)-F'(x_0)|\\ | &\leq ||F'(x_0)||+K||x_n-x_0||\leq ||F'(x_0)||+KR\eta,\end{aligned}\tag{9}$$

by applying hypothesis (iii) and Lemmas 1 and 2. Then by equ. (8),

$$\begin{aligned}\|F(x_n)\| &\leq ||F'(x_n)||\ ||\Gamma_n F(x_n)||\leq ||F'(x_n)||\ ||y_n-x_n||\\ &\leq ||F'(x_n)||\prod_{j=0}^{n-1} f(a_j)g(a_j)\eta.\end{aligned}$$

and as f and g are increasing and $\{a_n\}$ is decreasing,

$$||F(x_n)||\leq ||F'(x_n)||\Big(f(a_0)g(a_0)\Big)^n\eta.$$

Since $||F'(x_n)||$ is bounded (see (9)) and $\Big(f(a_0)g(a_0)\Big)^n$ tends to zero when $n\to\infty$, we conclude that $||F(x_n)||\to 0$. By continuity of F in Ω, $F(x^*)=0$. Let us observe that, if $a_0\in(0,0.1799...)$, $\frac{2}{K\beta}-R\eta>0$. So, we are going to prove the uniqueness of $x^*\in B\Big(x_0,\frac{2}{K\beta}-R\eta\Big)\cap\Omega$. Let us assume that y^* is a solution of $F(x)=0$ in $B\Big(x_0,\frac{2}{K\beta}-R\eta\Big)\cap\Omega$. Then, in order to prove that $y^*=x^*$ and taking

$$0=F(y^*)-F(x^*)=\int_0^1 F'^*+t(y^*-x^*))dt(y^*-x^*),$$

we must show that the operator $\int_0^1 F'^*+t(y^*-x^*))dt$ is invertible. So, by applying hypothesis (iii),

$$\begin{aligned}
& \|\Gamma_0\| \int_0^1 ||F'^{*} + t(y^* - x^*)) - F'(x_0)||dt \\
\leq \;& K\beta \int_0^1 ||x^* + t(y^* - x^*) - x_0||dt \\
\leq \;& K\beta \int_0^1 ((1-t)||x^* - x_0|| + t||y^* - x_0)||)dt \\
\leq \;& \frac{K\beta}{2}\Big(||x^* - x_0|| + ||y^* - x_0)||\Big) \\
< \;& \frac{K\beta}{2}\Big(R\eta + \frac{2}{K\beta} - R\eta\Big) = 1
\end{aligned}$$

by the Banach Lemma, the integral operator is invertible and hence $y^* = x^*$.

Another important aspect of this work is the comparative study of the efficiency of the proposed method with well known high-order methods, such as Jarratt's method [6] and the one recently introduced by Wang et al. [13] which are given below,
Jarratt's method (JM):

$$\begin{aligned}
x^{(k+1)} &= x^{(k)} - [6F'(y(x^{(k)})) - 2F'(x^{(k)})]^{-1}[3F'(y(x^{(k)})) + F'(x^{(k)})][F'(x^{(k)})]^{-1}F(x^{(k)}), \\
y(x^{(k)}) &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}).
\end{aligned}$$

Method of Wang et al. (Wang):

$$\begin{aligned}
x^{(k+1)} &= z(x^{(k)}) - [\frac{3}{2}F'(y(x^{(k)}))^{-1} - \frac{1}{2}F'(x^{(k)})^{-1}]F(z(x^{(k)})), \\
z(x^{(k)}) &= x^{(k)} - [6F'(y(x^{(k)})) - 2F'(x^{(k)})]^{-1}[3F'(y(x^{(k)})) + F'(x^{(k)})][F'(x^{(k)})]^{-1}F(x^{(k)}), \\
y(x^{(k)}) &= x^{(k)} - \frac{2}{3}[F'(x^{(k)})]^{-1}F(x^{(k)}).
\end{aligned}$$

4 Numerical Examples

The numerical experiments have been carried out using MATLAB software for the test problems given below. The approximate solutions are calculated correct to 1000 digits by using variable precision arithmetic. We use the following stopping criterion for the iterations:

$$err_{min} = \|x^{(k+1)} - x^{(k)}\|_2 < 10^{-100}.$$

We have used the approximated computational order of convergence p_c given by

$$p_c \approx \frac{\log\left(\|x^{(k+1)} - x^{(k)}\|_2 / \|x^{(k)} - x^{(k-1)}\|_2\right)}{\log\left(\|x^{(k)} - x^{(k-1)}\|_2 / \|x^{(k-1)} - x^{(k-2)}\|_2\right)}.$$

Let M be the number of iterations required for reaching the minimum residual err_{min}.

Test Problem 1 (TP1) We consider the following system:
$F(x_1, x_2) = 0$, where $F : (4, 6) \times (5, 7) \to \mathbb{R}^2$ and

$$F(x_1, x_2) = (x_1^2 - x_2 - 19, \quad x_2^3/6 - x_1^2 + x_2 - 17).$$

The Jacobian matrix is given by

$$F'(x) = \begin{pmatrix} 2x_1 & -1 \\ -2x_1 & \frac{1}{2}x_2^2 + 1 \end{pmatrix}.$$

The starting vector is $x^{(0)} = (5.1, 6.1)^T$ and the exact solution is $x^* = (5, 6)^T$.

Test Problem 2 (TP2) We consider the following system:

$$\begin{cases} \cos x_2 - \sin x_1 = 0, \\ x_3^{x_1} - \dfrac{1}{x_2} = 0, \\ \exp x_1 - x_3^2 = 0. \end{cases}$$

The solution is $x^* \approx (0.909569, 0.661227, 1.575834)^T$. We choose the starting vector $x^{(0)} = (1, 0.5, 1.5)^T$. The Jacobian matrix has 7 non-zero elements and it is given by

$$F'(x) = \begin{pmatrix} -\cos x_1 & -\sin x_2 & 0 \\ x_3^{x_1} \ln x_3 & 1/x_2^2 & x_3^{x_1} x_1/x_3 \\ \exp x_1 & 0 & -2x_3 \end{pmatrix}.$$

Test Problem 3 (TP3) We consider the following system:

$$\begin{aligned} x_2x_3 + x_4(x_2 + x_3) = 0, \\ x_1x_3 + x_4(x_1 + x_3) = 0, \\ x_1x_2 + x_4(x_1 + x_2) = 0, \\ x_1x_2 + x_1x_3 + x_2x_3 = 1. \end{aligned}$$

We solve this system using the initial approximation $x^{(0)} = (0.5, 0.5, 0.5, -0.2)^T$. The solution of this system is $x^* \approx (0.577350, 0.577350, 0.577350, -0.288675)^T$. The Jacobian matrix that has 12 non-zero elements is given by

$$F'(x) = \begin{pmatrix} 0 & x_3 + x_4 & x_2 + x_4 & x_2 + x_3 \\ x_3 + x_4 & 0 & x_1 + x_4 & x_1 + x_3 \\ x_2 + x_4 & x_1 + x_4 & 0 & x_1 + x_2 \\ x_2 + x_3 & x_1 + x_3 & x_1 + x_2 & 0 \end{pmatrix}.$$

Table 1 shows the results for the test problems (TP1, TP2, TP3), from which we conclude that $M6$ method is the most efficient method out of the methods compared with least number of iterations and residual error. Hence, proposed method $M6$ is preferable over some existing methods.

Table1: Comparison of different methods for system of nonlinear equations.

Methods	TP1			TP2			TP3		
	M	err_{min}	p_c	M	err_{min}	p_c	M	err_{min}	p_c
NM	10	1.0385e-103	1.99	8	3.9287e-145	2.00	9	8.9692e-179	1.99
JM	4	1.0270e-117	3.99	5	1.1522e-139	4.00	5	2.9883e-291	4.03
$Wang$	4	3.6801e-110	3.99	5	1.2669e-121	4.00	5	8.8962e-257	4.03
$M6$	3	5.0691e-054	5.88	5	5.5651e-280	6.00	4	3.8561e-175	6.11

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