

On The Concentration Of Mass In Generalized Unit Balls*

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Abstract

A property of the unit n -sphere is that most of its mass lies near the equator and that it is simultaneously concentrated near the surface. In this article, we show that this property also holds for generalized unit balls in Euclidian space. We also give exact formulas for the mass content of "ball-caps" and derive upper bounds for certain related volume content ratios.

1 Introduction

A well known fact about the unit sphere is that its volume shrinks to zero as the dimension increases, with the maximum occurring for $n = 5$ (e.g., see [6]). More recent analysis of the concentration of the mass within the sphere shows that most of the mass is contained in a thin slice near the equator (see [2]). Moreover, it is easy to show that the mass of the sphere in higher dimensions is concentrated in a narrow annulus near the surface. These observations are of interest in the study of the foundations of the theory of high dimensional data (see [3]). More generally, the interaction of higher dimensional geometry and probability theory is an older well established theory. In simple terms, this interaction is realized mainly in the study of inequalities. For example, the concentration of the mass of a cube near its equator can be explained using the weak law of large numbers. Important geometric inequalities arise from volume (or other measures) ratios for sets. Such inequalities may also extend to sets of functions and may be used to express probabilities. Bounds for the volumes or surface areas of spherical caps play a role in the study of the concentration of measure in geometry (see [2]). Looking at generalizations of such estimates for generalized balls can perhaps be justified at least from a pedagogical point of view (see Remark [2]). Generalized unit balls (GUB's) are defined by sets of the form

$$B_G(n, \mathbf{p}) = \{(x_1, \dots, x_n) : \|x\| := |x_1|^{p_1} + \dots + |x_n|^{p_n} \leq 1, 0 < p_i < \infty\}. \quad (1)$$

We refer to their surfaces as hyperspheres. GUB's are symmetrically centered objects as $-\mathbf{x} \in B_G$ if $\mathbf{x} \in B_G$. For $p_i \geq 1$, it is well known that the GUB's are convex. For $p_i \leq 1$, the GUB's are non-convex if at least one $p_j < 1$ (see Proposition [1]). In [8],

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it is shown that the volume of GUB's goes to zero as the dimension n goes to infinity. The purpose of this article is to show that the volumes of GUB's are concentrated in a thin slice near the equator, and that their mass lies mostly near the surface.

Now we give a proposition that characterizes the convexity of GUB's whenever at least some $p'_i < 1$. Recall that if a subset S of a vector space is convex and v_1, \dots, v_n belong to the subset S , then for $t_i \geq 0$ and $\sum_{i=1}^k t_i = 1$, the convex combination $\sum_{i=1}^k t_i v_i$ is in S , $1 < k \leq n$. Further, a set S in \mathbf{R}^n is star-convex with respect to a point $\mathbf{s}_0 \in S$, called the star-center of S , if for every $\mathbf{s} \in S$, the segment $[\mathbf{s}_0, \mathbf{s}]$ is contained in S .

PROPOSITION 1. For $B_G(n, \mathbf{p})$ to be a convex set, all $p_j \geq 1$, $j = 1, \dots, n$. If $p_j < 1$ for one j , then $B_G(n, \mathbf{p})$ is non-convex. However, it is star-convex with respect to the origin $\mathbf{0}$.

PROOF. Let $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$ denote the i^{th} unit vector in the direction of the i^{th} coordinate. According to the definition of B_G , \mathbf{e}_i and \mathbf{e}_j belong to B . Since B is convex then $(1-t)\mathbf{e}_i + t\mathbf{e}_j = (0, \dots, 1-t, \dots, t, \dots, 0) \in B$, $\forall t \in (0, 1)$. This implies that $|1-t|^{p_i} + |t|^{p_j} = (1-t)^{p_i} + t^{p_j} \leq 1$, $\forall t \in (0, 1)$. Hence,

$$t^{p_j} \leq 1 - (1-t)^{p_i} \Rightarrow p_j \geq \frac{\ln(1 - (1-t)^{p_i})}{\ln(t)} \quad \forall t \in (0, 1).$$

Take the limit of both sides, we obtain:

$$p_j \geq \lim_{t \rightarrow 0^+} \frac{\ln(1 - (1-t)^{p_i})}{\ln(t)} = 1.$$

So $p_j \geq 1$, $j = 1, \dots, n$. Therefore $B_G(n, \mathbf{p})$ is non-convex if there exists at least one j such that $p_j < 1$.

To show that B_G is star-convex with respect to $\mathbf{0} \in B_G$, let \mathbf{x} be any point in B_G , then the line segment from the origin to \mathbf{x} is given by $t\mathbf{x} + (1-t)\mathbf{0} = t\mathbf{x}$ for every t in $[0, 1]$. If $\mathbf{x} \in B_G$ and $t \in (0, 1)$, then $t\mathbf{x}$ is in B_G . This is so since we have $t^{p_i} < 1$ for $i = 1, \dots, n$, and thus $|tx_1|^{p_1} + |tx_2|^{p_2} + \dots + |tx_n|^{p_n} \leq |x_1|^{p_1} + |x_2|^{p_2} + \dots + |x_n|^{p_n} \leq 1$. Thus B_G is star-convex (or star-shaped) with respect to the origin.

2 Volumes of Ball-Caps

As hyperspheres are less symmetric than the n -sphere ($p=2$), we define the equator as the intersection of a GUB centered at the origin with a coordinate hyperplane. For instance, if we take the hyperplane $x_1 = 0$, the set $\{\mathbf{x} : |\mathbf{x}| \leq 1, x_1 = 0\}$ is called an equator. Now we describe our starting point to find the volume of the portion of a GUB above a certain hyperplane $x_1 = \varepsilon$. This is the volume of solid-cap T lying between $x_1 = \varepsilon$ and $x_1 = 1$, and we refer to it as the ball-cap. First we note that the volumes of the hyper-ellipsoids defined by

$$\left\{ (y_1, y_2, \dots, y_n) : \left| \frac{y_1}{c_1} \right|^{p_1} + \left| \frac{y_2}{c_2} \right|^{p_2} + \dots + \left| \frac{y_n}{c_n} \right|^{p_n} \leq 1, c_i, p_i > 0 \right\},$$

are given by

$$V(n, p) = 2^n \frac{\prod_{i=1}^n c_i \Gamma\left(1 + \frac{1}{p_i}\right)}{\Gamma\left(1 + \sum_{i=1}^n \frac{1}{p_i}\right)}; \quad p_i > 0, n = 1, 2, 3 \dots, \tag{2}$$

where $\Gamma(x)$ is the Euler gamma function. The formulas in (2) were obtained by G. P. Lejeune Dirichlet (see [4, 8]). An alternative derivation that is useful to us below is given in [1], which extends the method of P. Hein (see [5]). We recall the derivation briefly.

The Gauss hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

for $|z| < 1$, where a, b, c are complex numbers, $c \neq 0, -1, -2, \dots$, and $(\alpha)_k$ is the Pochhammer symbol defined as $(\alpha)_k = \alpha(\alpha + 1) \dots (\alpha + k - 1) = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}$, $n > 0$, $(\alpha)_0 = 1$ for $\alpha \neq 1$. For convenience, we denote ${}_2F_1$ by F . We have the following integral

$$\int (a - x^m)^{\frac{d}{m}} dx = a^{\frac{d}{m}} x F\left(\frac{1}{m}, -\frac{d}{m}; 1 + \frac{1}{m}; \frac{x^m}{a}\right), \tag{3}$$

for $|x^m| < a$. A well known result due to Gauss for F with unit argument is (Theorem 18 in [7], p. 47): $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, for $\Re(c - a - b) > 0$, and c is neither zero nor a negative integer. Also we obtain easily the equality for the product $\prod_{i=1}^k F\left(\frac{1}{m}, -\frac{i}{m}; 1 + \frac{1}{m}; 1\right) = \frac{\left(\Gamma\left(1 + \frac{1}{m}\right)\right)^{k+1}}{\Gamma\left(1 + \frac{k+1}{m}\right)}$.

A simple linear transformation converts a hyper-ellipsoid into a GUB (E.q (1)). Denote by

$$E(n, p) = \left\{ (x_1, \dots, x_n) : \sum_{i=1}^n x_i^{p_i} \leq 1, \text{ all } x_i \geq 0, p_i > 0 \right\}$$

the positive orthant of the GUB. Using the symmetry of the GUB's about the origin, we can start from the volume integral $2^n \int_{E(n, p)} dx_1 \dots dx_n$. A reduction can be effected by integrating first over x_n and using the integral in (3). Then continue the reduction process to arrive at the integral $\prod_{i=3}^n P_{i-1} \int_0^1 (1 - x_1^{p_1})^{\sum_{j=2}^n \frac{1}{p_j}} dx_1$, where P_{i-1} denotes $F\left(\frac{1}{p_{i-1}}, -\sum_{k=i}^n \frac{1}{p_k}; 1 + \frac{1}{p_{i-1}}; 1\right)$. Let T denote the set $\{\mathbf{x} \in B_G : \|\mathbf{x}\| \leq 1, x_1 \geq \varepsilon\}$. It is clear now that the last integral expression in the reduction can be used as a starting point for us to develop a formula for $V(T)$. To show that the mass of a GUB is near the equator, we compute the mass of the ball-cap T lying above the slice between $x_1 = 0$ and $x_1 = \varepsilon$. Let $\alpha = \sum_{i=2}^n \frac{1}{p_i}$. Thus, we have

$$\begin{aligned} V(T) &= V(n-1) \int_{\varepsilon}^1 (1 - x_1^{p_1})^\alpha dx_1 \\ &= V(n-1) \left(F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; 1\right) - \varepsilon F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \varepsilon^{p_1}\right) \right), \end{aligned} \tag{4}$$

where $V(n-1)$ is the volume of B_G of dimension $n-1$ given in (2).

Now we derive a lower bound for the volume of the upper-half of B_G . Define the slab $S := \{\mathbf{x} \in B_G : 0 < \|\mathbf{x}\| \leq \varepsilon\}$. Then the exact volume of S is given by the integral

$$V(S) = V(n-1) \int_0^\varepsilon (1 - x_1^{p_1})^\alpha dx_1 = V(n-1) \varepsilon F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \varepsilon^{p_1}\right). \quad (5)$$

Therefore, the exact ratio of the volume of the ball-cap T to the volume of the slab S is given by

$$\frac{V(T)}{V(S)} = \frac{\left(F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; 1\right) - \varepsilon F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \varepsilon^{p_1}\right)\right)}{\varepsilon F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \varepsilon^{p_1}\right)}.$$

Set $\varepsilon = \frac{c}{\sqrt[p_1]{n-1}}$ in $V(T)$ such that $c \leq \sqrt[p_1]{n-1}$, and $\varepsilon = \frac{1}{\sqrt[p_1]{n-1}}$ in $V(S)$. Then we have

$$\frac{V(T)}{V(S)} = \frac{\sqrt[p_1]{n-1} F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; 1\right) - c F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \frac{c^{p_1}}{n-1}\right)}{F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \frac{1}{n-1}\right)}. \quad (6)$$

Now the Gauss function is related to the incomplete beta function by

$$B_x(p, q) = \frac{x^p}{p} F(p, 1 - q, 1 + p; x),$$

where

$$B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt, \quad 0 \leq x \leq 1, \quad p, q > 0.$$

Take $p = \frac{1}{p_1}$, $q = 1 + \alpha$, and $x = \frac{1}{n-1}$. Then we can obtain an alternative expression for the ratio in terms perhaps of the more familiar beta function. We have

$$F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; \frac{c^{p_1}}{n-1}\right) = \frac{\sqrt[p_1]{n-1}}{c p_1} B_{\frac{c^{p_1}}{n-1}}\left(\frac{1}{p_1}, 1 + \alpha\right).$$

Also as $F(a, b; c, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, we have $F\left(\frac{1}{p_1}, -\alpha; 1 + \frac{1}{p_1}; 1\right) = \frac{\Gamma\left(1 + \frac{1}{p_1}\right)\Gamma(1+\alpha)}{\Gamma\left(1 + \alpha + \frac{1}{p_1}\right)} = \frac{1}{p_1} B\left(\frac{1}{p_1}, 1 + \alpha\right)$. Then

$$\frac{V(T)}{V(S)} = \frac{B\left(\frac{1}{p_1}, 1 + \alpha\right) - B_{\frac{c^{p_1}}{n-1}}\left(\frac{1}{p_1}, 1 + \alpha\right)}{B_{\frac{1}{n-1}}\left(\frac{1}{p_1}, 1 + \alpha\right)}. \quad (7)$$

Now the regularized beta function is defined as $I_x(a, b) = \frac{B_x(a, b)}{B(a, b)}$. Hence, we can express the ratio as

$$\frac{V(T)}{V(S)} = \frac{1 - I_{\frac{c^{p_1}}{n-1}}\left(\frac{1}{p_1}, 1 + \alpha\right)}{I_{\frac{1}{n-1}}\left(\frac{1}{p_1}, 1 + \alpha\right)}. \quad (8)$$

3 Bounds on Ball-Caps Volumes

There is a good number of different volume ratios that appear in the study of convex geometry and multiplicative geometric inequalities (e.g., see [2]). In section 2 of [3], a bound on the volume fraction of the hemi-spherical cap is given. The volume fraction is obtained by dividing $V(T)$ by a lower bound for the volume of the upper hemi-sphere. The lower bound is taken to be the volume of a certain inscribed cylinder in such a way that the bound on the volume fraction obtained is independent of n .

THEOREM A (Theorem 2.6 in [3]). For any $c \geq 1$, and $n \geq 3$, at least a $1 - \frac{2}{c}e^{-\frac{c^2}{2}}$ fraction of the volume of the n -dimensional unit ball has $|x_1| \leq \frac{c}{\sqrt{n-1}}$.

Estimates of this sort could also play a role in the further study of the concentration (or distribution) of measure in geometry. We now obtain bounds on the volume of ball-caps for GUB's.

We first find an upper bound on $V(T)$. As in [3], we make use of the inequalities $1 + x^m \leq e^{x^m}$, and $(\frac{x_1}{\varepsilon}) > 1$. The last integral in the reduction we discussed above is our starting point. Thus we have that

$$\begin{aligned} V(T) &= V(n-1) \int_{\varepsilon}^1 (1-x^{p_1})^{\sum_{i=2}^n \frac{1}{p_i}} dx_1 \\ &\leq V(n-1) \int_{\varepsilon}^{\infty} \frac{x_1^{p_1-1}}{\varepsilon^{p_1-1}} e^{-x_1^{p_1} \sum_{i=2}^n \frac{1}{p_i}} dx_1 \\ &= -\frac{V(n-1)}{p_1 \varepsilon^{p_1-1}} \frac{e^{-x_1^{p_1} \sum_{i=2}^n \frac{1}{p_i}}}{\sum_{i=2}^n \frac{1}{p_i}} \Big|_{\varepsilon}^{\infty}, \end{aligned}$$

which gives us an upper bound

$$V(T) \leq \frac{V(n-1)}{\left(\sum_{i=2}^n \frac{1}{p_i}\right) p_1 \varepsilon^{p_1-1}} e^{-\left(\sum_{i=2}^n \frac{1}{p_i}\right) \varepsilon^{p_1}} = \frac{V(n-1)}{\alpha p_1} \frac{e^{-\alpha \varepsilon^{p_1}}}{\varepsilon^{p_1-1}}. \tag{9}$$

where $\alpha = \sum_{i=2}^n \frac{1}{p_i}$.

Next, we derive a lower bound for the volume of the upper half of the GUB's. We first consider the case where $\alpha \geq 1$. A lower bound is given by the volume of the slab S between $x_1 = 0$ and $x_1 = \xi < 1$, $n > 2$. This volume is given by the following integral. Upon using the Weierstrass inequality $(1-x)^a \geq 1-ax$, which holds for $|x| < 1$ and $a \geq 1$, we have

$$\begin{aligned} V(S) &= V(n-1) \int_0^{\xi} (1-x_1^{p_1})^{\alpha} dx_1 \tag{10} \\ &\geq V(n-1) \int_0^{\xi} (1-\alpha x_1^{p_1}) dx_1 \\ &= V(n-1) \left(\xi - \frac{\alpha \xi^{p_1+1}}{p_1+1} \right). \end{aligned}$$

Then the fraction of the volume, $V_f := \frac{V(T)}{V(S)}$, above the hyperplane $x_1 = \varepsilon$ is bounded by the upper bound divided by the lower bound below the hyperplane $x_1 = \xi$. Thus, we obtain

$$V_f \leq \frac{\frac{1}{\alpha p_1} \frac{e^{-\alpha \varepsilon p_1}}{\varepsilon^{p_1-1}}}{\xi - \frac{\alpha \xi^{p_1+1}}{p_1+1}}. \tag{11}$$

Let $m = \min_{1 \leq i \leq n} p_i$, and $k = \max_{1 \leq i \leq n} p_i$. Then $\frac{n-1}{k} \leq \alpha \leq \frac{n-1}{m}$. If we pick $\varepsilon = \frac{c}{\sqrt[p_1]{n-1}}$, for some $0 < c < \sqrt[p_1]{n-1}$, and $\xi = \frac{m}{\sqrt[p_1]{n-1}}$, then from (11) we get that

$$\frac{\frac{k}{m(n-1)^{\frac{1}{p_1}}} \left(\frac{e^{-\frac{1}{k} c^{p_1}}}{c^{p_1-1}} \right)}{\left(\frac{m}{(n-1)^{\frac{1}{p_1}}} \right)^{\frac{1}{p_1+1}} \left(1 + p_1 - \frac{n-1}{m} \left(\frac{m}{(n-1)} \right) \right)} = km^{-\frac{p_1+1}{p_1}} \frac{p_1+1}{p_1} \left(\frac{e^{-\frac{1}{k} c^{p_1}}}{c^{p_1-1}} \right). \tag{12}$$

Hence we obtain the following theorem, which is a generalization of the Theorem A stated above.

THEOREM 1. For any $1 \leq c \leq \sqrt[p_1]{n-1}$, $\alpha \geq 1$, and $n \geq 3$, at least a $1 - km^{-\frac{p_1+1}{p_1}} \frac{p_1+1}{p_1} \left(\frac{e^{-\frac{1}{k} c^{p_1}}}{c^{p_1-1}} \right)$ fraction of the volume of the n -dimensional unit ball has $|x_1| \leq \frac{c}{\sqrt[p_1]{n-1}}$.

REMARK 1. Although the bound in Theorem 1 is independent of n , and the upper bound allows for having $c \geq 1$, Nevertheless, since $|x_1| \leq 1$, we have added the bound $1 \leq c < \sqrt[p_1]{n-1}$, and this upper bound on c helps satisfy the bound on x in the definition of $B_x(p, q)$.

Now we consider the case for $\alpha := \sum_{i=2}^n \frac{1}{p_i} < 1$, and thus $p_i < 1$ for all $i = 1, \dots, n$. We note first that (9) is still valid for $\alpha < 1$. So we only need to find a lower bound on $V(S)$ whenever $\alpha < 1$. Going back to (5), for $n > 2$, the volume of the slab S between $x_1 = 0$ and $x_1 = \varepsilon$ was given by

$$V(S) = V(n-1) \int_0^\varepsilon (1 - x_1^{p_1})^\alpha dx_1.$$

Recall the generalized binomial theorem is given by $(x + y)^r = \sum_{k=0}^\infty \frac{r(r-1)\dots(r-k+1)}{k!} x^{r-k} y^k$. From this we get that

$$(1 - x^\beta)^\alpha \geq 1 - \alpha x^\beta + \frac{\alpha(\alpha-1)}{2!} x^{2\beta}.$$

Hence we have

$$\int_0^\varepsilon (1 - x^{p_1})^\alpha dx \geq \int_0^\varepsilon \left(1 - \alpha x^{p_1} + \frac{\alpha(\alpha-1)}{2!} x^{2p_1} \right) dx,$$

where the last integral can be simply evaluated, and we have

$$I := \int_0^\varepsilon \left(1 - \alpha x^{p_1} + \frac{\alpha(\alpha-1)}{2!} x^{2p_1} \right) dx = \varepsilon - \frac{\varepsilon^{p_1+1}}{p_1+1} \alpha + \frac{1}{2} \frac{\varepsilon^{2p_1+1}}{2p_1+1} \alpha(\alpha-1).$$

$p_1 = \dots = p_n$	c	$V_f (E.q(8))$	$U.B. (E.q(12))$	$\frac{V(T)}{V(B_G)}$
$p_1 = \frac{1}{2}, n = 7$	1	0.4204	1.6240	0.1480
	30	3.0482e-13	0.0011	1.0730e-13
$p_1 = 4, n = 7$	1	0.2327	0.6884	0.0944
	1.5	0.0017	0.0739	7.0450e-4
$p_1 = \frac{1}{2}, n = 20$	1	0.5888	1.6240	0.1853
	100	7.529e-12	2.4734e-7	2.3769e-12
$p_1 = 4, n = 20$	1	0.3042	0.6884	0.1166
	2	2.6112e-6	0.0020	1.0011e-6

Table 1: Volume ratios with $\alpha \geq 1$.

Set $\varepsilon = \frac{1}{p\sqrt[n-1]{n-1}}$, then

$$I = \frac{1}{p\sqrt[n-1]{n-1}} - \frac{\left(\frac{1}{p\sqrt[n-1]{n-1}}\right)^{p_1+1}}{p_1+1} \alpha + \frac{1}{2} \frac{\left(\frac{1}{p\sqrt[n-1]{n-1}}\right)^{2p_1+1}}{2p_1+1} \alpha (\alpha - 1).$$

Then we obtain

$$V_f := \frac{V(T)}{V(S)} \leq \frac{\frac{1}{\alpha p_1 \left(\frac{c}{p\sqrt[n-1]{n-1}}\right)^{p_1-1}}}{\frac{1}{p\sqrt[n-1]{n-1}} - \frac{\left(\frac{1}{p\sqrt[n-1]{n-1}}\right)^{p_1+1}}{p_1+1} \alpha + \frac{1}{2} \frac{\left(\frac{1}{p\sqrt[n-1]{n-1}}\right)^{2p_1+1}}{2p_1+1} \alpha (\alpha - 1)} e^{-\alpha \frac{c p_1}{n-1}}.$$

If $p_i = m < 1$ for all $i = 1, \dots, n, n > 2, \alpha = \frac{n-1}{m}$, we obtain

$$\begin{aligned} V_f &\leq \frac{2(n-1)(m+1)m^2(2m+1)}{n-nm+6nm^3+4m^4n-2nm^2-4m^4-6m^3+m^2-1} \frac{e^{-\frac{c m}{m}}}{c^{m-1}} \quad (13) \\ &\leq \frac{2(m+1)m^2(2m+1)}{1-m+6m^3+4m^4-2m^2+\frac{-m^2-m}{2}} \frac{e^{-\frac{c m}{m}}}{c^{m-1}}. \end{aligned}$$

So we have the following theorem:

THEOREM 2. For any $1 \leq c \leq \sqrt[n]{n-1}, p_i = m, 1 \leq i \leq n, \alpha < 1$, and $n \geq 3$, at least a $1 - \frac{4(m+1)(2m+1)m^2}{2-3m-5m^2+12m^3+8m^4} \frac{e^{-\frac{c m}{m}}}{c^{m-1}}$ fraction of the volume of the n -dimensional unit ball has $|x_1| \leq \frac{c}{\sqrt[n]{n-1}}$.

For the numerical illustration of our results, we present in Tables 1 and 2 some estimates for the volume fractions from (8), and compute upper bounds for the volume fractions from (12) (c.f. Theorem [1]) and from (13) (c.f. Theorem [2]), and $V(T)/V(B_G)$ from (4) and (2), for various \mathbf{p} vectors with $\alpha < 1$ or $\alpha \geq 1$, and for different values of n and c . $U.B.$ stands for upper bound. The results in Tables 1 and 2 show clearly that $\frac{V(T)}{V(B_G)} < V_f < U.B.$, as expected. As the value of c approached its upper bound, smaller values for $U.B.$ were obtained.

$p_1 = \dots = p_n$	c	V_f (E.q(8))	$U.B.$ (E.q(13))	$\frac{V(T)}{V(B_G)}$
$p_1 = 8, n = 7$	1	0.1518	0.8962	0.0659
	1.2	0.0119	0.1656	0.0052
$p_1 = 25, n = 20$	1	0.0906	0.9624	0.0415
	1.12	5.2100e-4	0.0334	2.3886e-4

Table 2: Volume ratios with $\alpha < 1$.

Finally, we show that the volume of the GUB’s is near the surface. Consider the hyper-ellipsoid

$$\left\{ (y_1, y_2, \dots, y_n) : \left| \frac{y_1}{a_1} \right|^{p_1} + \left| \frac{y_2}{a_2} \right|^{p_2} + \dots + \left| \frac{y_n}{a_n} \right|^{p_n} \leq R, a_i, p_i > 0, 0 < R < 1 \right\}.$$

Then defining $c_i = a_i R^{\frac{1}{p_i}}$ and applying (2), we get $R^{\sum_{i=1}^n \frac{1}{p_i}} 2^n \frac{\prod_{i=1}^n a_i \Gamma\left(1 + \frac{1}{p_i}\right)}{\Gamma\left(1 + \sum_{i=1}^n \frac{1}{p_i}\right)}$. Then it follows that the ratio of the volume of a hyper-ellipsoid with $R := (1 - \delta) < 1$, to the volume of a hyper-ellipsoid with $R = 1$, and with the same set of $\{a_i\}$, equals $(1 - \delta)^{\sum_{i=1}^n \frac{1}{p_i}} \leq e^{-\sum_{i=1}^n \frac{1}{p_i} \delta} \leq e^{-\frac{n}{k} \delta}$, where $k = \max_{1 \leq i \leq n} p_i$, and thus $\sum_{i=1}^n \frac{1}{p_i} \geq \frac{n}{k}$. It follows that for fixed δ and k , the ratio will go to zero as n goes to infinity. This shows that most of the volume of a GUB (whenever, $a_i = 1$ for $i = 1, \dots, n$) lies in a thin shell near the surface. As the volume of GUB goes to zero as n goes to infinity, the volume of this shell will also shrink accordingly.

In [3] for the case of the n -sphere, the volume ratio for a ball with radius $(1 - \varepsilon)$ to a unit ball is $(1 - \varepsilon)^n \leq e^{-n\varepsilon}$. Picking $\varepsilon = \frac{c}{n}$ and drawing two random vectors from the unit ball, then we have with high probability these vectors have a length of $1 - O\left(\frac{1}{n}\right)$. For a GUB, pick $\delta = \frac{ck}{n}$, we see that the shell near the surface, that contains most of the volume, has a thickness $O\left(\frac{ck}{n}\right) = O\left(\frac{1}{n}\right)$. As we have shown that the volume is concentrated near the equator (according to our definition of the equator), this indicates that a good portion of the vectors picked at random from a GUB have length $1 - O\left(\frac{1}{n}\right)$. In 3D, for a convex ball, this means that most of the volume is concentrated near a 3D symmetric "cross-like" structure.

REMARK 2. In Chapter 2 of [3], the bounds on the volumes of spherical caps are shown to have several consequences. In [2], Lecture 2, upper and lower bounds on the area measure of spherical caps using volume ratios were derived. These bounds were employed to characterize bounds on the number of facets a symmetric polytope K in \mathbf{R}^n needs to have in order to approximate the Euclidean ball within a given distance. One can naively ask the same questions by replacing the Euclidean ball with a convex GUB, and attempt to formulate and prove similar results. For further possible generalizations and applications for considering such bounds, see Lectures 8,9 in [2].

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