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Lie Symmetry Analysis Of Early Carcinogenesis Model*

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Abstract

Lie group theory is applied to the model of a pre-cancerous cell population formulated by Bertolusso and Kimmel [1]. A complete symmetry analysis of the three dimensional PDE (partial differential equation) is performed to find invariant solutions and to construct new solutions as well as new solutions generated by the solution symmetry. Lie's equivalence transformation is used to transform the third equation into the heat transfer equation.

1 Introduction

In [1], Bertolusso and Kimmel developed a mathematical model which represents an early carcinogenesis. The development of cancer, by which normal cells convert to cancer cells is called oncogenesis or carcinogenesis. This transformation is identifed by the modification at the genetic and cellular levels and also at the abnormal cell division. The model explores the spatial effects steming. According to Bertolusso and Kimmel's model, the augmentation of pre-cancerous cells is due to the development of factor molecules. Bertolusso and Kimmel adopted elements of the model which was proposed in references [6] and [7]. Their model was governed by the following system of partial differential equations

$$\frac{\partial c}{\partial t} = (a(b,c) - d_c)c + \mu, \tag{1}$$

$$\frac{\partial b}{\partial t} = \alpha(c)g - d_b b - db, \tag{2}$$

$$\frac{\partial g}{\partial t} = \frac{1}{\gamma}\frac{\partial^2 g}{\partial x^2} - \alpha(c)g - d_g g + k(c) + db, \tag{3}$$

with the given boundary conditions for g

$$\partial_x g(0,t) = \partial_x g(1,t) = 0,$$

where c is the pre-cancerous cells; $a(b,c)$ is the proliferation rate; b is the growth rate; μ is the rate in which the pre-cancerous cells are provided; free factor g increases at rate

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$k(c)$; $1/\gamma$ is a diffusion rate; $\alpha(c)$ is a cell membrane receptor rate; d is a cell membrane dissociation rate; free factor g decreases at rate d_a and particles decrease at a rate d_b.

In this paper, the techniques of symmetry analysis is applied to identify the combinations of parameters insuring the linearisation of the nonlinear system (1)-(3) and produce the analytical solutions. Gazizov and Ibragimov [4] stated that Lie group analysis is a mathematical concept that incorporates symmetry of differential equations. The theory of Lie was developed by the Norwargian Mathematician, Sophus Lie. This method was the first mathematical theory to analyse nonlinear differential equations in terms of their symmetry groups. The method of Lie symmetry analysis uses group theoretic algorithms, by which a higher order differential equation is reduced to lower-order equations [4]. This paper is organised as follows. In Section 2, we analyse equation (3) from the Lie symmetry perspective. The eight-dimensional Lie symmetry algebra and the commutator table of the infinitesimal generators are obtained. In Section 3, we use a linear combination of the basic operator and found the equivalent invariant solutions. In Section 4, we construct news solutions from known ones. New solutions generated by the solution symmetry are find in Section 5. Transformations that map nonlinear PDE to linear PDE is constructed in Section 6. In Section 7, the general theorems on invertible mapping are used in order to map a nonlinear PDE into a heat equation.

2 Lie Group Analysis

A second order partial differential equation

$$u_t - F(t, x, u, u_{(1)}, u_{(2)}) = 0,$$

admits the one-parameter Lie group of transformations

$$\begin{aligned} \bar{t} &\approx t + a\xi^0(t,x,u), \\ \bar{x}^i &\approx x^i + a\xi^i(t,x,u), \\ \bar{u} &\approx u + a\eta(t,x,u), \end{aligned}$$

with infinitesimal generator

$$G = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{4}$$

if

$$\bar{u}_{\bar{t}} - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}) = 0, \quad \text{see [4]}.$$

The group transformations $\bar{t}$, $\bar{x}$ and $\bar{u}$ are obtained by solving the following Lie equations

$$\begin{aligned} \frac{d\bar{t}}{da} &= \xi^0(\bar{t}, \bar{x}, \bar{u}), \\ \frac{d\bar{x}^i}{da} &= \xi^i(\bar{t}, \bar{x}, \bar{u}), \\ \frac{d\bar{u}}{da} &= \eta(\bar{t}, \bar{x}, \bar{u}), \end{aligned} \tag{5}$$

with initial conditions

$$\bar{t}\mid_{a=0}= t, \quad \bar{x}^i \mid_{a=0}= x^i, \quad \bar{u}\mid_{a=0}= u.$$

The infinitesimal form of $\bar{u}_{\bar{t}}$, $\bar{u}_{(1)}$, $\bar{u}_{(2)}$ are found by the given formulas [4]:

$$\begin{aligned}
\bar{u}_{\bar{t}} &\approx u_t + a\zeta_0(t,x,u,u_t,u_{(1)}),\\
\bar{u}_{\bar{x}^i} &\approx u_{x^i} + a\zeta_i(t,x,u,u_t,u_{(1)}),\\
\bar{u}_{\bar{x}^i\bar{x}^j} &\approx u_{x^i} + a\zeta_{ij}(t,x,u,u_t,u_{(1)},u_{tx^k},u_{(2)}).
\end{aligned}$$

The functions ζ_0, ζ_i and ζ_{ij} are found by using the prolongation formulas below

$$\begin{aligned}
\zeta_0 &= D_t(\eta) - u_t D_t(\xi^0) - u_{x^i} D_t(\xi^i),\\
\zeta_i &= D_i(\eta) - u_t D_i(\xi^0) - u_{x^j} D_i(\xi^j),\\
\zeta_{ij} &= D_j(\zeta_i) - u_{x^i x^k} D_j(\xi^k) - u_{tx^i} D_j(\xi^0).
\end{aligned}$$

From equation (3), we have $n = 1$, $x^1 = x$ and the symbol of the Lie infinitesimal operator is set to be (See equation (4)):

$$G = \xi^0(t,x,g)\frac{\partial}{\partial t} + \xi^1(t,x,g)\frac{\partial}{\partial x} + \eta(t,x,g)\frac{\partial}{\partial g}. \tag{6}$$

Lie symmetry analysis of equation (3) is perfomed by using SYM package [3]. The analysis revealed that the obvious symmetries ∂_t and ∂_x are the only Lie point symmetry. Notwithstanding, if the dissociation rate d of the free growth factor g is negligible, then the coefficients of the Lie infinitesimal operator (6) is given by:

$$\begin{aligned}
\xi^0(x,t,g) &= \frac{xc_2}{2} + txc_3 + c_4 + tc_5,\\
\xi^1(x,t,g) &= xc_1 + t(c_2 + tc_3),\\
\eta(x,t,g) &= -\frac{1}{2}tc_3 - \frac{\gamma x^2 c_3}{4} - \alpha(c)t(c_2+tc_3) - d_g t(c_2+tc_3)\\
&\quad -\frac{1}{2}\gamma x c_5 + gc_6 + g + \phi(x,t).
\end{aligned}$$

Where c_1,...,c_6 are arbitrary constants and $\phi(x,t)$ is a solution of equation (3). Therefore we obtain the following Lie operators

$$\begin{aligned}
G_1 &= x\partial_x,\\
G_2 &= \frac{x}{2}\partial_t + t\partial_x - (\alpha(c)+d_g)t\partial_g,\\
G_3 &= tx\partial_t + t^2\partial_x - \frac{2t+\gamma x^2 + (4\alpha(c)+d_g)t^2}{4}\partial_g,\\
G_4 &= \partial_t,\\
G_5 &= t\partial_t - \frac{1}{2}\gamma x\partial_g,\\
G_6 &= g\partial_g,\\
G_7 &= \partial_g,
\end{aligned} \tag{7}$$

and

$$G_\phi = \phi(x,t)\partial_g. \tag{8}$$

Solving the Lie equations (5), we obtain the following corresponding basic generators:

$$\begin{aligned}
G_1 &: \bar{t} = t, \quad \bar{x} = xa_1, \quad \bar{g} = g,\\
G_2 &: \bar{t} = t + xa_2, \quad \bar{x} = x + ta_2, \quad \bar{g} = g - (\alpha c + d_g)ta_2, \quad a_2 \neq 0,\\
G_3 &: \bar{t} = te^{xa_3}, \quad \bar{x} = x + ta_3, \quad \bar{g} = g - \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}a_3,\\
G_4 &: \bar{t} = t + a_4, \quad \bar{x} = x, \quad \bar{g} = g,\\
G_5 &: \bar{t} = ta_5, \quad \bar{x} = x, \quad \bar{g} = g - \frac{1}{2}\gamma x a_5, \quad a_5 \neq 0,\\
G_6 &: \bar{g} = g + a_6,\\
G_7 &: \bar{t} = t, \quad \bar{x} = x, \quad \bar{g} = ga_7; \quad a_7 \neq 0,
\end{aligned}$$

and

$$G_\phi : \bar{t} = t, \quad \bar{x} = x, \quad \bar{g} = g + \phi(x,t).$$

	G_1	G_2	G_3	G_4	G_5	G_6	G_7
G_1	0	G_2	G_2	-G_2	G_2	0	G_2
G_2	$-G_2$	0	$(x-t)G_3$	0	G_2	$-(\alpha(c)+d_g)tG_7$	G_2
G_3	$-G_2$	$-(x-t)G_3$	0	G_2	$-\frac{t^2\gamma}{2}G_7$	G_3	G_2
G_4	G_2	0	$-G_2$	0	G_4	0	G_2
G_5	$-G_2$	$-G_2$	$-\frac{t^2\gamma}{2}G_7$	$-G_4$	0	$-\frac{\gamma x}{2}G_7$	G_2
G_6	0	$(\alpha(c)+d_g)tG_7$	$-G_3$	0	$\frac{\gamma x}{2}G_7$	0	G_2
G_7	$-G_2$	$-G_2$	$-G_2$	$-G_2$	$-G_2$	$-G_2$	0

The commutator table of the infinitesimal generator of equation (7).

3 Invariant Solutions

A linear combination of the infinitesimal generators (7) corresponds to a classification of a group-invariant solutions. These solutions are obtained from the reduced ordinary differential equation which depends on the particular subgroup under investigation [8]. Bearing in mind the one-parameter subgroup with the generator

$$G = G_4 + G_1 + G_6 \equiv \partial_t + x\partial_x + g\partial_g. \tag{9}$$

The Lagrange's system associated to equation (9) is given by:

$$\frac{dt}{1} = \frac{dx}{x} = \frac{dg}{g}. \tag{10}$$

Solving (10) we obtain the following two independent invariants

$$I_1 = t - \ln x \text{ and } I_2 = \frac{g}{x}. \tag{11}$$

Hence, the invariants of (9) will be in the form $I_2 = \phi(I_1)$ or from equation (11), the following invariant is obtained

$$g = x\psi(y), \tag{12}$$

where $y = t - \ln x$. The substituting of equation (12) into equation (3) gives

$$\frac{1}{\gamma}\psi''(y) - \zeta\psi'(y) = 0, \tag{13}$$

where $\psi'(y) = d\psi/dy$ and $\zeta = 1/\gamma + k(c) - (\alpha(c) + d_g)$. The solution of equation (13) is given by

$$\psi(y) = k_1 \exp(\zeta\gamma y) + k_2, \tag{14}$$

with k_1 and k_2 arbitrary constants. A tumor growth factor $g(x,t)$ is obtained by substituting equation (14) into (12)

$$g(x,t) = x\big[k_1 \exp(\zeta\gamma y) + k_2\big]. \tag{15}$$

Substituting the value of ζ and y into (15) we obtain

$$g(x,t) = x\big[k_1 \exp(1 + \frac{k(c) - (\alpha(c) + d_g)}{\gamma}(t - \ln x)) + k_2\big]. \tag{16}$$

The substitution of equation (15) into (2) gives

$$\frac{\partial b}{\partial t} + (d_b - d)b = \alpha(c)x[k_1 \exp(\zeta\gamma y) + k_2]. \tag{17}$$

Soving equation (17) we obtain

$$b(x,t) = \frac{\alpha(c)}{d_b - d}x\big[k_1 \exp(\zeta\gamma y) + k_2\big] + k_3. \tag{18}$$

The growth rate is obtain by substituting equation (16) into (18),

$$b(x,t) = \frac{\alpha(c)}{d_b - d}x\big[k_1 \exp(1 + \frac{k(c) - (\alpha(c) + d_g)}{\gamma}(t - \ln x)) + k_2\big] + k_3,$$

and the pre-cancerous cell $c(x,t)$ is given by

$$c(x,t) = -\frac{\mu}{a(b,c) - d_c} + k_4.$$

Sinkala [12] claimed that it is impractical to construct invariant solutions for all possible linear combinations of the basic operators (7). But one can determined a small representative set of symmetries (called an optimal system) and then calculates the corresponding invariant solutions [12].

4 New Solution From Known Ones

In this Section, the construction of new solutions of the differential equation is made from the known one.

THEOREM 1. If $g = f(x,t)$ is a solution of (3), so is $\bar{g} = f(\bar{x},\bar{t})$.

PROOF. The proof of Theorem 1 is outlined in [12].

In order to illustrate Theorem 1, we will be considering the one-parameter (δ) Lie group of transformation

$$G_3 : \bar{t} = te^{x\delta}, \quad \bar{x} = x + t\delta, \quad \bar{g} = g - \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}\delta, \delta \in \Re,$$

generated by the operator

$$G_3 = tx\partial_t + t^2\partial_x - \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}\partial_g.$$

Clearly,

$$g(x,t) = \exp{(\alpha(c) + d_g)t},$$

is a simple solution of equation (3) and by Theorem 1, so must

$$\bar{g}(\bar{x},\bar{t}) = \exp{(\alpha(c) + d_g)\bar{t}},$$

with $\bar{g}$, $\bar{x}$ and $\bar{t}$ given by

$$g(x,t) - \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}\delta = \exp{(\alpha(c) + d_g)te^{x\delta}}. \tag{19}$$

Setting $\delta = 1$ in (19) and make $g(x,t)$ the subject, we obtain tumor growth factor

$$g(x,t) = \exp{(\alpha(c) + d_g)te^x} + \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}. \tag{20}$$

The tumor growth rate, $b(x,t)$ is obtained by subtituting equation (20) into equation (3):

$$b(x,t) = \frac{\alpha(c)}{d_b - d}x\big[\exp{(\alpha(c) + d_g)te^x} + \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}\big],$$

and the pre-cancerous cell c is given by

$$c = -\frac{\mu}{a(b,c) - d_c}.$$

This method can be used to obtain more complicated solutions and possibly whole families of solutions.

5 New Solutions Generated By The Solution Symmetry

In this Section, a new solution from kown ones is found, even though using the direct method, the solution symmetry, G_φ (8), does not produce invariant solutions. Let $g = v(x,t)$ be a known solution of equation (3) so that G_v is admitted by (3). Then for any other infinitesimal generator G admitted by (3) and the Lie bracket is given by

$$[G_v, G] = G_{\bar{v}},$$

where $\bar{v}$ is another solution of (3), which in general is different from $v(x,t)$. It follows therefore that the family of operators, L_φ is an ideal of the Lie symmetry algebra of the form G_φ .

In order to illustrate this, we generate a new solution from the solution (15) by setting $k_1 = 1$ and $k_2 = 0$ into (15) to obtain

$$\varphi(x,t) = x\big[\exp(1 + \frac{k(c) - (\alpha(c) + d_g)}{\gamma}(t - \ln x))\big]. \tag{21}$$

Since (21) is a solution of (3), the symmetry G_φ is admitted by (3). Also

$$G_3 = tx\partial_t + t^2\partial_x - \frac{2t + \gamma x^2 + (4\alpha(c) + d_g)t^2}{4}\partial_g,$$

is a symmetry of equation (3). Taking the Lie bracket of G_φ and G_3 we obtain

$$[G_\varphi, G_3] = \bar{\phi}\partial_g,$$

where

$$\bar{\varphi}(x,t) = \varphi(x,t)\frac{(\alpha(c) + d_g)t + \ln x}{\gamma}.$$

Hence, $\bar{\varphi}(x,t)$ is the new solution of equation (3) with $\varphi(x,t)$ given by equation (21). The new solutions of equations (1) and (2) generated by the solution symmetry are given by

$$\bar{b}(x,t) = \frac{\alpha(c)}{d_b - d}x\big[\varphi(x,t)\frac{(\alpha(c) + d_g)t + \ln x}{\gamma}\big],$$

and

$$\bar{c}(x,t) = -\varphi(x,t)\frac{\mu}{a(b,c) - d_c},$$

respectively.

6 Mapping Nonlinear PDEs to Linear PDEs

Sophus Lie [5], in his theory of Lie group classification shown that a second-order partial differential equation can be reduced to the standard heat equation,

$$w_\tau = w_{zz}, \tag{22}$$

by means of the given Lie's transformation:

$$z = \alpha(x,t),\ \tau = \beta(t),\ w = \gamma(x,t)g,\ \alpha_x \neq 0,\ \beta_t \neq 0. \tag{23}$$

In this Section Sinkala's algorithm [12] is used to construct transformations that map equation (3) into heat equation (22). The general Theorems on invertible mappings are stated below [2]:

THEOREM 2. A mapping μ of the form [2]

$$\begin{aligned} z &= \phi(x, u, u_{(1)}, ..., u_l), \\ w &= \psi(x, u, u_{(1)}, ..., u_l), \end{aligned}$$

defines an invertible mapping from $(x, u, u_{(1)}, ..., u_p)$-space to $(z, w, w_{(1)}, ..., w_p)$-space for any fixed point p if and only if μ is a one-to-one contact transformation of the form

$$z = \phi(x, u, u_{(1)}), \tag{24}$$

$$w = \psi(x, u, u_{(1)}), \tag{25}$$

$$w_{(1)} = \psi_{(1)}(x, u, u_{(1)}). \tag{26}$$

Note that, if ϕ and ψ are independent of $u_{(1)}$, then (24)-(26) defines a point transformation.

PROOF. The proof of Theorem 2 is outlined in [12].

THEOREM 3. If there exists an invertible transformation μ which maps a given nonlinear partial differential equation $\Re\{x, u\}$ to a linear partial differential equation $S\{z, w\}$, then

(i) the mapping must be a point transformation of the form [2]

$$\begin{aligned} z_i &= \phi_j(x_1, x_2, u),\ j = 1, 2, \\ w &= \psi(x_1, x_2, u). \end{aligned}$$

(i) $\Re\{x, u\}$ must admit an infinite-parameter Lie group of point transformations with infinitesimal generator

$$G = \xi_1(x_1, x_2, u)\partial_{x_1} + \xi_2(x_1, x_2, u)\partial_{x_2} + \eta(x_1, x_2, u)\partial_{x_u} \tag{27}$$

with

$$\xi_1(x_1, x_2, u) = \alpha_1(x_1, x_2, u)F(x_1, x_2, u), \tag{28}$$

$$\xi_2(x_1, x_2, u) = \alpha_2(x_1, x_2, u)F(x_1, x_2, u), \tag{29}$$

$$\eta(x_1, x_2, u) = \beta(x_1, x_2, u)F(x_1, x_2, u), \tag{30}$$

where α_1, α_2 and β are some specific functions of (x_1, x_2, u) and F is an arbitrary solution of some linear partial differential equation

$$L[X]F = 0,$$

with LX representing a linear differential operator depending on independent variables

$$X = (X_1(x_1, x_2, u), X_2(x_1, x_2, u)),$$

of the same order as the order of the partial differential equation $\Re\{x, u\}$.

PROOF. The proof of Theorem 3 is outlined in [12].

THEOREM 4. Let a given nonlinear partial differential equation $\Re\{x, u\}$ admit an infinitesimal generator (27) the coefficients of which are of the form (28)–(30) with F being an arbitrary solution of a linear partial differential equation with specific independent variables

$$X = (X_1(x_1, x_2, u), X_2(x_1, x_2, u)).$$

If the linear homogeneous first-order partial differential equation for scalar Φ,

$$\alpha_1(x_1, x_2, u)\frac{\partial \psi}{\partial x_1} + \alpha_2(x_1, x_2, u)\frac{\partial \psi}{\partial x_2} + \beta(x_1, x_2, u)\frac{\partial \psi}{\partial u} = 1,$$

has a solution

$$\psi = (\psi^1(x_1, x_2, u), \psi^2(x_1, x_2, u)),$$

then the invertible mapping μ given by

$$\begin{aligned} z_1 &= \phi_1(x_1, x_2, u) = X_1(x_1, x_2, u), \\ z_2 &= \phi_2(x_1, x_2, u) = X_2(x_1, x_2, u), \\ w &= \psi(x_1, x_2, u), \end{aligned}$$

transforms $\Re\{x, u\}$ to a linear partial differential equation $S\{z, w\}$:

$$L[z]w = g(z),$$

for some nonhomogeneous term $g(z)$.

PROOF. The proof of Theorem 4 is outlined in [12].

Equation (23) is used to write the heat equation, (22), in terms of the variables x, t and g and we compare it with equation (3). Applying Theorems 2, 3 and 4, we obtain the following transformations:

$$\begin{aligned} z &= \frac{\ln x}{\gamma(L - Kt)} + \frac{M}{L - Kt} + N, \\ \tau &= -\frac{1}{2K(L - Kt)} + P, \\ w(z, \tau) &= Eg(x, t)\sqrt{L - KT}\exp\frac{M^2 K}{2(L - Kt)} - \frac{(\alpha(c) - d_g)}{\gamma}t \\ &\quad \times \frac{MK}{\gamma}(L - Kt) + K\frac{\ln x}{2\gamma^2(L - Kt)}, \end{aligned}$$

and

$$\begin{aligned} z &= \frac{L}{\gamma}\ln x + Mt + N, \\ \tau &= -\frac{L^2}{2}t + P, \\ w(z,\tau) &= Eg(x,t)\exp\frac{M^2}{2L^2} - \frac{(\alpha(c)-d_g)}{\gamma}t \times \frac{M}{\gamma}L, \end{aligned}$$

with E, K, L, M, N and P arbitrary constants.

7 Conversion of Equation (3) to the Heat Equation

We perceive in commutator Table 1 that the Lie bracket of the generators G_2 and G_4 is zero. We may use $X_1 := G_2$ and $X_2 := G_4$ to construct a transformation to map equation (3) invertibly into a constant coefficient partial differential equation [10]. Theorem 3 revealed that the nonlinear partial differential equation (3) admits the following Lie point transformations with infinitesimal generator:

$$\begin{aligned} X_1 &= \xi_{11}(x,t)\partial_x + \xi_{12}(x,t)\partial_t + f_1(x,t)\partial_g, \\ X_2 &= \xi_{21}(x,t)\partial_x + \xi_{22}(x,t)\partial_t + f_2(x,t)\partial_g. \end{aligned}$$

So that

$$\begin{aligned} \xi_{11} &= t,\ \xi_{12} = \frac{x}{2},\ f_1 = -(\alpha(c)-d_g)t, \\ \xi_{21} &= 0,\ \xi_{22} = 1,\ f_2 = 0. \end{aligned} \tag{31}$$

Since

$$det\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} = det\begin{pmatrix} t & \frac{x}{2} \\ 0 & 1 \end{pmatrix} = t \neq 0,$$

and from Theorem 4, there exists an invertible mapping of the form

$$z = \alpha(x,t),\ \tau = \beta(x,t),\ w = \nu(x,t)g, \tag{32}$$

to map the equation (3) into a constant coefficient partial differential equation. The mapping (32) must satisfy the following conditions:

$$\begin{aligned} \xi_{11}\alpha_x + \xi_{12}\alpha_t &= 1, \\ \xi_{21}\alpha_x + \xi_{22}\alpha_t &= 0, \\ \xi_{11}\beta_x + \xi_{12}\beta_t &= 0, \\ \xi_{21}\beta_x + \xi_{22}\beta_t &= 0, \\ \xi_{11}\nu_x + \xi_{12}\nu_t &= -f_1\nu, \\ \xi_{21}\nu_x + \xi_{22}\nu_t &= -f_2\nu. \end{aligned} \tag{33}$$

The substitution of (31) into (33) gives

$$\begin{aligned}\alpha(x,t) &= \frac{x}{t}+\zeta,\\ \beta(x,t) &= \chi,\\ \nu(x,t) &= \frac{\kappa}{1+(\alpha(c)+d_g)x}.\end{aligned}$$

where $\chi,\ \kappa \neq 0$ and ζ an arbitrary constant. For simplicity, we set $\chi = \kappa = 1$ and $\zeta = 0$, and obtain

$$\begin{aligned}z &= \frac{x}{t},\\ \tau &= 1,\\ w(x,t) &= \frac{1}{1+(\alpha(c)+d_g)x}g(x,t).\end{aligned} \tag{34}$$

For $a(b,c) \neq 0$, we have shown that the transformation (34) maps equation (3) invertible into the heat equation,

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial z^2}.$$

8 Conclusion

Nonlinear differential equations play an important role in the explanation of many physical models [9]. In order to arrive at a complete understanding of the phenomena which are modeled it is important to obtain closed form solutions [11]. In this paper, a model of a pre-cancerous cell population is analysed from the Lie symmetry perspective. The model was formulated by Marciniak and Kimmel [6]. In their model, they assume that a pre-cancerous cell increases quickly at a rate $a(b,c)$. In [10], Matadi claimed that Lie group analysis is the most useful method to obtain an analytical solution of nonlinear differential equations. In this paper, we integrated the pre-cancerous cell model by quadrature and obtain the general solution. The invariant solutions of the growth rate, the tumour growth and pre-cancerous cell are found. We also map a nonlinear partial differential equation (3) into a heat equation by the mean of the general theorems on invertible mapping.

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