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Asymptotic Stability For Cauchy Problem Of A Plate Equation With Distributed Delay Term*

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Abstract

In this paper, we exploit the weighted spaces and some density function under some conditions to establish a general decay rate property of solutions of a viscoelastic plate equation with distributed delay term in the whole space R^n.

1 Introduction

In this paper, we consider the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u_t(x,t) & \\ \quad + \phi(x)\left(\alpha\Delta^2 u(x,t) - \int_{-\infty}^{t} g(t-s)\Delta^2 u(x,s)ds\right) & (x,t) \in R^n \times R^+, \\ \quad + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(x,t-s)ds = 0, & \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in R^n, \\ u(x,-t) = u_0(x,t), & x \in R^n, \ t \geq 0, \\ u_t(x,-t) = f_0(x,t), & x \in R^n, \ t \geq 0, \end{cases} \tag{1}$$

where the space $D^{2,2}(R^n)$ $(n \geq 5)$ is defined in (2) and $\phi(x) > 0, \forall\, x \in R^n$, $(\phi(x))^{-1} = \rho(x)$ defined in (A_1), μ_1, α, τ_2 are positive constants, τ_1 is a nonnegative constant with $\tau_1 < \tau_2$ and $\mu_2 : [\tau_1, \tau_2] \to R$ is a bounded function and the initial data (u_0, u_1, f_0) belongs to a suitable space.

In this paper we consider the solutions in spaces weighted by the density function $\rho(x)$ in order to compensate for the lack of Poincaré's inequality. Equation (1) with the memory term $\int_{-\infty}^{t} g(t-s)\Delta^2 u(x,s)ds$ can be viewed as an elastoplastic flow equation with some kind of memory, the interaction of strong damping with other terms implies an exchange of the energy, which is physically transmitted from one place to another, there is a delay associated with the transmission which is denoted by τ_1 and τ_2. To motivate our work, let us start with the plate equation proposed in [2], the authors in

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[2] considered the following plate equation

$$\begin{aligned} &u_{tt}(x,t) + a\Delta^2 u(x,t) - \Delta_p u(x,t) - \int_0^t g(t-s)\Delta^2 u(x,s)ds - \Delta u_t(x,t) \\ &+ f(u(x,t)) = h\ (x,t) \in \Omega \times R^+, \end{aligned}$$

where Ω is a bounded domain in R^n with simply supported boundary condition, $g > 0$ is a memory kernel that decays exponentially and $f(u)$ is a nonlinear perturbation, they established a result about the global existence by using the Faedo-Galerkin's method and the energy decay by using the perturbed energy method.

The authors in [1] considered the more general problem

$$\begin{aligned} &u_{tt}(x,t) + a\Delta^2 u(x,t) - \Delta_p u(x,t) - \int_{-\infty}^t g(t-s)\Delta^2 u(x,s)ds - \Delta u_t(x,t) \\ &+ f(u(x,t)) = h\ (x,t) \in \Omega \times R^+, \end{aligned}$$

by applying the same procedure as above, they obtained the existence result and the asymptotic stability of the problem.

For the Cauchy problem with density, Karachalios and Stavrakikis [3] considered the following semilinear hyperbolic initial value problem

$$\begin{cases} u_{tt}(x,t) - \phi(x)\Delta u(x,t) + \delta u_t(x,t) + \lambda f(u(x,t)) = \eta(x), & (x,t) \in R^n \times R^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in R^n, \end{cases}$$

the authors proved local existence of solutions and established the existence of a global attractor in energy space $D^{1,2}(R^n) \times L^2_g(R^n)$ by using the compactness of embedding $D^{1,2}(R^n) \subset L^2_g(R^n)$ in the case where $(\phi(x))^{-1} := g(x) \in L^{\frac{n}{2}}(R^n)$ and $n \geq 3$. Very recently, Baowei Feng [5] studied the following problem

$$\begin{cases} u_{tt}(x,t) - \phi(x)\left(\Delta u(x,t) - \int_0^t g(t-s)\Delta u(x,s)ds\right) & \\ \quad + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & (x,t) \in R^n \times R^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in R^n, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in R^n,\ t \in (0,\tau), \end{cases}$$

where $u_0(x)$, $u_1(x)$ and $f_0(x,t-\tau)$ are given initial data belonging to appropriate spaces, the function $g(t)$ is the relaxation function. The constants μ_1, μ_2 are two real numbers, and $\tau > 0$ denote the time delay, the author proved a general decay result of solution for the initial value problem by using energy perturbation method, although he found the difficulties in non compactness of some operators. To overcome these difficulties, the main idea is to introduce some weighted spaces to compensate the lack of Poincaré's inequality in the whole space.

Motivated by the previous works, in the present paper, it is interesting to show more general decay rate property to that one obtained in [5], by exploiting the same procedure in [1–4] with the necessary modification imposed by the nature of our problem. Further, we omit the space variable x of $u(x,t)$, $u'(x,t)$ and for simplicity, we denote $u(x,t) = u$,

$u'(x,t) = u_t$, $|\nabla_x u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2$ and $\Delta_x u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. The constant c used through this paper takes on positive values, also the functions considered are all real valued, here $u' = \frac{\partial u(t)}{\partial t}$ and $u'' = \frac{\partial^2 u(t)}{\partial t^2}$.

2 Preliminary Results and Transformations

At first, we recall and make use of the following assumptions:

(A_0) We assume that the function $g : R^+ \to R^+$ is class C^1 satisfying

$$\int_0^\infty g(t)dt = g_0 > 0,\ g'^+ \leq 0, \quad \forall t \in R^+,$$

and there exists a constant $k > 0$ such that $g'(t) \leq -kg(t)$.

(A_1) The function $\rho : R^n \to R^n_+$, $\rho(x) \in C^{0,\gamma}(R^n)$ with $\gamma \in (0,1)$ and $\rho \in L^s(R^n) \cap L^\infty(R^n)$, where $s = \frac{2n}{2n-qn+4q}$.

The following technical lemmas are needed.

DEFINITION 1 ([3, 5]). We define the function spaces of our problem and its norm as follows:

$$D^{2,2}(R^n) = \left\{ f \in L^{\frac{2n}{n-4}}(R^n); \Delta_x f \in L^2(R^n) \right\}, \tag{2}$$

and the space $L^2_\rho(R^n)$ to be the closure of $C_0^\infty(R^n)$ functions with respect to the inner product

$$\langle f, h \rangle_{L^2_\rho(R^n)} = \int_{R^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on R^n we define

$$\|f\|_{L^2_\rho(R^n)} = \left(\int_{R^n} \rho |f|^q dx \right)^{\frac{1}{q}},$$

and $D^{2,2}(R^n)$ can be embedded continuously in $L^{\frac{2n}{n-4}}(R^n)$, i.e there exists $k > 0$ such that

$$\|u\|_{L^{\frac{2n}{n-4}}(R^n)} \leq k \|u\|_{D^{2,2}(R^n)}.$$

The separable Hilbert space $L^2_\rho(R^n)$ is equipped with the following inner product:

$$\langle f, f \rangle_{L^2_\rho(R^n)} = \|f\|^2_{L^2_\rho(R^n)}.$$

LEMMA 1. Let (A_1) hold, then for any $u \in D^{2,2}(R^n)$

$$\|u\|_{L^q_\rho(R^n)} \leq \|\rho\|_{L^s(R^n)} \|\Delta_x u\|_{L^2(R^n)}, \text{ with } s = \frac{2n}{2n-qn+4q}, \quad 2 \leq q \leq \frac{2n}{n-4},$$

where we can assume $\|\rho\|_{L^s(R^n)} = C_0 > 0$ to get

$$\|u\|_{L^q_\rho(R^n)} \leq C_0\|\Delta_x u\|_{L^2(R^n)}.$$

Following the idea of [5], we consider

$$\eta(x,t,\sigma) = u(x,t) - u(x,t-\sigma), \quad \text{in} \quad R^n \times R_+ \times R_+. \tag{3}$$

In order to consider the relative displacement η as a new variable, we introduce the weighted L^2-spaces

$$L^2_g(R^+; H^2(R^n)) = \left\{\zeta : R^+ \to H^2(R^n); \int_0^\infty g(\sigma)\|\zeta(\sigma)\|^2_V d\sigma < \infty\right\},$$

where $V = H^2(R^n)$ and $L^2_g(R^+; H^2(R^n))$ is the Hilbert space of $H^2(R^n)$-valued functions on R^+ endowed with the inner product

$$\langle\zeta,\zeta\rangle_{L^2_g} = \int_0^\infty g(\sigma)\langle\zeta(\sigma),\zeta(\sigma)\rangle_V d\sigma, \quad \|\zeta\|^2_{L^2_g} = \int_0^\infty g(\sigma)\|\zeta\|^2 d\sigma.$$

To deal with the delay term, we introduce the new variable z as in [7],

$$z(x,k,s,t) = u_t(x,t-ks), \quad (x,k,s,t) \in R^n \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

From (3), we obtain

$$\begin{cases} \eta(x,t,0) = 0, & \text{in} \quad R^n \times R_+, \\ \eta(x,t,\sigma) = 0, & \text{in} \quad R^n \times R_+ \times R_+, \\ \eta(x,\sigma) := \eta(x,0,\sigma) = u_0(x,0) - u_0(x,\sigma), & \text{in} \quad R^n \times R_+. \end{cases}$$

We denote $\eta_t = \frac{\partial\eta}{\partial t}$, $\eta_\sigma = \frac{\partial\eta}{\partial\sigma}$. So, (3) gives

$$\eta_t(x,t,\sigma) + \eta_\sigma(x,t,\sigma) = u_t(x,t), \quad in \quad R^n \times R_+ \times R_+.$$

Thus, the original memory term can be rewritten as

$$\begin{aligned} \int_{-\infty}^t g(t-\sigma)\Delta^2 u(\sigma)d\sigma &= \int_0^\infty g(\sigma)\Delta^2 u(t-\sigma)d\sigma \\ &= \left(\int_0^\infty g(\sigma)d\sigma\right)\Delta^2 u(t) - \int_0^\infty g(\sigma)d\sigma\Delta^2\eta(\sigma)d\sigma. \end{aligned}$$

Therefore, problem (1) is equivalent to:

$$\begin{cases} u_{tt} - \Delta u_t + \phi(x)\left(\alpha - \int_0^\infty g(\sigma)d\sigma\right)\Delta^2 u + \int_0^\infty g(\sigma)\phi(x)\Delta^2\eta(\sigma)d\sigma \\ \quad +\mu_1 u_t + \int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)ds = 0, \quad \text{in} \quad R^n \times (0,\infty), \\ z(x,k,s,0) = f_0(x,ks), \quad \text{in} \quad R^n \times (0,1) \times (\tau_1,\tau_2), \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \quad \text{in} \quad R^n. \end{cases} \tag{4}$$

Then assuming for simplicity that $\left(\alpha - \int_0^\infty g(\sigma)d\sigma\right) = 1$, problem (4) is transformed into the system

$$\begin{cases} u_{tt} - \Delta u_t + \phi(x)\Delta^2 u + \int_0^\infty g(\sigma)\phi(x)\Delta^2\eta(\sigma)d\sigma \\ \quad +\mu_1 u_t + \int_{\tau_1}^{\tau_2} \mu_2(s)z(x,1,s,t)ds = 0,, \quad \text{in} \quad R^n \times (0,\infty), \\ z(x,k,s,0) = f_0(x,ks),, \quad \text{in} \quad R^n \times (0,1) \times (\tau_1,\tau_2) \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x),, \quad \text{in} \quad R^n. \end{cases} \tag{5}$$

The energy associated with (5) is defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(R^n)} + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|\eta\|^2_{L^2_g} \\ &\quad + \frac{1}{2}\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2} \rho(x)s(|\mu_2(s)| + \xi)z^2(x,k,s,t)dsdkdx, \end{aligned} \tag{6}$$

where ξ is a positive constant such that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds + \frac{\xi(\tau_2 - \tau_1)}{2} < \mu_1. \tag{7}$$

LEMMA 2. The functional defined in (6) satisfies the following inequality:

$$\begin{aligned} E'(t) &\leq -\left[\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds - \frac{\xi(\tau_2-\tau_1)}{2}\right]\|u_t(t)\|^2_{L^2_\rho(R^n)} \\ &\quad -m_1\int_{R^n}\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\rho(x)dxds - \|\nabla u_t\|^2_{L^2_\rho(R^n)} - \delta g_0\|\Delta\eta_t\|_2^2 \\ &\quad -\frac{1}{4\delta}\|\eta\|^2_{L^2_g} + \frac{1}{2}\int_0^\infty g'(\sigma)\|\Delta\eta(\sigma)\|_2^2 d\sigma \leq 0, \ \forall\ t \geq 0, \end{aligned}$$

where μ_1 is a positive constant.

PROOF. By multiplying the first equation in (5) by $\rho(x)u_t(t)$, and integrating over R^n, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2_{L^2_\rho(R^n)} + \frac{1}{2}\|\nabla u_t(t)\|^2_{L^2_\rho(R^n)} + \frac{1}{2}\frac{d}{dt}\|\Delta u(t)\|_2^2 \\ &+ \int_{R^n}\int_{\tau_1}^{\tau_2} \mu_2(s)\rho(x)z(x,1,s,t)u_t(t)dxds + \mu_1\|u_t(t)\|_{L^2_\rho(R^n)} \\ &+ \int_0^\infty g(\sigma)\int_{R^n} \Delta\eta(\sigma)\Delta u_t(t)dxd\sigma = 0. \end{aligned}$$

Since

$$u_t(x,t) = \eta_t(x,\sigma) + \eta_\sigma(x,\sigma), \quad (x,\sigma) \in R^n \times R^+, \ t \geq 0,$$

we have

$$\begin{aligned}
&\int_0^\infty g(\sigma)\int_{R^n}\Delta\eta(\sigma)\Delta u_t(t)dxd\sigma\\
=\ &\int_0^\infty g(\sigma)\int_{R^n}\Delta\eta(\sigma)\Delta\eta_t(t)dxd\sigma+\int_0^\infty g(\sigma)\int_{R^n}\Delta\eta(\sigma)\Delta\eta_\sigma(t)dxd\sigma\\
=\ &\frac{1}{2}\int_0^\infty g(\sigma)\frac{d}{dt}\|\Delta\eta(\sigma)\|_2^2d\sigma-\frac{1}{2}\int_0^\infty g'(\sigma)\|\Delta\eta(\sigma)\|_2^2d\sigma\\
&+\int_0^\infty g(\sigma)\int_{R^n}\Delta\eta(\sigma)\Delta\eta_t(t)dxd\sigma.
\end{aligned}\tag{8}$$

Using Young's inequality, we have for any $\delta>0$

$$\begin{aligned}
&\int_0^\infty g(\sigma)\int_{R^n}\Delta\eta(\sigma)\Delta\eta_t(t)dxd\sigma\\
\le\ &\int_0^\infty g(\sigma)\left(\frac{1}{4\delta}\|\Delta\eta(\sigma)\|_2^2+\delta\|\Delta\eta_t\|_2^2\right)d\sigma\\
\le\ &\delta\left(\int_0^\infty g(\sigma)d\sigma\right)\|\Delta\eta_t\|_2^2+\frac{1}{4\delta}\int_0^\infty g(\sigma)\|\Delta\eta(\sigma)\|_2^2d\sigma\\
=\ &\delta g_0\|\Delta\eta_t\|_2^2+\frac{1}{4\delta}\|\eta\|_{L_g^2}^2.
\end{aligned}$$

We multiply the second equation in (5) by $\rho(x)z$ and integrate over $R^n\times(0,1)\times(\tau_1,\tau_2)$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}\rho(x)s(\mu_2(s)+\xi)z^2(x,k,s,t)dsdkdx\\
=\ &-\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}\rho(x)(\mu_2(s)+\xi)zz_k(x,k,s,t)dsdkdx\\
=\ &-\frac{1}{2}\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}\rho(x)(\mu_2(s)+\xi)\frac{\partial}{\partial k}z^2(x,k,s,t)dsdkdx\\
=\ &\frac{1}{2}\int_{R^n}\int_{\tau_1}^{\tau_2}\rho(x)(\mu_2(s)+\xi)(z^2(x,0,s,t)-z^2(x,1,s,t))dsdx\\
\le\ &\frac{1}{2}\left[\xi(\tau_2-\tau_1)+\int_{\tau_1}^{\tau_2}\mu_2(s)ds\right]\int_{R^n}\rho(x)u_t^2dx\\
&-\frac{1}{2}\int_{R^n}\int_{\tau_1}^{\tau_2}\rho(x)(\mu_2(s)+\xi)z^2(x,1,s,t)dsdx.
\end{aligned}\tag{9}$$

From (8) and (9), we obtain

$$\begin{aligned}
E'(t)\ \le\ &-\left[\mu_1-\frac{\xi(\tau_2-\tau_1)}{2}\right]\|u_t(t)\|_{L_\rho^2(R^n)}^2\\
&-m_1\int_{R^n}\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)\rho(x)u_tdxds-\|\nabla u_t\|_{L_\rho^2(R^n)}^2-\delta g_0\|\Delta\eta_t\|_2^2\\
&-\frac{1}{4\delta}\|\eta\|_{L_g^2}^2+\frac{1}{2}\int_0^\infty g'(\sigma)\|\Delta\eta(\sigma)\|_2^2d\sigma\le 0,\ \forall\ t\ge 0
\end{aligned}\tag{10}$$

and

$$\begin{aligned}
&\int_{R^n}\int_{\tau_1}^{\tau_2}\mu_2(s)\rho(x)z(x,1,s,t)u_t(x,t)dsdx\\
\leq\ &\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\|u_t(t)\|_{L^2_\rho(R^n)}ds+\frac{1}{2}\int_{R^n}\int_{\tau_1}^{\tau_2}\rho(x)z^2(x,1,s,t)dsdx. \qquad (11)
\end{aligned}$$

Inserting (11) into (10), we obtain

$$\begin{aligned}
E'(t)\ \leq\ & -\left[\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds-\frac{\xi(\tau_2-\tau_1)}{2}\right]\|u_t(t)\|^2_{L^2_\rho(R^n)}\\
&-m_1\int_{R^n}\int_{\tau_1}^{\tau_2}z^2(x,1,s,t)\rho(x)dxds-\|\nabla u_t\|^2_{L^2_\rho(R^n)}-\delta g_0\|\Delta\eta_t\|_2^2\\
&-\frac{1}{4\delta}\|\eta\|^2_{L^2_g}+\frac{1}{2}\int_0^\infty g'(\sigma)\|\Delta\eta(\sigma)\|_2^2d\sigma\leq 0,\ \forall\ t\geq 0.
\end{aligned}$$

3 Asymptotic Stability

In this section, we prove the energy decay result. Our procedure is based on the perturbed energy method. For some $\epsilon>0$, let us consider the perturbed energy functional

$$L(t)=E(t)+\epsilon\psi(t)+\varphi(t), \qquad (12)$$

where

$$\psi(t)=\int_{R^n}\rho(x)u_t(t)u(t)dx, \qquad (13)$$

$$\varphi(t)=\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}se^{-sk}(|\mu_2(s)|+\xi)z^2(x,k,s,t)dsdkdx. \qquad (14)$$

The next lemma will be useful, which means that there is an equivalence between the $L(t)$ and energy functional.

LEMMA 2. There exists a constant $N_1>0$ such that

$$|L(t)-E(t)|\leq\varepsilon N_1E(t),\quad \forall\ t\geq 0$$

where $\varepsilon\geq 0$.

PROOF. By applying Young's inequality to (13), we obtain

$$\begin{aligned}
|\psi(t)|\ &\leq\ \int_{R^n}|\rho(x)u_t(t)u(t)|\,dx\leq\frac{1}{2}\|u_t\|_{L^2_\rho(R^n)}+\frac{1}{2\lambda}\|u\|_{L^2_\rho(R^n)}\\
&\leq\ \frac{1}{2}\|u_t\|_{L^2_\rho(R^n)}+\frac{C_0}{2\lambda}\|\Delta u\|_2^2\leq\max\left\{1,\frac{C_0}{2\lambda}\right\}E(t). \qquad (15)
\end{aligned}$$

It follows from (14) that $\forall c>0$,

$$\begin{aligned}
|\varphi(t)| &= \left|\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}\rho(x)se^{-sk}(|\mu_2(s)|+\xi)z^2(x,k,s,t)dsdkdx\right| \\
&\leq c\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}\rho(x)s(|\mu_2(s)(s)|+\xi)z^2(x,k,s,t)dsdkdx \\
&\leq cE(t). \qquad (16)
\end{aligned}$$

Hence, combining (15) and (16), we get the desired result.

LEMMA 3. Under the assumptions (A_0)–(A_1), the functional ψ satisfies

$$\begin{aligned}
\frac{d\psi(t)}{dt} &\leq -E(t)+c_1\|\nabla u_t\|^2_{L^2_\rho(R^n)}+c_2\|\eta\|^2_{L^2_g}-c_3\|\Delta u(t)\|_2^2 \\
&\quad +c_4\int_{\tau_1}^{\tau_2}|\mu_2(s)|\|z(x,1,s,t)\|^2_{L^2_\rho(R^n)}ds,
\end{aligned}$$

where $c_1,\ c_2,\ c_3,\ c_4$, are positive constants.

PROOF. Differentiating (13), integrating by parts over R^n, using (5) and subtracting $E(t)$, we have

$$\begin{aligned}
\psi'(t) &= \int_{R^n}\rho(x)u_t^2dx+\int_{R^n}\rho(x)u_{tt}udx \\
&\quad +\int_{R^n}\left(\rho(x)u_t^2-\Delta^2u.u-\int_0^\infty g(s)\Delta^2\eta(s)uds+\rho(x)\Delta u_t.u\right) \\
&\quad -\int_{R^n}\left(\mu_1\rho(x)u_t.u+\int_{\tau_1}^{\tau_2}\mu_2(s)\rho(x)z(x,1,s,t).uds\right)dx \\
&= -E(t)+\frac{3}{2}\|u_t\|^2_{L^2_\rho(R^n)}-\frac{1}{2}\|\Delta_x u\|_2^2+\frac{1}{2}\|\eta\|^2_{L^2_g(R^n)}+I_1+I_2+I_3+I_4,
\end{aligned}$$

where

$$I_1=-\int_{R^n}\int_0^\infty g(s)\Delta\eta(s)\Delta u(t)dsdx,$$

$$I_2=-\int_{R^n}\rho(x)\nabla u(t)\nabla u_t(t)dx,$$

$$I_3=-\int_{R^n}\rho(x)u(t)u_t(t)dx,$$

$$I_4=-\int_{R^n}\int_{\tau_1}^{\tau_2}\mu_2(s)\rho(x)z(x,1,s,t)u(t)dsdx.$$

By using Young's and Poincaré's inequalities, we have for any $\delta > 0$,

$$\begin{aligned} I_1 &\leq \int_0^\infty g(s)\left(\frac{1}{4\delta}\|\Delta\eta(s)\|_2^2 + \delta\|\Delta u(s)\|_2^2\right)ds, \\ &+\delta\left(\int_0^\infty g(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{4\delta}\int_0^\infty g(s)\|\Delta\eta(s)\|_2^2 ds \\ &= \delta g_0\|\Delta u(t)\|_2^2 + \frac{1}{4\delta}\|\eta(s)\|_{L_g^2}^2, \end{aligned} \tag{17}$$

$$I_2 \leq \delta\|\nabla u\|_{L_\rho^2(R^n)}^2 + \frac{1}{\delta}\|\nabla u_t\|_{L_\rho^2(R^n)}^2 \leq \delta C_0\|\Delta u\|_2^2 + \frac{1}{\delta}\|\nabla u_t\|_{L_\rho^2(R^n)}^2, \tag{18}$$

$$I_3 \leq \delta\|u_t\|_{L_\rho^2(R^n)}^2 + \frac{C_0}{\delta}\|\Delta u\|_2^2, \tag{19}$$

$$\begin{aligned} &\int_{R^n}\int_{\tau_1}^{\tau_2}\rho(x)\mu_2(s)z(x,1,s,t)udx \\ \leq\ & \delta C_0\|\Delta u\|_2^2, + \underbrace{\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds}_{<\mu_1}\int_{R^n}\int_{\tau_1}^{\tau_2}\rho(x)^2|\mu_2(s)|z^2(x,1,s,t)dsdx. \end{aligned} \tag{20}$$

Then

$$\begin{aligned} &\int_{\tau_1}^{\tau_2}\int_{R^n}\rho(x)\mu_2(s)z(x,1,s,t)udxds \\ \leq\ & \delta C_0\|\Delta u\|_2^2 + \frac{\mu_1}{4\delta}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\|z(x,1,s,t)\|_{L_\rho^2(R^n)}^2 ds, \end{aligned}$$

which, together with (17)–(20), yield

$$\begin{aligned} \frac{d\psi(t)}{dt} \leq\ & -E(t) + \left(\frac{3}{2}C_0 + \frac{1}{\delta} + \delta C_0\right)\|\nabla u_t\|_{L_\rho^2(R^n)}^2 + \left(\frac{1}{2} + \frac{1}{4\delta}\right)\|\eta\|_{L_g^2}^2 \\ & -\left(\frac{1}{2} - \delta C_0 - \delta g_0 - 2\delta C_0\right)\|\Delta u(t)\|_2^2 \\ & +\frac{\mu_1}{4\delta}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\|z(x,1,s,t)\|_{L_\rho^2(R^n)}^2 ds. \end{aligned}$$

LEMMA 3. The functional defined by (14) satisfies for some $\gamma_0 > 0$,

$$\frac{d}{dt}\varphi(t) \leq c\int_{R^n}\rho(x)u_t^2dx - \gamma_0\int_{R^n}\int_{\tau_1}^{\tau_2}s(|\mu_2(s)| + \xi)\rho(x)z^2(x,s,k,t)dsdx.$$

PROOF. Differentiating (14) with respect to t and using the second equation in

(5), we have

$$
\begin{aligned}
\varphi'(t) &= -2\int_{R^n}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_0^1 e^{-sk}\rho(x)zz_\rho dkdsdx\\
&= -\int_{R^n}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_0^1 \rho(x)e^{-sk}\frac{\partial}{\partial k}z^2dkdsdx\\
&\quad -\int_{R^n}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\rho(x)Bigg\ [e^{-s}(z^2(x,1,s,t)-z^2(x,0,s,t))\\
&\qquad +s\int_0^1 e^{-sk}z^2dkBigg\ ]dsdx\\
&\leq c\int_{R^n}\rho(x)u_t^2dx-\gamma_0\int_{R^n}\int_0^1\int_{\tau_1}^{\tau_2}\rho(x)s(|\mu_2(s)|+\xi)z^2(x,s,k,t)dsdkdx.
\end{aligned}
$$

The proof is complete.

THEOREM 1. Let $(u_0(x), u_1(x), f_0(x, ks), \eta(x, \sigma)) \in (H^2(R^n)) \times L^2_\rho(R^n) \times L^2_\rho(R^n \times (0, 1) \times (\tau_1, \tau_2)) \times L^2_g(R^+, H^2(R^n))$. We suppose that (A_0)–(A_1) hold. Then, there exist positive constants C and β such that the energy of solution given by (6) satisfies

$$E(t) \leq CE(0)e^{-\beta t}.$$

PROOF. Taking the derivatives of (12) and exploiting Lemmas 1–3, we deduce that

$$
\begin{aligned}
L'(t) &= E'(t)+\epsilon\psi'(t)+\varphi'(t)\\
&\leq -\epsilon E(t)-\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)ds-\frac{\xi(\tau_2-\tau_2)}{2}-c\right)\|u_t\|^2_{L^2_\rho(R^n)}\\
&\quad -(1+\epsilon c_1)\|\nabla u_t\|^2_{L^2_\rho(R^n)}-\epsilon c_3\|\Delta u\|_2^2-\left(\frac{k}{2}-\epsilon c_2\right)\|\eta\|^2_{L^2_\rho(R^n)}\\
&\quad -\int_{\tau_1}^{\tau_2}(m+c_4\epsilon|\mu_2(s)|+\gamma_0 s(|\mu_2(s)|+\xi))\,\|z(x,1,s,t)\|^2_{L^2_\rho(R^n)}ds\\
&\leq -\epsilon E(t), \qquad (21)
\end{aligned}
$$

using Lemma 3, and integrating (21), we get for some $\beta > 0$,

$$L(t) \leq L(0)e^{-\epsilon\beta t}, \quad \forall\ t \geq 0,$$

using Lemma 3 again , we obtain

$$E(t) \leq CE(0)e^{-\epsilon\beta t}, \quad \forall\ t \geq 0.$$

This completes the proof of Theorem 1.

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