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Certain New Subclasses Of Analytic And m-Fold Symmetric Bi-Univalent Functions*

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Abstract

The purpose of the present paper is to introduce and investigate two new subclasses $SS^{*}_{\Sigma_m}(\lambda,\gamma;\alpha)$ and $S^{*}_{\Sigma_m}(\lambda,\gamma;\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions defined in the open unit disk U. We obtain upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to these subclasses. Many of the well-known and new results are shown to follow as special cases of our results.

1 Introduction

Let $\mathcal{A}$ denote the class of functions f that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Let S be the subclass of $\mathcal{A}$ consisting of the form (1) which are also univalent in U. The Koebe one-quarter theorem (see [4]) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of bi-univalent functions in U given by (1). For a brief history and interesting examples in the class Σ see [18], (see also [6, 7, 8, 10, 14, 15, 21, 22]).

For each function $f \in S$, the function $h(z) = \left(f(z^m)\right)^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [9, 12]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1},\ (z \in U, m \in \mathbb{N}). \tag{3}$$

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We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (3). In fact, the functions in the class S are one-fold symmetric.

In [19] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (3), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) \quad = \quad & w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ & - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} \\ & + \cdots , \end{aligned} \tag{4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \ \textit{and} \ \left[-\log\left(1-z^m\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [1, 2, 5, 16, 17, 19, 20]).

The aim of the present paper is to introduce the new subclasses $SS^*_{\Sigma_m}(\lambda, \gamma; \alpha)$ and $S^*_{\Sigma_m}(\lambda, \gamma; \beta)$ of Σ_m and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to prove our main results, we require the following lemma.

LEMMA 1 ([4]). If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in U for which

$$Re\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

2 Coefficient Estimates for the Functions Class $SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$

DEFINITION 1. A function $f \in \Sigma_m$ given by (3) is said to be in the class $SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$ if it satisfies the following conditions:

$$\left| arg\left[\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}+\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \frac{\alpha\pi}{2}, \quad (z\in U) \tag{5}$$

and

$$\left| arg\left[\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}+\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right]\right| < \frac{\alpha\pi}{2}, \quad (w\in U), \tag{6}$$

$$(0<\alpha\le 1, : 0<\lambda\le 1, : \gamma\ge 0, : m\in\mathbb{N}),$$

where the function $g = f^{-1}$ is given by (4).

REMARK 1. It should be remarked that the class $SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$is a generalization of well-known classes consider earlier. These classes are:

1. For $\gamma = 0$, the class $SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$ reduce to the class $S_{\Sigma_m}(\alpha,\lambda)$ which was introduced recently by Altinkaya and Yalcin [2];
2. For $\lambda = 1$ and $\gamma = 0$, the class $SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$ reduce to the class $S^{\alpha}_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [1];
3. For $\lambda = \gamma = 1$, the class $SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$ reduce to the class $\mathcal{H}^{\alpha}_{\Sigma,m}$ which was investigated by Srivastava et al. [19].

REMARK 2. For one-fold symmetric bi-univalent functions, we denote the class $SS^*_{\Sigma_1}(\lambda,\gamma;\alpha) = SS^*_{\Sigma}(\lambda,\gamma;\alpha)$. Special cases of this class illustrated below:

1. For $\lambda = 1$, the class $SS^*_{\Sigma}(\lambda,\gamma;\alpha)$ reduce to the class $P_{\Sigma}(\alpha,\gamma)$ which was introduced by Prema and Keerthi [13];
2. For $\lambda = 1$ and $\gamma = 0$, the class $SS^*_{\Sigma}(\lambda,\gamma;\alpha)$ reduce to the class $S^*_{\Sigma}(\alpha)$ which was given by Brannan and Taha [3];
3. For $\lambda = \gamma = 1$, the class $SS^*_{\Sigma}(\lambda,\gamma;\alpha)$ reduce to the class $\mathcal{H}^{\alpha}_{\Sigma}$ which was investigated by Srivastava et al. [18].

THEOREM 1. Let $f \in SS^*_{\Sigma_m}(\lambda,\gamma;\alpha)$ $(0<\alpha\le 1,\ 0<\lambda\le 1,\ \gamma\ge 0,\ m\in\mathbb{N})$ be given by (3). Then

$$|a_{m+1}| \le \frac{4\lambda\alpha}{(m+\gamma)\sqrt{(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right)+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)}} \tag{7}$$

and

$$|a_{2m+1}| \leq \frac{8\lambda^2\alpha^2(m+1)}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\lambda\alpha}{(2m+\gamma)(\lambda+1)}. \tag{8}$$

PROOF. It follows from conditions (5) and (6) that

$$\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = [p(z)]^{\alpha} \tag{9}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = [q(w)]^{\alpha}, \tag{10}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{11}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{12}$$

Comparing the corresponding coefficients of (9) and (10) yields

$$\frac{(m+\gamma)(\lambda+1)}{2\lambda} a_{m+1} = \alpha p_m, \tag{13}$$

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left(2a_{2m+1} + (\gamma-1)a_{m+1}^2\right) + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2} a_{m+1}^2$$
$$= \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2, \tag{14}$$

$$-\frac{(m+\gamma)(\lambda+1)}{2\lambda} a_{m+1} = \alpha q_m \tag{15}$$

and

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left[(2m+\gamma+1)a_{m+1}^2 - 2a_{2m+1}\right] + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2} a_{m+1}^2$$
$$= \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \tag{16}$$

Making use of (13) and (15), we obtain

$$p_m = -q_m \tag{17}$$

and

$$\frac{(m+\gamma)^2(\lambda+1)^2}{2\lambda^2} a_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \tag{18}$$

Also, from (14), (16) and (18), we find that

$$\left(\frac{(2m+\gamma)(m+\gamma)(\lambda+1)}{2\lambda}+\frac{(m+\gamma)^2(1-\lambda)}{2\lambda^2}\right)a_{m+1}^2$$
$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right)$$
$$=\alpha(p_{2m}+q_{2m})+\frac{(\alpha-1)(m+\gamma)^2(\lambda+1)^2}{4\lambda^2\alpha}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{4\lambda^2\alpha^2(p_{2m}+q_{2m})}{(m+\gamma)^2\left[(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right)+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)\right]}. \tag{19}$$

Now, taking the absolute value of (19) and applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\le\frac{4\lambda\alpha}{(m+\gamma)\sqrt{(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right)+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (7). In order to find the bound on $|a_{2m+1}|$, by subtracting (16) from (14), we get

$$\frac{(2m+\gamma)(\lambda+1)}{\lambda}a_{2m+1}-\frac{(2m+\gamma)(m+1)(\lambda+1)}{2\lambda}a_{m+1}^2$$
$$=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{20}$$

It follows from (17), (18) and (20) that

$$a_{2m+1}=\frac{\lambda^2\alpha^2(m+1)\left(p_m^2+q_m^2\right)}{(m+\gamma)^2(\lambda+1)^2}+\frac{\lambda\alpha\left(p_{2m}-q_{2m}\right)}{(2m+\gamma)(\lambda+1)}. \tag{21}$$

Taking the absolute value of (21) and applying Lemma 1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\le\frac{8\lambda^2\alpha^2(m+1)}{(m+\gamma)^2(\lambda+1)^2}+\frac{4\lambda\alpha}{(2m+\gamma)(\lambda+1)},$$

which completes the proof of Theorem 1.

REMARK 3. In Theorem 1, if we choose

1. $\gamma=0$, then we obtain the results which was proven by Altinkaya and Yalcin [[2], Theorem 1];

2. $\lambda = \gamma = 1$, then we obtain the results which was proven by Srivastava et al. [[19], Theorem 2].

For one-fold symmetric bi-univalent functions, Theorem 1 reduces to the following corollary:

COROLLARY 1. Let $f \in SS_{\Sigma}^{*}(\lambda, \gamma; \alpha)$ $(0 < \alpha \leq 1,\ 0 < \lambda \leq 1,\ \gamma \geq 0)$ be given by (1). Then

$$|a_2| \leq \frac{4\lambda\alpha}{(1+\gamma)\sqrt{(\lambda+1)\left(\frac{2\lambda\alpha(2+\gamma)}{1+\gamma} + (1-\alpha)(\lambda+1)\right) + 2\alpha(1-\lambda)}}$$

and

$$|a_3| \leq \frac{16\lambda^2\alpha^2}{(1+\gamma)^2(\lambda+1)^2} + \frac{4\lambda\alpha}{(2+\gamma)(\lambda+1)}.$$

REMARK 4. In Corollary 1, if we choose

1. $\lambda = 1$, then we have the results which was given by Prema and Keerthi [[13], Theorem 2.2];
2. $\lambda = 1$ and $\gamma = 0$, then we have the results obtained by Murugusundaramoorthy et al. [[11], Corollary 6];
3. $\lambda = \gamma = 1$, then we obtain the results obtained by Srivastava et al. [[18], Theorem 1].

3 Coefficient Estimates for the Functions Class $S_{\Sigma_m}^{*}(\lambda, \gamma; \beta)$

DEFINITION 2. A function $f \in \Sigma_m$ given by (3) is said to be in the class $S_{\Sigma_m}^{*}(\lambda, \gamma; \beta)$ if it satisfies the following conditions:

$$Re\left\{\frac{1}{2}\left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta, \quad (z \in U) \tag{22}$$

and

$$Re\left\{\frac{1}{2}\left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right)\right\} > \beta, \quad (w \in U), \tag{23}$$

$$(0 \leq \beta < 1, : 0 < \lambda \leq 1, : \gamma \geq 0, : m \in \mathbb{N}),$$

where the function $g = f^{-1}$ is given by (4).

REMARK 5. It should be remarked that the class $S_{\Sigma_m}^{*}(\lambda, \gamma; \beta)$ is a generalization of well-known classes consider earlier. These classes are:

1. For $\gamma = 0$, the class $S^*_{\Sigma_m}(\lambda, \gamma; \beta)$ reduce to the class $S_{\Sigma_m}(\beta, \lambda)$ which was introduced recently by Altinkaya and Yalcin [2];
2. For $\lambda = 1$ and $\gamma = 0$, the class $S^*_{\Sigma_m}(\lambda, \gamma; \beta)$ reduce to the class $S^{\beta}_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [1];
3. For $\lambda = \gamma = 1$, the class $S^*_{\Sigma_m}(\lambda, \gamma; \beta)$ reduce to the class $\mathcal{H}_{\Sigma,m}(\beta)$ which was investigated by Srivastava et al. [19].

REMARK 6. For one-fold symmetric bi-univalent functions, we denote the class $S^*_{\Sigma_1}(\lambda, \gamma; \beta) = S^*_{\Sigma}(\lambda, \gamma; \beta)$. Special cases of this class illustrated below:

1. For $\lambda = 1$, the class $S^*_{\Sigma}(\lambda, \gamma; \beta)$ reduce to the class $P_{\Sigma}(\beta, \gamma)$ which was introduced by Prema and Keerthi [13];
2. For $\lambda = 1$ and $\gamma = 0$, the class $S^*_{\Sigma}(\lambda, \gamma; \beta)$ reduce to the class $S^*_{\Sigma}(\beta)$ which was given by Brannan and Taha [3];
3. For $\lambda = \gamma = 1$, the class $S^*_{\Sigma}(\lambda, \gamma; \beta)$ reduce to the class $\mathcal{H}_{\Sigma}(\beta)$ which was investigated by Srivastava et al. [18].

THEOREM 2. Let $f \in S^*_{\Sigma_m}(\lambda, \gamma; \beta)$ $(0 \le \beta < 1,\ 0 < \lambda \le 1,\ \gamma \ge 0,\ m \in \mathbb{N})$ be given by (3). Then

$$|a_{m+1}| \le \frac{2\lambda}{m+\gamma} \sqrt{\frac{2(1-\beta)}{\left(\frac{m}{m+\gamma}+1\right)\lambda^2 + \frac{m}{m+\gamma}\lambda + 1}} \tag{24}$$

and

$$|a_{2m+1}| \le \frac{8\lambda^2 (m+1)(1-\beta)^2}{(m+\gamma)^2 (\lambda+1)^2} + \frac{4\lambda(1-\beta)}{(2m+\gamma)(\lambda+1)}. \tag{25}$$

PROOF. It follows from conditions (22) and (23) that there exist $p, q \in \mathcal{P}$ such that

$$\frac{1}{2}\left(\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \left(\frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = \beta + (1-\beta)p(z) \tag{26}$$

and

$$\frac{1}{2}\left(\frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \left(\frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}}\right)^{\frac{1}{\lambda}}\right) = \beta + (1-\beta)q(w), \tag{27}$$

where $p(z)$ and $q(w)$ have the forms (11) and (12), respectively. Equating coefficients (26) and (27) yields

$$\frac{(m+\gamma)(\lambda+1)}{2\lambda} a_{m+1} = (1-\beta)p_m, \tag{28}$$

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left(2a_{2m+1} + (\gamma-1)a_{m+1}^2\right) + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2} a_{m+1}^2 = (1-\beta)p_{2m}, \tag{29}$$

$$-\frac{(m+\gamma)(\lambda+1)}{2\lambda}a_{m+1}=(1-\beta)q_m \tag{30}$$

and

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda}\left((2m+\gamma+1)a_{m+1}^2-2a_{2m+1}\right)+\frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2}a_{m+1}^2 = (1-\beta)q_{2m}. \tag{31}$$

From (28) and (30), we get

$$p_m=-q_m \tag{32}$$

and

$$\frac{(m+\gamma)^2(\lambda+1)^2}{2\lambda^2}a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{33}$$

Adding (29) and (31), we obtain

$$\left(\frac{(2m+\gamma)(m+\gamma)(\lambda+1)}{2\lambda}+\frac{(m+\gamma)^2(1-\lambda)}{2\lambda^2}\right)a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{34}$$

Therefore, we have

$$a_{m+1}^2=\frac{2\lambda^2(1-\beta)(p_{2m}+q_{2m})}{(m+\gamma)^2\left[\left(\frac{m}{m+\gamma}+1\right)\lambda^2+\frac{m}{m+\gamma}\lambda+1\right]}.$$

Applying Lemma 1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\frac{2\lambda}{m+\gamma}\sqrt{\frac{2(1-\beta)}{\left(\frac{m}{m+\gamma}+1\right)\lambda^2+\frac{m}{m+\gamma}\lambda+1}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (24). In order to find the bound on $|a_{2m+1}|$, by subtracting (31) from (29), we get

$$\frac{(2m+\gamma)(\lambda+1)}{\lambda}a_{2m+1}-\frac{(2m+\gamma)(m+1)(\lambda+1)}{2\lambda}a_{m+1}^2=(1-\beta)\left(p_{2m}-q_{2m}\right),$$

or equivalently

$$a_{2m+1}=\frac{(m+1)}{2}a_{m+1}^2+\frac{\lambda(1-\beta)\left(p_{2m}-q_{2m}\right)}{(2m+\gamma)(\lambda+1)}.$$

Upon substituting the value of a_{m+1}^2 from (33), it follows that

$$a_{2m+1}=\frac{\lambda^2(m+1)\left(1-\beta\right)^2\left(p_m^2+q_m^2\right)}{(m+\gamma)^2\left(\lambda+1\right)^2}+\frac{\lambda\left(1-\beta\right)\left(p_{2m}-q_{2m}\right)}{(2m+\gamma)(\lambda+1)}.$$

Applying Lemma 1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{8\lambda^2(m+1)\left(1-\beta\right)^2}{(m+\gamma)^2\left(\lambda+1\right)^2}+\frac{4\lambda\left(1-\beta\right)}{(2m+\gamma)(\lambda+1)}.$$

which completes the proof of Theorem 2.

REMARK 7. In Theorem 2, if we choose

1. $\gamma = 0$, then we obtain the results which was proven by Altinkaya and Yalcin [[2], Theorem 2];
2. $\lambda = \gamma = 1$, then we obtain the results which was proven by Srivastava et al. [[19], Theorem 3].

For one-fold symmetric bi-univalent functions, Theorem 2 reduces to the following corollary:

COROLLARY 2. Let $f \in S^*_\Sigma(\lambda, \gamma; \beta)$ $(0 \leq \beta < 1,\ 0 < \lambda \leq 1,\ \gamma \geq 0)$ be given by (1). Then

$$|a_2| \leq \frac{2\lambda}{1+\gamma}\sqrt{\frac{2(1-\beta)}{\frac{2+\gamma}{1+\gamma}\lambda^2 + \frac{1}{1+\gamma}\lambda + 1}}$$

and

$$|a_3| \leq \frac{16\lambda^2\left(1-\beta\right)^2}{\left(1+\gamma\right)^2\left(\lambda+1\right)^2} + \frac{4\lambda\left(1-\beta\right)}{(2+\gamma)(\lambda+1)}.$$

REMARK 8. In Corollary 2, if we choose

1. $\lambda = 1$, then we have the results which was given by Prema and Keerthi [[13], Theorem 3.2];
2. $\lambda = 1$ and $\gamma = 0$, then we have the results obtained by Murugusundaramoorthy et al. [[11], Corollary 7];
3. $\lambda = \gamma = 1$, then we obtain the results obtained by Srivastava et al. [[18], Theorem 2].

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References

[1] S. Altinkaya and S. Yalçin, Coefficient bounds for certain subclasses of m-fold symmetric bi-univalent functions, Journal of Mathematics, Art. ID 241683, (2015), 1–5.

[2] S. Altinkaya and S. Yalçcin, On some subclasses of m-fold symmetric bi-univalent functions, Commun. Fac. Sci. Univ. Ank. Series A1, 67(1)(2018), 29–36.

[3] D. A. Brannan and T. S. Taha, On Some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math., 31(2)(1986), 70–77.

[4] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[5] S. S. Eker, Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions, Turk. J. Math., 40(2016), 641–646.

[6] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24(2011), 1569–1573.

[7] S. P. Goyal and P. Goswami, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, J. Egyptian Math. Soc., 20(2012), 179–182.

[8] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, Pan Amer. Math. J., 22(4)(2012), 15–26.

[9] W. Koepf, Coefficients of symmetric functions of bounded boundary rotations, Proc. Amer. Math. Soc., 105(1989), 324–329.

[10] N. Magesh and J. Yamini, Coefficient bounds for certain subclasses of bi-univalent functions, Int. Math. Forum, 8(27)(2013), 1337–1344.

[11] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent function, Abstr. Appl. Anal., Art. ID 573017, (2013), 1–3.

[12] C. Pommerenke, On the coefficients of close-to-convex functions, Michigan Math. J., 9(1962), 259–269.

[13] S. Prema and B. S. Keerthi, Coefficient bounds for certain subclasses of analytic function, J. Math. Anal., 4(1)(2013), 22–27.

[14] H. M. Srivastava and D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions, J. Egyptian Math. Soc., 23(2015), 242–246.

[15] H. M. Srivastava, S. Bulut, M. Caglar and N. Yagmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat, 27(5)(2013), 831–842.

[16] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Universitatis Apulensis, 41 (2015), 153–164.

[17] H. M. Srivastava, S. Gaboury and F. Ghanim, Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, Acta Mathematica Scientia, 36B(3)(2016), 863–871.

[18] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23(2010), 1188–1192.

[19] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, Tbilisi Math. J., 7(2)(2014), 1–10.

[20] H. Tang, H. M. Srivastava, S. Sivasubramanian and P. Gurusamy, The Fekete-Szegő functional problems for some subclasses of m-fold symmetric bi-univalent functions, J. Math. Ineq., 10(2016), 1063–1092.

[21] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett., 25(2012), 990–994.

[22] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput., 218(2012), 11461–11465.