

Lyapunov-Schmidt Reduction In The Study Of Bifurcation Solutions Of Nonlinear Fractional Differential Equation*

Mudhir Abdul Wahid Abdul Hussain[†]

Received 05 August 2017

Abstract

In this article the bifurcation of periodic travelling wave solutions of nonlinear fractional differential equation is studied by using Lyapunov-Schmidt reduction and He's fractional derivative. The fractional complex transform is used to convert the fractional differential equation into partial differential equation. The reduced equation corresponding to the main problem is found as a system of two nonlinear algebraic equations. The existence of the linear approximation solutions of the nonlinear fractional differential equation is discussed.

1 Introduction

Many of the nonlinear problems that appear in Mathematics and Physics can be written in the operator equation form

$$F(x, \lambda) = b, \quad x \in O \subset E, \quad b \in M, \quad \lambda \in R^n \quad (1)$$

where F is a smooth Fredholm map of index zero and E, M are Banach spaces and O is open subset of E . For these problems, the method of reduction to finite dimensional equation,

$$\theta(\xi, \lambda) = \tilde{\beta}, \quad \xi \in \tilde{M}, \quad \tilde{\beta} \in \tilde{N} \quad (2)$$

can be used, where \tilde{M} and \tilde{N} are smooth finite dimensional manifolds. A passage from (1) into (2) (variant local scheme of Lyapunov-Schmidt) with the conditions that equation (2) has all the topological and analytical properties of (1) (multiplicity, bifurcation diagram, etc) can be found in [1, 7, 12, 13]. Vainberg [7], Loginov [1] and Sapronov [11–13] are dealing with equation (1) into equation (2) by using local method of Lyapunov-Schmidt with the conditions that, equation (2) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, etc).

In recent years, researchers have been interested in the study of fractional differential equations because they are of great importance in many fields. For example, these

*Mathematics Subject Classifications: 34K18, 93C10.

[†]Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq

equations appear in physics, biology, finance, engineering, system identification, control theory, fluid flow, fractional dynamics and signal processing. It has been the interest of more researchers to find exact solutions for this type of equations using some analytical methods such as the method of fractional sub-equations [9, 10], the first integral method [2] and other methods. Li and He [14, 15] proposed the fractional complex transform to transform fractional differential equations into partial differential equations so that the last equations can be easily studied using the previous methods. In this paper we studied the bifurcation of periodic travelling wave solutions of the following nonlinear fractional differential equation using the local method of Lyapunov-Schmidt and He's fractional derivative

$$\frac{\partial^{2\alpha} w}{\partial t^{2\alpha}} + \frac{\partial^4 w}{\partial y^4} + \frac{\partial^2 w}{\partial y^2} + w + w^2 \frac{\partial^2 w}{\partial y^2} + w \left(\frac{\partial w}{\partial y} \right)^2 = 0, \quad w = w(y, t) \quad (3)$$

where, $\frac{\partial^\alpha}{\partial t^\alpha}$ is the He's fractional derivative and $\alpha \in (0, 1]$.

THEOREM 1 ([8]). Suppose E and M are real Banach spaces and $F(x, \lambda)$ is a C^1 map defined in a neighborhood U of a point (x_0, λ_0) with range in M such that $F(x_0, \lambda_0) = 0$ and $F_x(x_0, \lambda_0)$ is a linear Fredholm operator. Then all solutions (x, λ) of $F(u, \lambda) = 0$ near (x_0, λ_0) (with λ fixed) are in one-to-one correspondence with the solutions of a finite-dimensional system of N_1 real equations in a finite number N_0 of real variables. Furthermore, $N_0 = \dim(\ker L)$ and $N_1 = \dim(\operatorname{coker} L)$, ($L = F_x(x_0, \lambda_0)$).

DEFINITION 2 ([13]). The discriminate set Σ of equation (1) is defined to be the union of all $\lambda = \bar{\lambda}$ for which the equation (1) has degenerate solution $\bar{x} \in O$:

$$F(\bar{x}, \bar{\lambda}) = b, \quad \operatorname{codim} \left(\operatorname{Im} \frac{\partial F}{\partial x}(\bar{x}, \bar{\lambda}) \right) > 0$$

where, Im is the image of the operator $\frac{\partial F}{\partial x}(\bar{x}, \bar{\lambda})$.

DEFINITION 3 ([3]). He's fractional derivative is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_{t_0}^t (s - t)^{n - \alpha - 1} (f_0(s) - f(s)) ds$$

where $f_0(x)$ is a known function.

The first step to study equation (3) by Lyapunov-Schmidt method is to convert the equation into partial differential equation. By the following fractional complex transformation [15]:

$$T = \frac{t^\alpha}{\Gamma(1 + \alpha)}. \quad (4)$$

Equation (3) is converted into the following partial differential equation:

$$\frac{\partial^2 w}{\partial T^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\partial^2 w}{\partial y^2} + w + w^2 \frac{\partial^2 w}{\partial y^2} + w \left(\frac{\partial w}{\partial y} \right)^2 = 0, \quad w = w(y, t). \quad (5)$$

To study the travelling wave solutions of equation (5) we use the following transformation

$$u(x) = w(y, t), \quad x = y - cT, \quad c = \kappa\alpha \quad (6)$$

where $\kappa \in R$ is a constant and α is a parameter. By substituting (6) into equation (5), equation (5) is reduced into fourth order nonlinear ODE

$$u'''' + \lambda u'' + u + u^2 u'' + u(u')^2 = 0, \quad \lambda = 1 + \kappa^2 \alpha^2, \quad ' = d/dx. \quad (7)$$

In this study we assume that u is a periodic function, $u(x) = u(x + 2\pi)$. Abdul Hussain [4-6] studied the bifurcation solutions of equation (7) using classical Lyapunov-Schmidt method and modify Lyapunov-Schmidt method. In the next section we will use the Local method of Lyapunov-Schmidt to reduce equation (7) into an equivalent finite dimensional system of nonlinear algebraic equations.

2 Reduction to Bifurcation Equation

In order to study the bifurcation of periodic travelling wave solutions of equation (5) it is convenient to set equation (7) in the form of operator equation,

$$F(u, \lambda) = u'''' + \lambda u'' + u + u^2 u'' + u(u')^2, \quad (8)$$

where $F : E \rightarrow M$ is a nonlinear Fredholm operator of index zero, $E = \Pi_4([0, 2\pi], R)$ is the space of all periodic continuous functions that have derivative of order at most four, $M = \Pi_0([0, 2\pi], R)$ is the space of all periodic continuous functions, R is the real space, $u = u(x)$, $x \in [0, 2\pi]$. From this we note that the bifurcation solution of equation (8) is equivalent to the bifurcation solution of operator equation

$$F(u, \lambda) = 0. \quad (9)$$

Theorem 1 tells us that the solutions of equation (4) are equivalent to the solutions of finite dimensional system with $2 = \dim(\ker F_u(0, \lambda))$ variables and $2 = \dim(\text{coker } F_u(0, \lambda))$ equations. The first step in this reduction is to determine the linearized equation corresponding to the equation (9), which is given by the following equation. The first step in this reduction is to determine the linearized equation corresponding to the equation (9), which is given by the following equation

$$Ah = 0, \quad h \in E,$$

$$A = \frac{\partial F}{\partial u}(0, \lambda) = \frac{d^4}{dx^4} + \lambda \frac{d^2}{dx^2} + 1.$$

The periodic solution of linearized equation is given by

$$e_p(x) = a_p \sin(px) + b_p \cos(px), \quad p = 1, 2, 3, \dots$$

and the characteristic equation corresponding to this solution is

$$p^4 - \lambda p^2 + 1 = 0.$$

It follows that the point $\lambda = 2$ is a bifurcation point of equation (9). Localized parameter λ as follows,

$$\lambda = 2 + \delta, \quad \delta \text{ is a small parameter}$$

leads to bifurcation along the modes

$$e_1(x) = a_1 \sin(x), \quad e_2(x) = a_2 \cos(x),$$

where $\|e_i\|_H = 1$ and $a_i = \sqrt{2}$ for $i = 1, 2$, (H is a Hilbert space $L^2([0, 2\pi], R)$). Let $N = \ker(A) = \text{span}\{e_1, e_2\}$, then the space E can be decomposed into the direct sum of two subspaces, N and the orthogonal complement to N ,

$$E = N \oplus N^\perp, \quad N^\perp = \{v \in E : v \perp N\}.$$

Similarly, the space M can be decomposed into the direct sum of two subspaces, N and the orthogonal complement to N

$$M = N \oplus \tilde{N}^\perp, \quad \tilde{N}^\perp = \{g \in M : g \perp N\}.$$

From the above decompositions we get two projections $P : E \rightarrow N$ and $I - P : E \rightarrow N^\perp$ such that $Pu = w$, $(I - P)u = v$. Hence every vector $u \in E$ can be written in the form of

$$u = z + v, \quad z = \sum_{i=1}^2 \xi_i e_i \in N, \quad v \in N^\perp, \quad \xi_i = \langle u, e_i \rangle_H.$$

Similarly, there exists projections $Q : M \rightarrow N$ and $I - Q : M \rightarrow \tilde{N}^\perp$ such that

$$QF(u, \lambda) = F_1(u, \lambda), \quad (I - Q)F(u, \lambda) = F_2(u, \lambda),$$

and hence,

$$\begin{aligned} F(u, \lambda) &= F_1(u, \lambda) + F_2(u, \lambda), \\ F_1(u, \lambda) &= \sum_{i=1}^2 v_i(u, \lambda) e_i \in N, \quad F_2(u, \lambda) \in \tilde{N}^\perp, \\ v_i(u, \lambda) &= \langle F(u, \lambda), e_i \rangle_H. \end{aligned}$$

Accordingly, equation (9) can be written in the form of

$$QF(u, \lambda) = 0, \quad (I - Q)F(u, \lambda) = 0,$$

or

$$QF(z + v, \lambda) = 0, \quad (I - Q)F(z + v, \lambda) = 0.$$

By the implicit function theorem, there exists a smooth map $\Phi : N \rightarrow N^\perp$ such that $v = \Phi(u, \lambda)$ and

$$(I - Q)F(z + \Phi(z, \lambda), \lambda) = 0$$

so to find the solutions of equation (9) in the neighborhood of the point $u = 0$ it is sufficient to find the solutions of the equation,

$$QF(z + \Phi(z, \lambda), \lambda) = 0. \tag{10}$$

The bifurcation equation (10) has the form

$$\Theta(\xi, q) = 0, \quad \xi = (\xi_1, \xi_2)$$

where

$$\Theta(\xi, q) = F_1(z + \Phi(z, \lambda), \lambda).$$

Equation (9) can be written in the form of

$$F(z + v, \lambda) = A(z + v) + T(z + v) = Az + z^2z'' + z(z')^2 + \dots,$$

where $T(z + v) = (z + v)^2(z + v)'' + (z + v)((z + v)')^2$ and the dots denote the terms containing the element v . Since

$$\Theta(\xi, q) = F_1(z + v, \lambda) = 0, \tag{11}$$

we get from equation (11)

$$\sum_{i=1}^2 \langle Az + z^2z'' + z(z')^2, e_i \rangle_H e_i + \dots = 0. \tag{12}$$

After some calculations of equation (12) and by using the property $Ae_1 = q(\lambda)e_1, Ae_2 = q(\lambda)e_2$ we have the following system

$$\begin{cases} -\xi_1(\xi_1^2 + \xi_2^2) + q\xi_1 = 0, \\ -\xi_2(\xi_1^2 + \xi_2^2) + q\xi_2 = 0, \end{cases} \tag{13}$$

where $q = 1 - \kappa^2\alpha^2, \kappa \in R$ and

$$\langle g(x), h(x) \rangle_H = \frac{1}{2\pi} \int_0^{2\pi} g(x)h(x)dx.$$

From the above results we conclude the following theorem,

THEOREM 2. The bifurcation equation

$$\Theta(\xi, q) = F_1(z + \Phi(z, \lambda), \lambda) = 0$$

corresponding to the equation (2.2) has the following form

$$\Theta(\xi, q) = \begin{pmatrix} -\xi_1(\xi_1^2 + \xi_2^2) + q\xi_1 \\ -\xi_2(\xi_1^2 + \xi_2^2) + q\xi_2 \end{pmatrix} + o(|\xi|^2 + O(|\xi|^2))O(\delta) = 0,$$

where $\xi = (\xi_1, \xi_2)$.

We note that system (13) is contact equivalent to the system (14). From theorem (1) the solutions of equation (9) are in one-to-one correspondence with the solutions of system (13). Also, system (13) has all the topological and analytical properties of equation (9). Thus to study the bifurcation solutions of equation (9) it is sufficient to study the bifurcation solutions of system (13).

3 Bifurcation Analysis of System (13)

From section (2) we have that the point $a \in E$ is a solution of equation (9) if and only if

$$a = \sum_{i=1}^2 \bar{\eta}_i e_i + \Phi(\bar{\eta}, \bar{\lambda}).$$

where $\bar{\eta}$ is a solution of system (14). It is easy to see that system (13) has only the following five real solutions

$$(0, 0), \quad (0, \pm\sqrt{q}), \quad (\pm\sqrt{q}, 0).$$

These solutions are degenerate only when $q = 0$, it follows that $1 - \kappa^2 \alpha^2 = 0$, hence $\alpha = \frac{1}{\kappa}$, $\kappa \neq 0$. We note that $\alpha \in (0, 1]$ if $|\kappa| > 1$ and for $|\kappa| \leq 1$, α does not belong to $(0, 1]$. Thus we can find $\alpha \in (0, 1]$ such that the solutions of system (13) are vary when α change. By fixing the value of κ the system (13) has five regular real solutions if $\alpha < 1/\kappa$ and has only one regular real solution $(0, 0)$ if $\alpha > 1/\kappa$. From theorem (1) the solutions

$$(0, \sqrt{q}), \quad (0, -\sqrt{q}), \quad (\sqrt{q}, 0), \quad (-\sqrt{q}, 0)$$

are in one-to-one correspondence with the following linear approximation solutions of equation (9) respectively,

$$u_1 = \sqrt{2q} \cos\left(y - \frac{\kappa\alpha t^\alpha}{\Gamma(\alpha+1)}\right), \quad u_2 = -\sqrt{2q} \cos\left(y - \frac{\kappa\alpha t^\alpha}{\Gamma(\alpha+1)}\right),$$

$$u_3 = \sqrt{2q} \sin\left(y - \frac{\kappa\alpha t^\alpha}{\Gamma(\alpha+1)}\right), \quad u_4 = -\sqrt{2q} \sin\left(y - \frac{\kappa\alpha t^\alpha}{\Gamma(\alpha+1)}\right).$$

These solutions can be described geometrically by fixing the value of κ , so we can choose $\kappa = 2$ to get five regular solutions for $\alpha < \frac{1}{2}$, since $(\kappa > 1)$. To do this we select any value of $\alpha < \frac{1}{2}$ say $\alpha = \frac{1}{4} \in (0, 1]$. The solutions u_1, u_2, u_3, u_4 have the following geometric descriptions

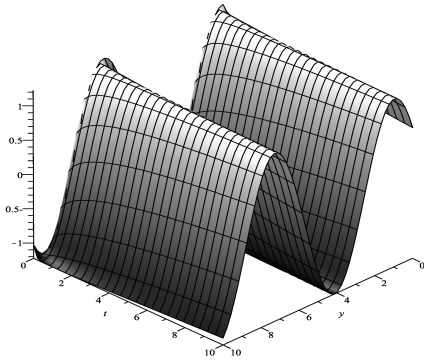


Figure 1: The graph of the solution u_1 .

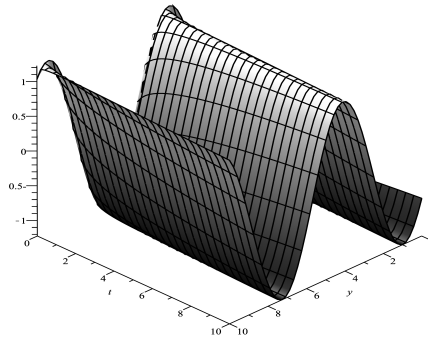
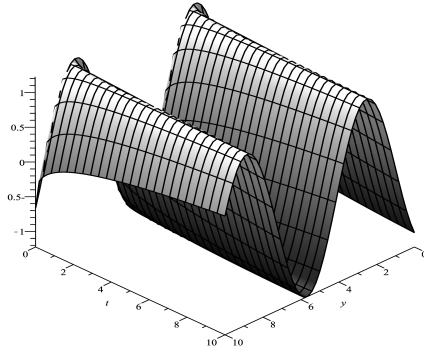
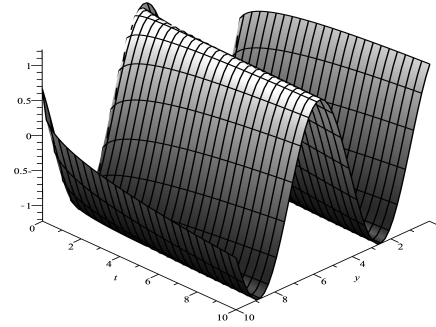


Figure 2: The graph of the solution u_2 .

Figure 2: The graph of the solution u_3 .Figure 2: The graph of the solution u_4 .

4 Conclusions

In this article we studied the bifurcation of periodic travelling wave solutions of nonlinear fractional differential equation using He's fractional derivative and Lyapunov-Schmidt reduction. The fractional complex transform is adopted to convert the fractional differential equation into partial differential equation and the travelling wave transformation is used to reduce the partial differential equation into ODE. The reduced equation corresponding to the ODE has been found as a system of two nonlinear algebraic equations. We described geometrically the linear approximation solutions of the nonlinear fractional differential equation. Finally, we shows that we can use the topological methods in the study of nonlinear fractional differential equation.

Acknowledgment. The author is grateful to the reviewers for their constructive comment.

References

- [1] B. V. Loginov, Theory of Branching Nonlinear Equations in The conditions of Invariance Group-Tashkent: Fan, 1985.
- [2] B. Lu., The first integral method for some time fractional differential equation, J. Math. Anal. Appl., 395(2012), 684–693.
- [3] F. J. Liu, Z.B. Li, S. Zhang and H. Y. Liu, He's fractional derivative for heat conduction in a fFractal medium arising in silkworm cocoon hierarchy, Thermal Science, 19(2015), 4, pp.1155–1159.
- [4] M. A. Abdul Hussain, Bifurcation solutions of elastic beams equation with small perturbation, Int. J. Math. Anal., 3(2009), 879–888.

- [5] M. A. Abdul Hussain, Two-Mode bifurcation in solution of a perturbed nonlinear fourth order differential equation, *Arch. Math.*, 48(2012), 27–37.
- [6] M. A. Abdul Hussain, Nonlinear Ritz approximation for Fredholm functionals, *Electron. J. Differential Equations*, 2015, No. 294, 11 pp.
- [7] M. M. Vainberg and V. A. Trenogin, *Theory of Branching Solutions of Nonlinear Equations*, M.-Science , 1969.
- [8] M. S. Berger, *Nonlinearity and Functional Analysis, Lectures on Nonlinear Problems in Mathematical Analysis*, Academic Press, Inc., 1977.
- [9] S. Zhang and H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A* 375 (2011), no. 7, 1069–1073.
- [10] S. Guo, L.Q. Mei, Y. Li and Y. F. Sun, The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics, *Phys. Lett. A* 376 (2012), no. 4, 407–411.
- [11] Yu. I. Sapronov, Regular Perturbation of Fredholm Maps and Theorem about odd Field, *Works Dept. of Math., Voronezh Univ.*, 1973. V. 10, 82–88.
- [12] Yu. I. Sapronov, Finite-dimensional reductions in smooth extremal problems, *Usp. Mat. Nauk*, 51(1996), 101–132.
- [13] Yu. I. Sapronov and V. R. Zachepa, *Local Analysis of Fredholm Equation*, Voronezh Univ., 2002.
- [14] Z. B. Li and J.-H. He, Fractional complex transform for fractional differential equations, *Math. Comput. Appl.*, 15(2010), 970–973.
- [15] Z. B. Li and J. H. He, Converting fractional differential equations into partial differential equations, *Ther. Sci.*, 16(2012), 331–334.