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Korovkin Subsets In The Sense Of Summation Process*

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Abstract

In this paper we introduce Korovkin subsets for a positive linear operator T in the sense of summation process. The characterization of Korovkin subsets of $C_0(X)$, closure of the set of all functions with compact support on X, for a positive linear operator in the sense of summation process is obtained where X is a locally compact Hausdorff space and has a countable base. We also provide some examples that are Korovkin subsets for the identity operator in the sense of summation process.

1 Introduction

The key moment in the development of approximation theory is Weierstrass' theorem. The problem has been studied by many famous mathematicians. Bernstein [7] has succeeded to give the most elegant and short proof of this theorem via Bernstein polynomials. Another important instrument in approximation theory by positive linear operators is the Korovkin theory. In 1953, Korovkin [9] proved a well known approximation theorem: if $\{T_n\}$ is a sequence of positive linear operators on $C[0,1]$, the set of all continuous functions on $[0,1]$, such that $\left\|T_n e_k - e_k\right\| \to 0$ as $n \to \infty$ for $k = 0, 1, 2,$ where $e_k(x) = x^k$ and $\|f\| := \max_{x\in[0,1]} |f(x)|$, then $\{T_n\}$ converges strongly to the identity operator. Takahasi [13, 14] has studied Korovkin type theorems with a different view. Namely, he has answered the question of convergence of $\{T_n\}$, a sequence of positive linear operators, to a positive linear operator T different from identity operator. This theory has close connections with real analysis, functional analysis and summability theory. Especially classical Korovkin theory has been generalized with the use of different convergences in summability theory [11, 12, 15]. Korovkin type theorems have also been extended by various authors [2, 4, 8, 10] with the aim of finding other subsets of functions, called Korovkin subsets, which satisfy the same property as $\{1, e_1, e_2\}$.

In this paper, we follow the idea of Altomare [3] with the use of summability theory, especially summation process. We define the Korovkin subsets for a positive linear operator T in the sense of summation process. The characterization of such subsets of $C_0(X)$, where X is a locally compact Hausdorff space and has a countable base, is

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obtained. Finally, we provide examples of Korovkin subsets for the identity operator in the sense of summation process.

Let $\mathscr{A} := \{A^{(n)}\} = \{a_{kj}^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. A sequence $\{T_j\}$ of positive linear operators from $C_0(X)$ into $C_0(Y)$ is called an $\mathscr{A}$-summation process in $C_0(X)$ if $\{T_j f\}$ is $\mathscr{A}$-summable to f for every $f \in C_0(X)$, i.e.,

$$\lim_{k\to\infty} \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} T_j f - f \right\| = 0, \text{ uniformly in } n, \tag{1}$$

where it is assumed that the series in (1) converges for each k, n and f. Recall that a sequence of real numbers $\{x_j\}$ is said to be $\mathscr{A}$-summable [6] to L if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L, \quad \text{uniformly in } n \in \mathbb{N}.$$

Also, $\mathscr{A}$ is said to be a regular method of matrices if $\lim\limits_{j\to\infty} x_j = L$ implies $\lim\limits_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$, uniformly in $n \in \mathbb{N}$. It has been shown in [6] that the method $\mathscr{A} := \{A^{(n)}\}$ is regular if and only if

(i) for each $j \in \mathbb{N}$, $\lim\limits_{k\to\infty} a_{kj}^{(n)} = 0$, uniformly in n.

(ii) $\lim\limits_{k\to\infty} \sum\limits_j a_{kj}^{(n)} = 1$, uniformly in n.

(iii) for each $n, k \in \mathbb{N}$, $\sum\limits_j \left| a_{kj}^{(n)} \right| < \infty$, and there exist integers N, M such that $\sum\limits_j \left| a_{kj}^{(n)} \right| < M$ for $k \geq N$ and all $n = 1, 2, ...$

DEFINITION 1. Let $f : E \to \mathbb{R}$ be a real function on a topological space E. The set

$$supp(f) := \overline{\{x : \ f(x) \neq 0\}}$$

is called the support of f, where $\overline{K}$ is the closure of $K \subset E$.

Let $C(E)$ be the set of all continuous functions on E. If E is locally compact, then we will designate by $C_c(E)$ the set of all $f \in C(E)$ with compact support $supp(f)$. A function $f \in C(E)$ lies in $C_c(E)$ just if there is some compact subset of E in the complement of which f is identically zero. We denote by $C_b(E)$ and $C_0(E)$ all bounded continuous real functions on E and the closure of $C_c(E)$ with respect to the usual sup-norm, respectively.

Clearly,

$$C_c(E) \subset C_0(E) \subset C_b(E) \subset C(E)$$

since an $f \in C_c(E)$ is bounded on its compact support, hence throughout E.

Now let us recall positive bounded Radon measures which play key role in our main theorem. A positive bounded Radon measure is a positive linear functional on $C_0(X)$ where X is a locally compact Hausdorff space. The set of all of positive bounded Radon measures is denoted by M_b^+. It is obvious that every $\mu \in M_b^+$, that is, every positive linear functional $\mu : C_0(X) \to \mathbb{R}$ is continuous with respect to the norm which is given by

$$\|\mu\| := \sup\left\{|\mu(f)| : \ f \in C_0(X), \ \ |f| \le 1\right\}.$$

The following result is known as Urysohn's lemma [5].

PROPOSITION 1. Let E be a locally compact space and let U be an open neighbourhood of the compact subset B. Then $C_c(E)$ contains a function ψ which satisfies

$$0 \le \psi \le 1, \ \ \psi(B) = \{1\} \ \text{ and } \ supp(\psi) \subset U.$$

We now give the definition of Korovkin subsets of E for a positive linear operator T in the sense of summation process which is our main definition in the paper.

DEFINITION 2. Let $\mathscr{A} := \{A^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. Let E and F be Banach lattices and consider a positive linear operator $T : E \to F$. A subset M of E is said to be a Korovkin subset of E for T in the sense of summation process if for every sequence $\{L_j\}$ of positive linear operators from E into F satisfying

(i) $\sup\limits_{n,k} \sum\limits_j a_{kj}^{(n)} \|L_j\| < \infty$.

(ii) $\lim\limits_{k\to\infty} \sum\limits_j a_{kj}^{(n)} L_j(g) = T(g)$ uniformly in n, for every $g \in M$, it follows that, for every $f \in E$,

$$\lim_{k\to\infty} \sum_j a_{kj}^{(n)} L_j(f) = T(f) \ \text{ uniformly in } n.$$

2 Main Results

In this section we give our main theorem which characterizes the Korovkin subsets of $C_0(X)$ for a positive linear operator T in the sense of summation process and give some results which can be deduced from our main theorem. We also provide examples of Korovkin subsets for the identity operator in the sense of summation process.

Now we are ready to give our main theorem.

THEOREM 1. Let $\mathscr{A} := \{A^{(n)}\}$ be a regular method of infinite matrices with non-negative real entries. Let X and Y be locally compact Hausdorff spaces. Further, assume that X has a countable base and Y is metrizable. Given a positive linear operator $T : C_0(X) \to C_0(Y)$ and a subset M of $C_0(X)$, the following statements are

equivalent:
(a) M is a Korovkin subset of $C_0(X)$ for T in the sense of summation process.
(b) If $\mu \in M_b^+(X)$ and $y \in Y$ satisfy $\mu(g) = T(g)(y)$ for every $g \in M$, then $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$.

PROOF. Assume that $\mu \in M_b^+(X)$ and $y \in Y$ such that $\mu(g) = T(g)(y)$ for every $g \in M$. Let us take a decreasing countable base (U_j) of open neighbourhoods of y in Y. From Proposition 1 if we consider the compact set $\{y\}$, we choose $\psi_j \in C_c(Y)$ such that: $0 \leq \psi_j \leq 1$, $\psi_j(y) = 1$ and also $supp(\psi_j) \subset U_j$. Let us define $L_j : C_0(X) \to C_0(Y)$ by

$$L_j(f) := \mu(f)\psi_j + v_j T(f)(1 - \psi_j) \text{ for every } f \in C_0(X)$$

where $v = (v_j)$ is non-negative, bounded sequence such that the sequence $\{|v_j - 1|\}$ is $\mathscr{A}$-summable to 0, but not ordinary convergent to 0. Observe that $\{L_j\}$ is a sequence of positive linear operators and also

$$\sum_j a_{kj}^{(n)} \|L_j\| \leq \sum_j a_{kj}^{(n)} (\|\mu\| + |v_j|\|T\|) \leq (\|\mu\| + \|v\|\|T\|) \sum_j a_{kj}^{(n)}$$

which implies $\sup_{n,k} \sum_j a_{kj}^{(n)} \|L_j\| < \infty$. On the other hand, since $T(g) \in C_0(Y)$ for every $g \in M$, for every $\varepsilon > 0$, there exists $v \in \mathbb{N}$ such that for every $z \in U_v$

$$\big|T(g)(z) - T(g)(y)\big| \leq \varepsilon.$$

Thus one can get for every $z \in U_v$ that

$$\begin{aligned} |T(g)(y) - v_j T(g)(z)| &= |T(g)(y) - v_j T(g)(z) - T(g)(z) + T(g)(z)| \\ &\leq |T(g)(z) - T(g)(y)| + |T(g)(z)||v_j - 1| \\ &\leq \varepsilon + |T(g)(z)||v_j - 1|. \end{aligned}$$

Moreover for every $j \geq v$ and $z \in Y$, we have

$$\begin{aligned} &\Big|\sum_j a_{kj}^{(n)} L_j(g)(z) - T(g)(z)\Big| \\ &= \Big|\mu(g) \sum_j a_{kj}^{(n)} \psi_j(z) + T(g)(z) \sum_j a_{kj}^{(n)} v_j - T(g)(z) \sum_j a_{kj}^{(n)} v_j \psi_j(z) - T(g)(z)\Big| \\ &= \Big|\sum_j a_{kj}^{(n)} \psi_j(z)\big[\mu(g) - v_j T(g)(z)\big] + T(g)(z)\Big[\sum_j a_{kj}^{(n)} v_j - 1\Big]\Big| \\ &\leq \sum_j a_{kj}^{(n)} \psi_j(z)\big|T(g)(y) - v_j T(g)(z)\big| + \big|T(g)(z)\big|\Big|\sum_j a_{kj}^{(n)} v_j - 1\Big|. \end{aligned}$$

Hence using the last inequality we get

$$\Big|\sum_j a_{kj}^{(n)} L_j(g)(z) - T(g)(z)\Big| \leq \begin{cases} \big|T(g)(z)\big|\big|\sum_j a_{kj}^{(n)} v_j - 1\big| & , z \notin U_j \\ \sum_j a_{kj}^{(n)} (\varepsilon + |T(g)(z)||v_j - 1|) + \big|T(g)(z)\big|\big|\sum_j a_{kj}^{(n)} v_j - 1\big| & , z \in U_j \end{cases}$$

and therefore we have

$$\lim_k \Big\| \sum_j a_{kj}^{(n)} L_j(g) - T(g) \Big\| = 0, \quad \text{uniformly in } n,$$

i.e., $\lim_k \sum_j a_{kj}^{(n)} L_j(g) = T(g)$ uniformly in n on M. Since M is a Korovkin subset for T in the sense of summation process, it is obtained that for every $f \in C_0(X)$,

$$\lim_{k\to\infty} \sum_j a_{kj}^{(n)} L_j(f) = T(f) \;\; \text{uniformly in } n.$$

Therefore we get $\lim_{k\to\infty} \sum_j a_{kj}^{(n)} L_j(f)(y) = T(f)(y)$ uniformly in n and for every $k \in \mathbb{N}$. On the other hand observe that $\mu(f) = L_j(f)(y)$ for every $f \in C_0(X)$. It follows that we obtain $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$. This completes the proof of (b).

Conversely if $\mu \in M_b^+(X)$ and $y \in Y$ satisfy $\mu(g) = T(g)(y)$ for every $g \in M$, then $\mu(f) = T(f)(y)$ for every $f \in C_0(X)$. Observe that

$$if \;\; \mu \in M_b^+(X) \text{ and } \mu(g) = 0 \text{ for every } g \in M, \text{ then } \mu = 0. \tag{2}$$

Since X has a countable base, every bounded sequence in $M_b^+(X)$ has a vaguely convergent subsequence. Consider now a sequence $\{L_j\}$ of positive linear operators from $C_0(X)$ into $C_0(Y)$ satisfying properties (i) and (ii) of Definition 2 and suppose that for some $f_0 \in C_0(X)$ the sequence $\{\sum_j a_{kj}^{(n)} L_j(f_0)\}_k^n$ does not converge for k uniformly in n. So there exist $\varepsilon_0 > 0$, a sequence $\sum_j a_{r_k,j}^{(n_k)} L_j$ and a sequence (y_k) in Y and $\{n_k\} \subset \mathbb{N}$ such that

$$\Big| \sum_j a_{r_k,j}^{(n_k)} L_j(f_0)(y_k) - T(f_0)(y_k) \Big| \geq \varepsilon_0 \text{ for every } k \geq 1. \tag{3}$$

We have two cases: (y_k) is converging to the point at infinity of Y or not (see [1], p. 18). In the first case, since (y_k) converges to the point at infinity of Y we get $\lim_{k\to\infty} h(y_k) = 0$ for every $h \in C_0(Y)$. For every $k \geq 1$, define $\mu_k \in M_b^+(X)$ by

$$\mu_k(f) := \sum_j a_{r_k,j}^{(n_k)} L_j(f)(y_k) \quad (f \in C_0(X)).$$

From hypothesis, we have $\|\mu_k\| \leq \sum_j a_{r_k,j}^{(n_k)} \|L_j\| < \infty$. Since (μ_k) is bounded, we may assume that there exists $\mu \in M_b^+(X)$ such that $\mu_k \to \mu$ vaguely (If necessary the sequence μ_k is replaced with a suitable subsequence). On the other hand if $g \in M$, then for each $k \geq 1$ we get

$$\begin{aligned} |\mu_k(g)| &\leq \Big| \sum_j a_{r_k,j}^{(n_k)} L_j(g)(y_k) - T(g)(y_k) \Big| + \big| T(g)(y_k) \big| \\ &\leq \Big\| \sum_j a_{r_k,j}^{(n_k)} L_j(g) - T(g) \Big\| + \big| T(g)(y_k) \big| \end{aligned}$$

which implies $\mu(g) = \lim_k \mu_k(g) = 0$. From (2) we obtain $\mu(f_0) = 0$ as well and hence for any $k \geq 1$ we have

$$\Big|\sum_j a^{(n_k)}_{r_k,j} L_j(f_0)(y_k) - T(f_0)(y_k)\Big| = \big|\mu_k(f_0) - T(f_0)(y_k)\big| \to 0.$$

This contradicts (3). In the second case we assume that the sequence (y_k) does not converge to the point at infinity of Y. By replacing it with a suitable subsequence, we may assume that it converges to some $y \in Y$. Let us consider for any $k \geq 1$

$$\mu_k(f) := \sum_j a^{(n_k)}_{r_k,j} L_j(f)(y_k) \quad (f \in C_0(X)).$$

As in the first case, by the same reasoning we may assume that there exists $\mu \in M^+_b(X)$ such that $\mu_k \to \mu$ vaguely. Moreover since for every $g \in M$,

$$\Big|\mu_k(g) - T(g)(y_k)\Big| \leq \sum_j a^{(n_k)}_{r_k,j} \|L_j(g) - T(g)\| \to 0,$$

we have $\mu(g) = T(g)(y)$. So (b) implies $\mu(f_0) = T(f_0)(y_k)$, i.e.

$$\lim_{k\to\infty} \Big[\sum_j a^{(n_k)}_{r_k,j} L_j(f_0)(y_k) - T(f_0)(y_k)\Big] = 0$$

which contradicts (3).

If we replace $T : X \to Y$ with the identity operator $I_X : X \to X$ in the above theorem, one can observe the following

COROLLARY 1. Let $\mathscr{A} := \{A^{(n)}\}$ be a regular method of infinite matrices with non-negative real entries. Let X be a locally compact Hausdorff space with a countable base, which is then metrizable as well. For a given subset M of $C_0(X)$, the following statements are equivalent:

(i) M is a Korovkin subset of $C_0(X)$ for identity operator I_X in the sense of summation process.

(ii) If $\mu \in M^+_b(X)$ and $x \in X$ satisfying $\mu(g) = g(x)$ for every $g \in M$, then $\mu(f) = f(x)$ for every $f \in C_0(X)$ i.e. $\mu = I_X$.

Combining our main theorem and Theorem 5.5 of [3], we have the following:

COROLLARY 2. Under the assumptions of Theorem 1, the following statements are equivalent:

(i) M is a Korovkin subset of $C_0(X)$ for T.

(ii) M is a Korovkin subset of $C_0(X)$ for T in the sense of summation process.

Let X and Y be locally compact Hausdorff spaces. A mapping $\varphi : Y \to X$ is said to be proper if for every compact subset $K \subset X$, $\varphi^{-1}(K) := \{y \in Y : \varphi(y) \in K\}$, the pre-image of K, is compact in Y. In this case, $fo\varphi \in C_0(Y)$ for every $f \in C_0(X)$. Now we can give the following as a consequence of Theorem 1.

COROLLARY 3. Let $\mathscr{A} := \{A^{(n)}\}$ be a regular method of infinite matrices with non-negative real entries. Let Y be a metrizable locally compact Hausdorff space. If M is a Korovkin subset of $C_0(X)$ for I_X in the sense of summation process, then M is a Korovkin subset in the sense of summation process for any positive linear operator $T : C_0(X) \to C_0(Y)$ of the form

$$T(f) := \lambda(fo\varphi), \quad (f \in C_0(X))$$

where $\lambda \in C_b(Y)$, $\lambda \geq 0$ and $\varphi : Y \to X$ is a proper mapping.

Now we can give some examples of Korovkin subsets for the identity operator in the sense of summation process following our Corollary 1 and Corollary 6.7 and Proposition 6.8 of [3]. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_2 < \lambda_3$. Then

- $\{e_{\lambda_1}, e_{\lambda_2}, e_{\lambda_3}\}$ is a Korovkin subset of $C_0(X)$ in the sense of summation process where $e_{\lambda_k}(x) := x^{\lambda_k}$ for every $x \in X := (0, 1]$ and $k = 1, 2, 3$.
- $\{e_{-\lambda_1}, e_{-\lambda_2}, e_{-\lambda_3}\}$ is a Korovkin subset of $C_0(X)$ in the sense of summation process where $e_{-\lambda_k}(x) := x^{-\lambda_k}$ for every $x \in X := [1, \infty)$ and $k = 1, 2, 3$.
- $\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}$ is a Korovkin subset of $C_0(X)$ in the sense of summation process where $f_{\lambda_k}(x) := e^{-\lambda_k x}$ for every $x \in X := [0, \infty)$ and $k = 1, 2, 3$.

Using Corollary 2 one can obtain all results given in Chapter 6 of [3] for Korovkin subset in the sense of summation process.

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