Applied Mathematics E-Notes, 18(2018), 124-126 © ISSN 1607-2510
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A Simple Derivation Of Faulhaber's Formula*

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Received 3 August 2017

Abstract

In this article, a simple derivation of Faulhabers formula is introduced. The results are based on series expansions of some appropriate functions. Moreover, the derivation is easy to understand and is accessible to students familiar with introductory level calculus.

1 Introduction

Faulhaber's formula gives an expression for the sum of any positive integer power, p, of the first n positive integers:

$$S^p(n) = 1^p + 2^p + ... + n^p \tag{1}$$

as a $(p+1)$th-degree polynomial of n. Specifically, it is given by the following equation:

$$S^p(n) = \sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j \binom{p+1}{j} B_j n^{p+1-j} \tag{2}$$

where B_j is the jth Bernoulli number with the convention of $B_1 = -\frac{1}{2}$.

The expression in (2) is named after Johann Faulhaber who published formulas for $S^1(n)$, $S^3(n)$,...,$S^2(17)$ in 1631. However, he did not include proofs for these, and a method to derive Faulhaber's formula was first introduced by Jakob Bernoulli in 1713. Moreover, as it is pointed out in [1], a rigorous proof of Faulhaber's assertion was published by Carl Jacobi in 1834.

A typical proof of (2) uses complex generating functions, double sums, and relatively complicated algebraic steps (see, for example, [2]). Consequently, the common derivation is not accessible to students unfamiliar with these topics. Nonetheless, as we show in the next section, (2) can be derived using techniques from introductory calculus.

*Mathematics Subject Classifications: 41A58, 30K05.

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2 The Novel Derivation

First, note that given

$$G(x) = 1 + e^x + e^{2x} + \cdots + e^{nx} = \sum_{i\,=\,1}^{n} e^{(i-1)x} = \frac{e^{(n+1)x} - 1}{e^x - 1}, \tag{3}$$

we have

$$G'(0) = 1 + 2 + ... + n = S^1(n),$$
$$G''(0) = 1^1 + 2^2 + ... + n^2 = S^2(n),$$

and

$$G^{(p)}(0) = 1^p + 2^p + ... + n^p = S^p(n).$$

Therefore, the Taylor expansion of $G(x)$ is given by

$$S^0(n) + \frac{S^1(n)}{1!}x + \frac{S^2(n)}{2!}x^2 + ... + \frac{S^p(n)}{p!}x^p.... \tag{4}$$

Moreover,

$$G(x) = \left(\frac{x}{e^x - 1}\right)\left(\frac{e^{Nx} - 1}{x}\right),$$

where $N = n + 1$ and $\frac{x}{e^x-1}$ is the generating function of the Bernoulli numbers. Since, by definition,

$$\frac{x}{e^x - 1} = B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + ... + B_k\frac{x^k}{k!}..., \tag{5}$$

and the Taylor expansion of $\frac{e^{Nx}-1}{x}$ is given by

$$\frac{e^{Nx} - 1}{x} = \frac{\sum_{k=0}^{\infty}\frac{(Nx)^k}{k!} - 1}{x} = N + \frac{N^2}{2!}x + \frac{N^3}{3!}x^2 + ... + \frac{N^{k+1}}{(k+1)!}x^k + ..., \tag{6}$$

$G(X)$ can be expressed as a product of the above two series. Multiplying the series in (5) and (6) gives

$$\begin{aligned} &B_0N + \left(B_0\frac{N^2}{2!} + \frac{B_1}{1!}N\right)x + \left(B_0\frac{N^3}{3!} + \frac{B_1}{1!}\frac{N^2}{2!} + \frac{B_2}{2!}N\right)x^2 + ... \\ &+ \left(B_0\frac{N^{p+1}}{(p+1)!} + \frac{B_1}{1!}\frac{N^p}{p!} + \frac{B_2}{2!}\frac{N^{p-1}}{(p-1)!} + ... + \frac{B_p}{p!}N\right)x^p + ... \end{aligned} \tag{7}$$

Comparing the coefficients corresponding of x^p in (4) and (7) yields

$$\frac{S^p(n)}{p!} = \left(B_0\frac{N^{p+1}}{(p+1)!} + \frac{B_1}{1!}\frac{N^p}{p!} + \frac{B_2}{2!}\frac{N^{p-1}}{(p-1)!} + ... + \frac{B_p}{p!}N\right). \tag{8}$$

Rearranging the above gives

$$\begin{aligned} S^p(n) &= \left(B_0\frac{N^{p+1}}{(p+1)!} + \frac{B_1}{1!}\frac{N^p}{p!} + \frac{B_2}{2!}\frac{N^{p-1}}{(p-1)!} + ... + \frac{B_p}{p!}N\right)\frac{(p+1)!}{(p+1)} \\ &= \frac{1}{(p+1)}\left(\frac{(p+1)!}{(p+1)!}B_0N^{p+1} + \frac{(p+1)!}{p!1!}B_1N^p + ... + \frac{(p+1)!}{1!(p)!}B_pN\right). \end{aligned}$$

Therefore,

$$S^p(n) = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j N^{p+1-j}. \tag{9}$$

Finally, it remains to be shown that the above is equivalent to (2). Since $B_j = 0$ for all odd $j > 1$ and

$$B_1 = -\frac{1}{2} = -(1 + B_1),$$

we obtain

$$\begin{aligned} S^p(N) &= \sum_{k=1}^{N} k^p = S^p(n) + N^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j N^{p+1-j} + N^p \\ &= \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j N^{p+1-j}. \end{aligned}$$

Conclusions

In this paper, a novel derivation of Faulhaber's formula is presented. Our approach does not rely on advanced mathematical topics, and hence it is accessible to students with knowledge of introductory level calculus.

References

[1] D. E. Knuth, Johann Faulhaber and sums of powers, Mathematics of Computation, 61(1993), 277–294.

[2] J. H. Conway and R. K. Guy, The Book of Numbers, Springer-Verlag, New York, 1996, p. 106.