On Arbitrary Powers Of Double Band Matrices*

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Abstract

In the present note, by defining two new fractional difference operators, we determine the explicit formulas for any arbitrary power of double band matrices. Subsequently, adaptive algorithms for finding the arbitrary powers of both upper and lower double band matrices have been developed. Respective programming codes for the new algorithms have been constructed and verified by implementing them in MATLAB. Some counter examples are also given in support to the new programming codes.

1   Introduction

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of all real numbers and positive integers, respectively. Let $A = (a_{ij})$ $(i, j \in \mathbb{N})$ be a non singular matrix of order n, $(n \in \mathbb{N})$ i.e.,

$$A := \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix}.$$

Then, matrix A can be factorized as the product of a unit lower triangular matrix L and an upper triangular matrix U, i.e., $A = LU$.
Suppose the matrices $L(a, b) = (l_{ij})$ and $U(a, b) = (u_{ij})$ denote the lower and upper double band matrices, respectively, then for $a \neq 0$, we write

$$l_{ij} = \begin{cases} a & \text{for } j = i, \\ b & \text{for } j = i - 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad u_{ij} = \begin{cases} a & \text{for } j = i, \\ b & \text{for } i = j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let w be the space of all real or complex valued sequences, and X and Y be two subspaces of w, then we define a matrix mapping $A : X \to Y$, as

$$(Ax)_n := \sum_k a_{nk} x_k; (n \in \mathbb{N}). \tag{1}$$

In fact, for $x = (x_k) \in X$, Ax is called the A-transform of x provided the series in (1) converges for each $n \in \mathbb{N}$. Moreover, the matrix $A = (a_{ij})$ $(i, j \in \mathbb{N})$ is also regarded as a linear operator. One of the most effective and powerful tools for the study of summability and matrix theory is the development of several difference operators, their related sequence spaces and their various applications. The applications of difference operators become more apparent in study of several summable matrices and related properties mostly involving inversion, powers and norm of matrices in Linear algebra and the study of derivatives of arbitrary orders and their dynamic natures in Fractional calculus. These operators are also being used in study of spectral properties of different matrices and related eigenvalue problems in operator theory and many others. The idea of difference operators of order one was initially provided by Kızmaz [1] and Altay and Basar [2] and further these were extended to the case of positive integer m by Et and Colak [3] and Ahmad and Mursaleen [4]. On generalization of the difference matrix Δ, Altay and Başar [10] (see also [11–14]) defined the double band matrix $B(r, s)$, where $0 \neq r, s \in \mathbb{R}$ and studied the related sequence spaces and their spectral properties. For more detail on difference sequence spaces one may refer [15–28]. Recently, for a proper fraction α, Baliarsingh [6](see also [5, 7, 8]) defined fractional difference operator Δ^α, which not only generalizes most of difference operators defined earlier, but it also provides some new and interesting ideas regarding fractional power of certain matrices, fractional derivatives of some functions and many others. Motivated by the earlier works, the main objective of this note is to define a fractional difference operator analog to the double band matrix $B(r, s)$ and establish certain results on finding powers of these matrices. Let $x = (x_k)$ be any sequence in w and $a(\neq 0), b$ be two real numbers, then we define the generalized difference operators which generate matrices $L(a, b)$ and $U(a, b)$ as

$$(L(a, b)x)_k = ax_k + bx_{k+1}; (k \in \mathbb{N})$$

and

$$(U(a, b)x)_k = ax_k + bx_{k-1,}; (k \in \mathbb{N}).$$

Particularly, the matrices $L(a, b)$ and $U(a, b)$ generalize the difference operators of order one Δ and $\Delta^{(1)}$ (see [1, 2]), respectively under the case $a = 1$ and $b = -1$, where

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \Delta^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is also observed that the mth power of matrices Δ and $\Delta^{(1)}$ are being calculated by taking difference operators Δ^m and $\Delta^{(m)}$ (see[3, 4, 5]) for all $m \in \mathbb{N}$, respectively. One of the interesting calculations involving arbitrary power $(\alpha \in \mathbb{R})$ of these matrices are also being calculated by taking difference operators Δ^α and $\Delta^{(\alpha)}$ (see[6, 7, 8]). Now,

it is trivial to check that

$$(\Delta)^\alpha = \begin{pmatrix} 1 & -\alpha & \frac{\alpha(\alpha-1)}{2!} & -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} & \cdots \\ 0 & 1 & -\alpha & \frac{\alpha(\alpha-1)}{2!} & \cdots \\ 0 & 0 & 1 & -\alpha & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } (\Delta^{(1)})^\alpha = ((\Delta)^\alpha)^T,$$

where A^T represents the transposition of A. On generalizing all the difference operators discussed above, we define

$$(U^\alpha(a,b)x)_k = \sum_{i=0}^\infty \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} a^{\alpha-i} b^i x_{k+i}, \ (k \in \mathbb{N}) \tag{2}$$

$$(L^\alpha(a,b)x)_k = \sum_{i=0}^\infty \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} a^{\alpha-i} b^i x_{k-i}, \ (k \in \mathbb{N}), \tag{3}$$

where $\Gamma(\alpha)$ denotes the well known Gamma function of a real number α and $\alpha \notin \{0,-1,-2,-3\ldots\}$. For any integral values of α, Eqns.(2) and (3) reduce to the finite sums. It is remarked that the difference sequences defined in (2) and (3) generate the operator Δ for $\alpha = 1$, $a = 1$, $b = -1$, $B(r,s)$ for $\alpha = 1$, $a = r$, $b = s$, Δ^α for $a = 1$, $b = -1$ and Δ^m for $\alpha = m$, $a = 1$, $b = -1$. Now, we state some numerical examples on certain sequences via these operators.

EXAMPLES.

- Let us take a sequence $x = (x_k) = e = (1,1,1,\ldots)$. Then the difference sequences $(U^\alpha(a,b)x)_k \to (a+b)^\alpha$ and $(L^\alpha(a,b)x)_k \to (a+b)^\alpha$ as $k \to \infty$.

- Suppose we consider a sequence $x = (x_k)$ with $x_k = k$ for all $k \in \mathbb{N}$ and $b = -a$, then by straight forward calculations, we have

$$(U^\alpha(a,-a)x)_k = \sum_{i=0}^\infty \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} a^{\alpha-i}(-a)^i(k+i)$$

$$= a^\alpha \sum_{i=0}^\infty \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} (-1)^i(k+i)$$

$$= ka^\alpha \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} + a^\alpha \sum_{i=0}^\infty (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} i.$$

In fact, both of the difference sequences $(U^\alpha(a,-a)x)_k$ and $(L^\alpha(a,-a)x)_k$ go to zero as $k \to \infty$ provided α is either an integer or a real number greater than one.

- Let us consider a sequence $x = (x_k)$ with $x_k = (-1)^k$ for all $k \in \mathbb{N}$. Then the difference sequences $(U^\alpha(a,b)x)_k = 0$ for $a = b$ and $(U^\alpha(a,b)x)_k = (-1)^k(2a)^\alpha$ for $b = -a$.

Recently, Baliarsingh et al. [9] studied the fractional powers of double band matrices using fractional difference operators. Also, in that paper certain theoretical results on finding arbitrary powers of a matrix have been discussed. The main objective of this paper is to give a technical and numerical treatment to these results.

Using [9] now, we state following theorems involving the integral and non integral powers of double band matrices $U(r,s)$ and $L(r,s)$.

LEMMA 1. Let the lower double band matrix $L = (l_{nk})$ be defined by

$$l_{nk} = \begin{cases} r & \text{for } k = n, \\ s & \text{for } k = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

then for $\alpha \in \mathbb{R}$, $L^\alpha = (l_{nk}^\alpha)$ is given by

$$l_{nk}^\alpha = \begin{cases} r^\alpha & \text{for } k = n, \\ \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} r^{\alpha-n+k} s^{n-k} & \text{for } 0 \le k < n, \\ 0, & \text{for } k > n. \end{cases}$$

LEMMA 2. Let the upper double band matrix $U = (u_{nk})$ be defined by

$$u_{nk} = \begin{cases} r & \text{for } k = n, \\ s & \text{for } k = n+1, \\ 0 & \text{otherwise,} \end{cases}$$

then for $\alpha \in \mathbb{R}$, $U^\alpha = (u_{nk}^\alpha)$ is given by

$$u_{nk}^\alpha = \begin{cases} r^\alpha & \text{for } k = n, \\ \frac{\Gamma(\alpha+1)}{(k-n)!\Gamma(\alpha-k+n+1)} r^{\alpha-k+n} s^{k-n} & \text{for } k > n, \\ 0 & \text{for } 0 \le k < n. \end{cases}$$

2 Main Results

In this section, we provide some applications of Lemmas 1 and 2.

THEOREM 1. The arbitrary power α of the lower double band matrix $L(1,c)$ is given by $L^\alpha(1,c) = (l_{nk}^\alpha)$, where

$$l_{nk}^\alpha = \begin{cases} 1 & \text{for } k = n, \\ \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} c^{n-k} & \text{for } 0 \le k < n, \\ 0 & \text{for } k > n. \end{cases}$$

PROOF. The proof is a direct consequence of Lemma 1.

THEOREM 2. The arbitrary power α of the lower double band matrix $U(a,b)$ is given by $U^\alpha(a,b) = (u_{nk}^\alpha)$, where

$$u_{nk}^\alpha = \begin{cases} a^\alpha & \text{for } k = n, \\ \frac{\Gamma(\alpha+1)}{(k-n)!\Gamma(\alpha-k+n+1)}a^{\alpha-k+n}b^{k-n} & \text{for } k > n, \\ 0 & \text{for } 0 \le k < n. \end{cases}$$

PROOF. The proof follows from Lemma 2.

THEOREM 3. Let A be a nonsingular matrix of order n and $A = LU$, then for any real $\alpha \neq 0, -1$, we have

$$A^\alpha \neq U^\alpha L^\alpha. \tag{4}$$

However, if $A = L(1,c)U(a,b)$, then the explicit formula for $A^{-1} = a_{ij}^{-1}$ is given by

$$a_{ij}^{-1} = \begin{cases} (-1)^{i+j}a^{-1-j+i}b^{j-i} + \sum_{k=j+1}^{n}(-1)^{i+j}a^{-1-k+i}b^{k-i}c^{k-j} & \text{for } i \le j, \\ (-1)^{i+j}c^{i-j}a^{-1} + \sum_{k=i+1}^{n}(-1)^{(i+j)}a^{-1-k+i}b^{k-i}c^{k-j} & \text{for } i > j. \end{cases}$$

PROOF. The entire proof is divided into two parts. In the first part, supporting to Eqn (4) we have mentioned the following counter example: Consider a non singular matrix A of order 5, where

$$A = \begin{pmatrix} 9 & 2 & 0 & 0 & 0 \\ 45 & 19 & 2 & 0 & 0 \\ 0 & 45 & 19 & 2 & 0 \\ 0 & 0 & 45 & 19 & 2 \\ 0 & 0 & 0 & 45 & 19 \end{pmatrix}.$$

On LU factorization of the tridiagonal type matrix A, we have $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 9 & 2 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 & 0 \\ 0 & 0 & 9 & 2 & 0 \\ 0 & 0 & 0 & 9 & 2 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix}.$$

Clearly, using Theorems 1 and 2, square roots of the matrix L and U are being calculated directly and

$$L^{1/2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/2 & 1 & 0 & 0 & 0 \\ -25/8 & 5/2 & 1 & 0 & 0 \\ 125/16 & -25/8 & 5/2 & 1 & 0 \\ -3225/128 & 125/16 & -25/8 & 5/2 & 1 \end{pmatrix}$$

and

$$U^{1/2} = \begin{pmatrix} 3 & 1/3 & -1/54 & 1/486 & -5/17496 \\ 0 & 3 & 1/3 & -1/54 & 1/486 \\ 0 & 0 & 3 & 1/3 & -1/54 \\ 0 & 0 & 0 & 3 & 1/3 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

It can be easily shown that

$$A^{1/2} \neq U^{1/2} L^{1/2}.$$

However, the equality holds in Eq.(4) for $\alpha = -1$. In fact, for any natural number α the equality sign holds provided the matrices L and U commute. Secondly, from Theorems 1 and 2, we can calculate the explicit inverses of L and U which are mentioned below:

$$l_{nk}^{-1} = \begin{cases} 1 & \text{for } k = n, \\ (-1)^{n-k} c^{n-k} & \text{for } 0 \leq k < n, \\ 0 & \text{for } k > n, \end{cases}$$

and

$$u_{nk}^{-1} = \begin{cases} 1/a & \text{for } k = n, \\ (-1)^{k-n} a^{-1-k+n} b^{k-n} & \text{for } k > n, \\ 0 & \text{for } 0 \leq k < n. \end{cases}$$

Now, using above results in Eq.(4), it is calculated that the inverse of A as $A^{-1} = (a_{ij}^{-1})$, where

$$a_{ij}^{-1} = \sum_{k=0}^{n} u_{ik}^{-1} l_{kj}^{-1} = \sum_{k=j}^{n} u_{ik}^{-1} l_{kj}^{-1} = u_{ij}^{-1} l_{jj}^{-1} + \sum_{k=j+1}^{n} u_{ik}^{-1} l_{kj}^{-1}$$

$$= (-1)^{i+j} a^{-1-j+i} b^{j-i} + \sum_{k=j+1}^{n} (-1)^{i+j} a^{-1-k+i} b^{k-i} c^{k-j} \quad \text{for } i \leq j.$$

Similarly,

$$a_{ij}^{-1} = \sum_{k=0}^{n} u_{ik}^{-1} l_{kj}^{-1} = \sum_{k=i}^{n} u_{ik}^{-1} l_{kj}^{-1} = u_{ii}^{-1} l_{ij}^{-1} + \sum_{k=i+1}^{n} u_{ik}^{-1} l_{kj}^{-1}$$

$$= (-1)^{i+j} c^{i-j} a^{-1} + \sum_{k=i+1}^{n} (-1)^{(i+j)} a^{-1-k+i} b^{k-i} c^{k-j} \quad \text{for all } i > j.$$

This completes the proof.

3   Applications

As an application of above theorems, we have developed algorithms for finding arbitrary powers of double band matrices. Subsequently, required MATLAB programming for both the algorithms have been constructed.

   Algorithm 1:

Step 1 : Input n, a, b, α and $A = (a_{ij})$ where $a_{ij} = a$ for $i = j$, b for $j = i - 1$ and 0 otherwise.

Step 2 : Compute $S_{n-k}(L) = \frac{\Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha-n+k+1)} a^{\alpha-n+k} b^{n-k}$.

Step 3 : Compute $L^{-1}(a,b) = (l_{nk}^{-1})$, where $l_{nk}^{-1} = S_{n-k}(L)$ for $1 \leq k < n$ and a^{α} for $k = n$ and 0 for $k > n$.

Algorithm 2:

Step 1 : Input n, a, b, α and $A = (a_{ij})$ where $a_{ij} = a$ for $i = j$, b for $i = j - 1$ and 0 otherwise.

Step 2 : Compute $S_{k-n}(U) = \frac{\Gamma(\alpha+1)}{(k-n)!\Gamma(\alpha-k+n+1)} a^{\alpha-k+n} b^{k-n}$.

Step 3 : Compute $U^{-1}(a,b) = (u_{nk}^{-1})$, where $u_{nk}^{-1} = S_{k-n}(U)$ for $k > n$ and a^{α} for $k = n$ and 0 for for $1 \leq k < n$.

Total number of operations for both of the algorithms is $\mathcal{O}(n^2)$ in worse case, where n is the size of the matrix. Especially, step 2 in both of the algorithms are involved $\mathcal{O}(n^2)$ multiplication operations. Since these algorithms are unstable under the case $a = 0$, it is strictly suggested to choose the matrices with non zero diagonal elements. From both of the algorithms it is calculated that

$$\|L(a,b)\| = \|U(a,b)\| \leq |a^{\alpha}| \sum_{k=0}^{n} \frac{\Gamma(\alpha + 1)}{(k)!\Gamma(\alpha - k + 1)} \left|\frac{b}{a}\right|^{k}.$$

In fact, if $|a| < |b|$ the above matrix norms are unbounded for larger matrices ($n >>$). Therefore, above algorithms are numerically stable under the suitable condition $|a| \geq |b|$.

Programming for lower double band matrices:

```
 n=input('Enter  the  value  of  n  :  ');
a=input('Enter  the  value  of  a  :  ');
 b=input('Enter  the  value  of  b  :  ')
 alpha=input('Enter  the  value  of  alpha  :  ');
matrix_power=zeros(n);  for  row=1:n
    for  column=1:n
        if(column==row)
            matrix_power(row,column)=a.^alpha;
         elseif  (column>row)
            matrix_power(row,column)=0;
        else
            z=row-column;
            z1=factorial(z);
 matrix_power(row,column)=
  gamma(alpha+1)/(z1*gamma(alpha-z+1))*a.^(alpha-z)*b.^z;
        end
    end
end  matrix_power
```

Programming for upper double band matrices:

```
clc
  n=input('Enter the value of n : ');
a=input('Enter the value of a : ');
 b=input('Enter the value of b : ');
 alpha=input('Enter the value of alpha : ');
matrix_power=zeros(n);
 for row=1:n
     for column=1:n
         if(column==row)
             matrix_power(row,column)=a.^alpha;
          elseif (row>column)
              matrix_power(row,column)=0;
         else
             z=column-row;
             z1=factorial(z);
  matrix_power(row,column)=
  gamma(alpha+1)/(z1*gamma(alpha-z+1))*a.^(alpha-z)*b.^z;
         end
     end
end matrix_power
```

In supporting to both of the above codings, here two examples are mentioned which have been verified by MATLAB software.

Example 1:

```
 Enter the value of n : 5
Enter the value of a : 4
 Enter the value of b : 9
 Enter the value of alpha : 1/2
matrix_power =
     2.0000          0          0          0          0
     2.2500     2.0000          0          0          0
    -1.2656     2.2500     2.0000          0          0
     1.4238    -1.2656     2.2500     2.0000          0
    -2.0023     1.4238    -1.2656     2.2500     2.0000
```

Example 2:

```
 Enter the value of n : 5
 Enter the value of a : 18
Enter the value of b : 16 Enter the value of alpha : 1/5
matrix_power =
     1.7826     0.3169    -0.1127     0.0601    -0.0374
          0     1.7826     0.3169    -0.1127     0.0601
          0          0     1.7826     0.3169    -0.1127
          0          0          0     1.7826     0.3169
          0          0          0          0     1.7826
```

Conclusion: Fractional difference operators analog to double band matrices have been introduced using which new algorithms for integral and non integral powers of double band matrices have been proposed. The corresponding MATLAB programming has been constructed. As one of its applications, it is being used for finding the inverse of tridiagonal type matrices. Furthermore, using the proposed difference operators, one may also define related sequence spaces and study their topological and geometrical properties. The spectral properties of these operators are yet to be studied which may generalize the notion of fine spectra of all double band matrices and difference operators of any arbitrary orders.

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