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A New Way Of Computing The Orthogonal Projection Onto The Intersection Of Two Hyperplanes In A Finite-Dimensional Hilbert Space*

Williams López†

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Abstract

In this work, we present and analyze a new way for computing the orthogonal projection onto the intersection of two hyperplanes in a finite-dimensional Hilbert space via the method of alternating projections and a line search.

1 Introduction

Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$ and let M be a closed subspace of H. The orthogonal projection onto M will be denoted by P_M. In particular, P_M is a linear, self-adjoint, idempotent operator, and $P_M(x)$ is the best approximation or the nearest point to x from M:

$$\|x - P_M(x)\| = d(x, M),$$

where $d(x, M) = \inf\{\|x - y\| : y \in M\}$, $\|z\|^2 = \langle z, z\rangle$, for any $z \in H$. P_M is called the orthogonal projection onto M because of the characterising property

$$\langle x - P_M(x), y\rangle = 0, \quad \text{for any} \quad y \in M.$$

In other words, $x - P_M(x)$ is orthogonal to M. Using the notation

$$M^\perp = \{y \in H : \langle y, x\rangle = 0, \text{ for any } x \in M\}, \tag{1}$$

we see that $x - P_M(x) \in M^\perp$. The set $M^\perp$ defined in (1) is a subspace of H and it is called the orthogonal complement of M.

LEMMA 1 (von Neumann [17]). Let M_1 and M_2 be two closed subspaces in H. Then

$$P_{M_2}P_{M_1} = P_{M_1}P_{M_2} \iff P_{M_2}P_{M_1} = P_{M_1\cap M_2}.$$

In words, P_{M_1} and P_{M_2} commute if and only if their composition is also an orthogonal projection. In particular, von Neumann was interested in the case in which P_{M_1} and P_{M_2} dit not commute, proving the following:

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†Departamento de Física y Matemáticas, Núcleo Universitario "Rafael Rangel", Universidad de Los Andes, Trujillo, Venezuela

THEOREM 1 (von Neumann [17]). If M_1 and M_2 are closed subspaces in H, then for each $x_0 \in H$,

$$\lim_{k\to\infty}(P_{M_2}P_{M_1})^k(x_0) = P_{M_1\cap M_2}(x_0).$$

This result was later extended to more than 2 subspaces.

THEOREM 2 (I. Halperin [12]). If $M_1, \ldots, M_p$ are closed subspaces in H, then for each $x_0 \in H$,

$$\lim_{k\to\infty}(P_{M_p}P_{M_{p-1}}\cdots P_{M_1})^k(x_0) = P_{\cap_{i=1}^p M_i}(x_0).$$

Theorem 2 suggests an algorithm, called the method of alternating projections (or MAP for short); see [7, 8], which can be described as follows: for any $x_0 \in H$, set

$$\begin{cases} x_0^k = x_p^{k-1} \\ x_i^k = P_{M_i}(x_{i-1}^k) \quad i = 1, 2, \ldots, p, \end{cases} \tag{2}$$

for $k \in Z^+$, with initial value $x_p^0 = x_0$. It follows from Theorem 2 that for any $i = 1, 2, \ldots p$, the sequence $\{x_i^k\}$ generated by (2) converges to $P_M(x_0)$.

Algorithm 1 MAP

Require: $x_0 \in H$
Ensure: $P_{\cap_{i=1}^p M_i}(x_0)$
  for $k = 0, 1, \ldots$ **do**
    $x_{k+1} = P_{M_p}P_{M_{p-1}}\cdots P_{M_1}(x_k)$
  end for

The MAP has an r-linear rate of convergence that can be very slow when the angles between the subspaces are small (see, e.g., [8]). In fact, for the case of two subspaces, Franchetti and Light [9] and Bauschke, Borwein, and Lewis [1] gave examples to illustrate the possible slowness of MAP. Consequently, several acceleration schemes have been proposed (see, e.g. [15, 16, 2, 4, 5, 3, 10, 13]).

In Hilbert spaces of dimension n the hyperplanes are special cases of translated closed subspaces of dimension $n-1$. Thus, in this paper we proof that if M_1 and M_2 are closed subspaces of the Hilbert space H, and if the dimension of $M_2^\perp$ is 1, then in order to know the projection $P_{M_1\cap M_2}(x_0)$ it is enough to know the iterates x_2^1 and x_2^2 generated by (2), which is equivalent to knowing $x_1 = P_{M_2}P_{M_1}(x_0)$ and $x_2 = P_{M_2}P_{M_1}(x_1)$ generated by Algorithm 1. This result is obtained by using the Gearhart and Koshy scheme given in [10].

The rest of this paper is organized as follows. In section 2 we describe the Gearhart and Koshy scheme. In section 3 we give a closed formula depending of the two iterates from the Algorithm 1 to find the projection on the intersection of two closed subspaces, then we apply this result to find the intersection of two hyperplanes in a finite-dimensional Hilbert space.

2 The Gearhart and Koshy Scheme

We will give a brief explanation of the Gearhart and Koshy scheme. Let us denote x_0 the given starting point and by Q the composition of the projection operators, i.e, $Q = P_{M_p} P_{M_{p-1}} \cdots P_{M_1}$, where P_{M_i} is the projection operator onto M_i for all i. Let x_k the k th iterate, and let Qx_k be the next iterate after applying a sweep of MAP. The idea is to search along the line through the points x_k and Qx_k to obtain the point closest to the solution $x^* = P_{\cap_{i=1}^p M_i}(x_0)$. Let us represent any point on this line as

$$x^k(\alpha) = \alpha Q x_k + (1-\alpha) x_k = x_k + \alpha(Qx_k - x_k),$$

for some real number α. Let α_k be the value of α for which this point is closest to x^*. Then,

$$\left\langle x^k(\alpha_k) - x^*, x_k - Qx_k \right\rangle = 0. \tag{3}$$

Now, since $x^* \in \cap_{i=1}^p M_i$ and the projections P_{M_i} are self-adjoint, then

$$\langle x^*, Qx_k \rangle = \left\langle P_{M_1} P_{M_2} \cdots P_{M_p} x^*, x_k \right\rangle = \langle x^*, x_k \rangle .$$

Consequently, $\langle x^*, x_k - Qx_k \rangle = 0$, and so x^* can be eliminated from (3) to obtain

$$\left\langle x^k(\alpha_k), x_k - Qx_k \right\rangle = 0.$$

Solving for α_k gives

$$\alpha_k = \frac{\langle x_k, x_k - Qx_k \rangle}{\|x_k - Qx_k\|^2}. \tag{4}$$

Therefore

$$x_k + \alpha_k(Qx_k - x_k), \tag{5}$$

where α_k is given in (4), is the point in the line connecting Qx_k and x_k closest to the solution x^*.

3 Projecting onto the Intersection of Two Subspaces

Let M_1 and M_2 be two closed subspaces in the Hilbert space H. In this section we will always assume that $P_{M_2} P_{M_1}(x) \neq P_{M_1 \cap M_2}(x)$ for all $x \in H$. As a consequence, the iterates generated by (2) are all different.

THEOREM 3. Let M_1 and M_2 be two closed subspaces in H, and $x_0 \in H$. Let $x_1^1 = P_{M_1}(x_0), x_2^1 = P_{M_2}(x_1^1), x_1^2 = P_{M_1}(x_2^1)$ and $x_2^2 = P_{M_2}(x_1^2)$ be generated by (2). If $x_1^1 + \alpha(x_1^2 - x_1^1) = x_2^1 + \alpha(x_2^2 - x_2^1)$ for some $\alpha \in \mathbb{R}$, then

$$P_{M_1 \cap M_2}(x_0) = x_1^1 + \alpha(x_1^2 - x_1^1) = x_2^1 + \alpha(x_2^2 - x_2^1).$$

PROOF. Let $\alpha \in \mathbb{R}$ be such that

$$z = x_1^1 + \alpha(x_1^2 - x_1^1) = x_2^1 + \alpha(x_2^2 - x_2^1). \tag{6}$$

Therefore $z \in M_1 \cap M_2$. We will establish that $z = P_{M_1 \cap M_2}(x_0)$. For it is enough to show that $z - x_0$ is orthogonal to $M_1 \cap M_2$. We have that $x_1^1 - x_0 = P_{M_1}(x_0) - x_0$ is orthogonal to M_1, and so in particular $x_1^1 - x_0$ is orthogonal to $M_1 \cap M_2$. Since $x_2^1 - x_1^1 = P_{M_2}(x_1^1) - x_1^1$ is orthogonal to M_2, then $x_2^1 - x_1^1$ is orthogonal to $M_1 \cap M_2$. Since $x_1^2 - x_2^1 = P_{M_1}(x_2^1) - x_2^1$ is orthogonal to M_1, then $x_1^2 - x_2^1$ is orthogonal to $M_1 \cap M_2$.

Therefore, for any $w \in M_1 \cap M_2$, we have that

$$\langle x_1^2 - x_1^1, w\rangle = \langle x_1^2 - x_2^1, w\rangle + \langle x_2^1 - x_1^1, w\rangle = 0,$$

and hence $x_1^2 - x_1^1$ is orthogonal to $M_1 \cap M_2$. Now, using (6),

$$z - x_1^1 = \alpha(x_1^2 - x_1^1),$$

and so $z - x_1^1$ is also orthogonal to $M_1 \cap M_2$. Finally, for any $w \in M_1 \cap M_2$,

$$\langle z - x_0, w\rangle = \langle z - x_1^1, w\rangle + \langle x_1^1 - x_0, w\rangle = 0,$$

and the result is established.

THEOREM 4. Let M_1 and M_2 be two closed subspaces in H, and $x_0 \in H$. Let x_1^1, x_2^1, x_1^2 and x_2^2 be generated by (2). Then $x_1^1 + \alpha(x_1^2 - x_1^1) = x_2^1 + \alpha(x_2^2 - x_2^1)$ for some $\alpha \in \mathbb{R}$, if and only if $\{x_2^1 - x_1^1, x_2^2 - x_1^2\}$ is a linearly dependent set of vectors.

PROOF. Let us consider the lines

$$l_1 = \{x_1^1 + \alpha(x_1^2 - x_1^1) : \alpha \in \mathbb{R}\} \text{ and } l_2 = \{x_2^1 + \alpha(x_2^2 - x_2^1) : \alpha \in \mathbb{R}\}.$$

Then

$$x_1^1 + \alpha(x_1^2 - x_1^1) = x_2^1 + \alpha(x_2^2 - x_2^1) \text{ for some } \alpha \in \mathbb{R} \iff l_1 \cap l_2 \neq \emptyset.$$

But

$$\begin{aligned} x_1^1 + \alpha(x_1^2 - x_1^1) = x_2^1 + \alpha(x_2^2 - x_2^1) \quad &\iff \quad x_2^1 - x_1^1 = \alpha(x_1^2 - x_1^1) - \alpha(x_2^2 - x_2^1) \\ &\iff \quad x_2^1 - x_1^1 = \alpha(x_1^2 - x_1^1 - x_2^2 + x_2^1). \end{aligned} \tag{7}$$

On the other hand, observe that

$$l_1 = \{x_1^2 + \beta(x_1^2 - x_1^1) : \beta \in \mathbb{R}\} \text{ and } l_2 = \{x_2^2 + \beta(x_2^2 - x_2^1) : \beta \in \mathbb{R}\}.$$

Then

$$x_1^2 + \beta(x_1^2 - x_1^1) = x_2^2 + \beta(x_2^2 - x_2^1) \text{ for some } \beta \in \mathbb{R} \iff l_1 \cap l_2 \neq \emptyset.$$

But

$$\begin{aligned} x_1^2 + \beta(x_1^2 - x_1^1) = x_2^2 + \beta(x_2^2 - x_2^1) \quad &\iff \quad x_2^1 - x_1^1 = \alpha(x_2^2 - x_1^2) - \beta(x_2^2 - x_2^1) \\ &\iff \quad x_2^2 - x_1^2 = \beta(x_1^2 - x_1^1 - x_2^2 + x_2^1). \end{aligned} \tag{8}$$

If $x_1^1+\alpha(x_1^2-x_1^1)=x_2^1+\alpha(x_2^2-x_2^1)$ for some $\alpha\in\mathbb{R}$, then $l_1\cap l_2\neq\emptyset$ and from (7) and (8) it follows that $\{x_2^1-x_1^1, x_2^2-x_1^2\}$ is a linearly dependent set of vectors. If $\{x_2^1-x_1^1, x_2^2-x_1^2\}$ is a linearly dependent set of vectors, then $x_2^1-x_1^1=\lambda(x_2^2-x_1^2)$, for some $\lambda\in\mathbb{R}$. We have that $\lambda\neq 0$, otherwise $P_{M_2}P_{M_1}(x_0)=P_{M_1\cap M_2}(x_0)$, and $\lambda\neq 1$, otherwise the lines l_1 and l_2 would be parallel and as a consequence the sequence $\{x_2^k\}$ does not converge. Then

$$\begin{aligned} x_1^2-x_1^1-x_2^2+x_2^1 &= x_2^1-x_1^1-(x_2^2-x_1^2)\\ &= \lambda(x_2^2-x_1^2)-(x_2^2-x_1^2)\\ &= (\lambda-1)(x_2^2-x_1^2). \end{aligned}$$

Hence $x_2^2-x_1^2=1/(\lambda-1)(x_1^2-x_1^1-x_2^2+x_2^1)$ and from (8) the result is established.

COROLLARY 1. Let M_1 and M_2 be two closed subspaces in H, and $x_0\in H$. Suppose that $M_2^\perp$ is a one-dimensional subspace. Let x_2^1 and x_2^2 be generated by (2). Then

$$P_{M_1\cap M_2}(x_0)=x_2^1+\alpha(x_2^2-x_2^1)\ \text{ for some } \alpha\in\mathbb{R}.$$

PROOF. We have that $x_2^1-x_1^1=P_{M_2}(x_1^1)-x_1^1\in M_2^\perp$ and $x_2^2-x_1^2=P_{M_2}(x_1^2)-x_1^2\in M_2^\perp$. Since $M_2^\perp$ is a one-dimensional subspace, then $\{x_2^1-x_1^1, x_2^2-x_1^2\}$ is a linearly dependent set of vectors. From Theorem 4 and Theorem 3 it follows that $P_{M_1\cap M_2}(x_0)=x_2^1+\alpha(x_2^2-x_2^1)$, for some $\alpha\in\mathbb{R}$.

THEOREM 5. Let M_1 and M_2 be two closed subspaces in H, and $x_0\in H$. Suppose that $M_2^\perp$ is a one-dimensional subspace. Let x_2^1 and x_2^2 be generated by (2). Then

$$P_{M_1\cap M_2}(x_0)=x_2^1+\left(\langle x_2^1, x_2^1-x_2^2\rangle/\|x_2^1-x_2^2\|^2\right)(x_2^2-x_2^1).$$

PROOF. From Corollary 1 it follows that $P_{M_1\cap M_2}(x_0)\in\{x_2^1+\alpha(x_2^2-x_2^1):\alpha\in\mathbb{R}\}$. From (4) and (5) it follows that $x_2^1+\left(\langle x_2^1, x_2^1-x_2^2\rangle/\|x_2^1-x_2^2\|^2\right)(x_2^2-x_2^1)$ is the point in the line $\{x_2^1+\alpha(x_2^2-x_2^1):\alpha\in\mathbb{R}\}$ closest to $P_{M_1\cap M_2}(x_0)$. Therefore

$$P_{M_1\cap M_2}(x_0)=x_2^1+\left(\langle x_2^1, x_2^1-x_2^2\rangle/\|x_2^1-x_2^2\|^2\right)(x_2^2-x_2^1).$$

EXAMPLE 1. We consider the following two closed subspaces in the space of square real matrices $\mathbb{R}^{2\times 2}$, with the Frobenius norm $\|A\|_F^2=\langle A,A\rangle=trace(A^TA)$:

$$M_1=\left\{A=(A_{ij})\in\mathbb{R}^{2\times 2}: A^T=A\right\}\ \text{ and }\ M_2=\left\{A=(A_{ij})\in\mathbb{R}^{2\times 2}: A_{12}=0\right\}.$$

Clearly $M_1^\perp$ and $M_2^\perp$ are one-dimensional closed subspaces. In that case the projections onto each of the individual subspaces M_i are simple to compute. If $A=(A_{ij})\in\mathbb{R}^{2\times 2}$ then

$$P_{M_1}(A)=(A^T+A)/2\ \text{ and }\ P_{M_2}(A)=\begin{pmatrix} A_{11} & 0\\ A_{21} & A_{22}\end{pmatrix}.$$

We will use the Theorem 5 for finding the projection of

$$A_0 = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$

onto $M_1 \cap M_2$. After the first cycle of MAP, we have that

$$A_2^1 = P_{M_2} P_{M_1}(A_0) = P_{M_2} P_{M_1} \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} = P_{M_2} \begin{pmatrix} 3 & 4.5 \\ 4.5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 4.5 & 6 \end{pmatrix},$$

and after the second cycle we obtain

$$A_2^2 = P_{M_2} P_{M_1}(A_2^1) = P_{M_2} P_{M_1} \begin{pmatrix} 3 & 0 \\ 4.5 & 6 \end{pmatrix} = P_{M_2} \begin{pmatrix} 3 & 2.25 \\ 2.25 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2.25 & 6 \end{pmatrix}.$$

From Theorem 5 it follows that

$$\begin{aligned} P_{M_1 \cap M_2}(A_0) &= A_2^1 + \left(\langle A_2^1, A_2^1 - A_2^2 \rangle / \| A_2^1 - A_2^2 \|^2 \right) \left(A_2^2 - A_2^1 \right) \\ &= \begin{pmatrix} 3 & 0 \\ 4.5 & 6 \end{pmatrix} + 2 \left(\begin{pmatrix} 3 & 0 \\ 2.25 & 6 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 4.5 & 6 \end{pmatrix} \right) \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}. \end{aligned}$$

3.1 Projecting onto the Intersection of Two Hyperplanes

Let H be a n-dimensional Hilbert space with inner product $\langle \, , \, \rangle$. For each $a \in H$, $a \neq 0$, and b (scalar), let M be the subset of H defined by

$$M = \{ x \in H : \langle a, x \rangle = b \}. \tag{9}$$

The closet and convex set M is called hyperplane in H. For each $x \in H$, the projection onto M is given by

$$P_M(x) = x - \frac{b - \langle a, x \rangle}{\langle a, a \rangle} a.$$

If $b = 0$ in (9) then M is a closed subspace of dimension $n - 1$ and $M^\perp$ is a one-dimensional closed subspace. In that case, we can use the Theorem 5 for finding the best approximation to x onto the intersection of two hyperplanes.

COROLLARY 2. Let M_1 and M_2 be two hyperplanes in H, and $x_0 \in H$. Let $Q = P_{M_2} P_{M_1}$ be the composition of the P_{M_i}, $i = 1, 2$. Then

$$P_{M_1 \cap M_2}(x_0) = Q(x_0) + \left(\langle Q(x_0), Q(x_0) - Q^2(x_0) \rangle / \| Q(x_0) - Q^2(x_0) \|^2 \right) \left(Q^2(x_0) - Q(x_0) \right).$$

PROOF. This is a consequence of Theorem 5.

EXAMPLE 2. We consider the following two hyperplanes in $\mathbb{R}^3$:

$M_1 = \{(x_1, x_2, x_3) : x_1{+}2x_2{-}4x_3 = 0\}$ and $M_2 = \{(x_1, x_2, x_3) : 5x_1{+}11x_2{-}21x_3 = 0\}$.

We will use the Corollary 2 for finding the projection of $x_0 = (1, 1, 1)$ onto $M_1 \cap M_2$. After the first cycle of MAP, we have that

$$\begin{aligned} P_{M_1}(x_0) &= x_0 - \frac{\langle (1,2,-4), x_0\rangle}{\langle (1,2,-4),(1,2,-4)\rangle}(1,2,-4) \\ &= 1/21(20,\ 19,\ 25), \end{aligned}$$

and

$$\begin{aligned} Q(x_0) &= P_{M_2}P_{M_1}(x_0) \\ &= P_{M_1}(x_0) - \frac{\langle (5,11,-21), P_{M_1}(x_0)\rangle}{\langle (5,11,-21),(5,11,-21)\rangle}(5,11,-21) \\ &= 1/12327(12820,\ 13529,\ 10139). \end{aligned}$$

After the second cycle we obtain

$$\begin{aligned} P_{M_1}(Q(x_0)) &= P_{M_1}(x_2^1) \\ &= Q(x_0) - \frac{\langle (1,2,-4), Q(x_0)\rangle}{\langle (1,2,-4),(1,2,-4)\rangle}(1,2,-4) \\ &= 1/86289(89966,\ 95155,\ 70069), \end{aligned}$$

and

$$\begin{aligned} Q^2(x_0) &= P_{M_2}P_{M_1}(Q(x_0)) \\ &= P_{M_1}(Q(x_0)) - \frac{\langle (5,11,-21), P_{M_1}(Q(x_0))\rangle}{\langle (5,11,-21),(5,11,-21)\rangle}(5,11,-21) \\ &= 1/50651643(52684612,\ 55580039,\ 41657309). \end{aligned}$$

Then by Corollary 2 we have that

$$\begin{aligned} P_{M_1\cap M_2}(x_0) &= Q(x_0) + \left(\left\langle Q(x_0), Q(x_0) - Q^2(x_0)\right\rangle / \|Q(x_0) - Q^2(x_0)\|^2\right)\left(Q^2(x_0) - Q(x_0)\right) \\ &= (4/3,\ 2/3,\ 2/3). \end{aligned}$$

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