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A Note On k-Derangements*

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Abstract

Let $D_{k,n}$ denote the set of k-derangements in S_n. In this paper, we determine the maximum of $\Psi_\pi = \sum_{i=1}^{n} |\pi(i) - i|$, over all elements π of $D_{k,n}$. Moreover, the structure of $\pi \in D_{k,n}$ that maximizes Ψ_π is a particular bipartite graph.

1 Introduction

Suppose that S_n is the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$. Let $[n]^k$ $(1 \leq k \leq n)$ denote the set of all subsets containing k distinct elements of $[n]$. The group S_n acts in the natural way on $[n]^k$. In other words, for each $\pi \in S_n$,

$$\{i_1, \ldots, i_k\}^\pi = \{\pi(i_1), \ldots, \pi(i_k)\}.$$

A *k-derangement* in S_n is a permutation π on $[n]$ that leaves no k-subset of elements fixed. In other words, $x^\pi \neq x$ for all $x \in [n]^k$. Let $D_{k,n}$ denote the set of k-derangements of S_n. Specifically, if $k = 1$, then $D_n = D_{1,n}$ is the set of *derangements* in S_n, that is, the set of permutations in S_n without fixed points. Suppose that $\pi \in S_n$. Construct a bipartite graph $\Gamma_\pi = (X \cup Y, E)$, corresponding to π, where $X = \{x_i : i \in [n]\}$, $Y = \{y_i : i \in [n]\}$ and $E = \{(x_i, y_j) : i, j \in [n], \pi(i) = j\}$.
In the current paper, we measure how much derangement is actually disordered. For this, the following term is defined:

$$\Psi_\pi = \sum_{i=1}^{n} |\pi(i) - i|.$$

Then, let Ψ denote the maximum of Ψ_π over all elements π of $D_{k,n}$. Ψ is determined and it is shown that Ψ is independent of k.

The following proposition determines all permutations that belong to $D_{k,n}$.

PROPOSITION 1 ([2]). A permutation $\sigma \in S_n$ is a k-derangement if and only if the cycle decomposition of σ does not contain a set of cycles whose lengths partition k.

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2 Computing Ψ for k-Derangements

The main result of this paper is the following theorem.

THEOREM 1. Suppose k and n are integers and $1 \leq k < n$. Then Ψ is independent of k and we have

$$\Psi = \begin{cases} \frac{1}{2}n^2 & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2-1) & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. Let us first prove a reduced form of the theorem for $k = 1$. Consider the case that n is even, i.e., $n = 2m$ for an integer $m \geq 1$. Let $\pi \in D_{2m}$ be a derangement such that $\pi(i) > m$ if and only if $i \leq m$. We claim that Ψ_π is maximized over all elements π of D_{2m} and its value is equal to Ψ. Suppose by the contrary that Ψ_σ is maximized for some $\sigma \in D_{2m}$ such that Γ_σ contains the edge $(x_i, y_{\sigma(i)})$ where $\sigma(i) \leq m$ for some $i \leq m$. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ be a partition of vertices of Γ_σ such that $X_1 = \{x_1, \ldots, x_m\}$, $X_2 = \{x_{m+1}, \ldots, x_{2m}\}$, $Y_1 = \{y_1, \ldots, y_m\}$ and $Y_2 = \{y_{m+1}, \ldots, y_{2m}\}$. So $x_i \in X_1$ and there exists $j \in [m]$ such that $y_j \in Y_1$ and $\sigma(i) = j$. Since each vertex of Γ_σ has degree 1 and $|X_2| = |Y_1 \backslash \{y_j\}| + 1$, by the pigeonhole principle, there exists an edge from a vertex x_r in X_2 to a vertex y_s in Y_2. Now consider a new permutation π' such that $\pi'(i) = s, \pi'(r) = j$ and $\pi'(t) = \sigma(t)$ for other $t \neq i, r$. Since $1 \leq i, j \leq m$ and $m + 1 \leq r, s \leq 2m$, we have

$$|i - j| + |r - s| \leq |i - s| + |r - j|.$$

From the above inequality, it is easy to deduce that $\Psi_\sigma < \Psi_{\pi'}$, which is a contradiction. Now we show that the value of Ψ_π is constant for each $\pi \in D_{2m}$ such that $\pi(i) > m$ if and only if $i \leq m$. This condition implies that $\pi(i) - i$ is positive if $i \in [m]$, otherwise it is negative. Compute Ψ_π as

$$\begin{aligned} \Psi_\pi &= \sum_{i=1}^{m} |\pi(i) - i| + \sum_{i=m+1}^{2m} |\pi(i) - i| = \sum_{i=1}^{m} (\pi(i) - i) + \sum_{i=m+1}^{2m} (i - \pi(i)) \\ &= \sum_{i=1}^{m} \pi(i) + \sum_{i=m+1}^{2m} i - \sum_{i=1}^{m} i - \sum_{i=m+1}^{2m} \pi(i). \end{aligned}$$

By the structure of π, it is easy to see that

$$\sum_{i=1}^{m} \pi(i) = \sum_{i=m+1}^{2m} i \quad \text{and} \quad \sum_{i=m+1}^{2m} \pi(i) = \sum_{i=1}^{m} i.$$

Hence

$$\Psi_\pi = 2 \sum_{i=m+1}^{2m} i - 2 \sum_{i=1}^{m} i = 2m^2 = \frac{n^2}{2}.$$

This completes the proof in the case that n is even.

Consider the case that n is odd, i.e., $n = 2m + 1$ for an integer $m \geq 1$. Let $\pi \in D_{2m+1}$ be a derangement such that Γ_π has an edge (x_i, y_{2m+1}). Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ be a partition of vertices of Γ_π such that $X_1 = \{x_1, \ldots, x_{m+1}\}$, $X_2 = \{x_{m+2}, \ldots, x_{2m+1}\}$, $Y_1 = \{y_1, \ldots, y_m\}$ and $Y_2 = \{y_{m+1}, \ldots, y_{2m+1}\}$. Then vertex x_i must belong to X_1. Otherwise, by the pigeonhole principle, there exists an edge from a vertex x_r in X_1 to a vertex y_s in Y_1. For the same reason as above, we can also get a contradiction. Let $\Gamma_{\pi'}$ be the graph obtained from Γ_π by deleting the two vertices x_i and y_{2m+1}. Then $\Gamma_{\pi'}$ is a bipartite graph such that it has an even number of vertices on both sides. Clearly, Ψ_π is maximized if and only if $\Psi_{\pi'}$ is also. As in the even case, $\Psi_{\pi'}$ is maximized exactly when π' is a derangement such that $\pi'(i) > m$ if and only if $i \leq m + 1$. Obviously, $\Psi_\pi = \Psi_{\pi'} + 2m + 1 - i$. Moreover, the same of argument as in the even case we can show that $\Psi_{\pi'} = (n^2 - 1)/2 - n + i$. So $\Psi_\pi = (n^2 - 1)/2$ and its value is equal to Ψ. This completes the proof in the case that n is odd.

Now let k be an arbitrary positive integer. Let

$$\pi = (1\ 2n\ 2\ 2n-1\ \ldots\ n-1\ n+2\ n\ n+1)$$

be a cyclic permutation of length $2n$. In other words, the permutation $\pi \in S_{2n}$ is representing the following mapping:

$$\pi(i) = \begin{cases} 2n + 1 - i & \text{for } 1 \leq i \leq n, \\ 1 & \text{for } i = n + 1, \\ 2n + 2 - i & \text{for } n + 1 < i \leq 2n. \end{cases}$$

Since the cycle structure of the permutation π is one cycle of length $2n$, so by Proposition 1, π is a k-derangement for $k \in [2n - 1]$. It is easy to see that $\pi(i) > n$ if and only if $i \leq n$. Similarly, if

$$\pi(i) = \begin{cases} n + i & \text{for } 1 \leq i \leq n + 1, \\ i - n - 1 & \text{for } n + 1 < i \leq 2n + 1, \end{cases}$$

then $\pi = (1\ n+1\ 2n+1\ n\ \ldots\ 3\ n+3\ 2\ n+2)$ is a cyclic permutation of length $2n+1$ in S_{2n+1}. So by Proposition 1, π is a k-derangement permutation for $k \in [2n]$. Also, Ψ_π satisfies the maximum condition in the odd case. Hence, this completes the proof of the theorem.

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