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An Existence Result For p-Kirchhoff-Type Problems With Singular Nonlinearity*

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Abstract

This work is devoted to study the existence of positive solutions for a class of p-Kirchhoff-type problems with singular nonlinearity. Our approach relies on the variational method and some analysis techniques.

1 Introduction

Consider the p-Kirchhoff equation

$$\begin{cases} -\Big(a+b\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = m(x)u^{-\gamma} - \lambda u^q, & x\in\Omega, \\ u>0, & x\in\Omega, \\ u=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$($N\geq 3$), $\Delta_p u = div\,(|\nabla u|^{p-2}\nabla u)$, for $1<p<N$ denotes the p-Laplacian operator, and $\lambda>0$ is a real parameter. Here $\gamma\in(0,1)$ is a constant, $0<q<p^*-1$, $a,b\geq 0$, $a+b>0$ are parameters. The weight function $m:\Omega\to\mathbb{R}$ is in $L^{\frac{p^*}{p^*+\gamma-1}}$ with $m(x)>0$ for almost every $x\in\Omega$, and $p^*=\frac{pN}{N-p}$ denote the critical Sobolev exponent.

Problem (1) is called nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^p dx)\Delta_p u$ which implies that the equation in (1) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which makes the study of such a class of problems particularly interesting. Besides, such a problem has physical motivation. Moreover, problem (1) is related to the stationary version of *Kirchhoff* equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

presented by Kirchhoff [8]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in Eq. (2) have the following meanings: L is the length

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of the string, h is the area of cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert's equation. Later on, Woinowsky-Krieger (Nash-Modeer) incorporated this correction in the classical Euler–Bernoulli equation for the beam (plate) with hinged ends. See, for example, [4] and [5] and the references therein.

Moreover, nonlocal problems also appear in other fields as, for example, biological systems where (u, v) describes a process which depends on the average of itself (for instance, population density). See, for example, [1, 2, 7, 14, 15] and the references therein.

In the recent decades, the Kirchhoff type problems have been extensively investigated, and a lot of classical results have been obtained on a bounded domain or unbounded domain, see for example [3, 9, 10, 11, 13]. See [12] where the authors discussed the problem (1) when $p = 2$. Here we focus on extending the study in [12]. In fact this paper is motivated, in part, by the mathematical difficulty posed by the degenerate quasilinear elliptic operator compared to the Laplacian operator ($p = 2$). This extension is nontrivial and requires more careful analysis of the nonlinearity. Combining constraint variational method we prove that the problem possesses a global minimizer solution.

The plan of the paper is as follows. Section 2 is devoted to our variational setting and our main result. Section 3 is dedicated to the proof of main result.

2 Variational Setting

Let $W = W_0^{1,p}(\Omega)$ be the usual Sobolev space, equipped with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}}$$

and $|u|_\sigma = \Big(\int_\Omega |u|^\sigma dx\Big)^{\frac{1}{\sigma}}$ denotes the norm in $L^\sigma(\Omega)$.

A function $u \in W$ is said to be a weak solution of problem (1) if $u > 0$ in Ω and there holds

$$\Big(a + b\int_\Omega |\nabla u|^p dx\Big)\int_\Omega |\nabla u|^{p-2}\nabla u\,\nabla\varphi dx + \lambda\int_\Omega u^q\varphi dx - \int_\Omega m(x)u^{-\gamma}\varphi dx = 0,$$

for all $\varphi \in W$. We shall look for (weak) solutions of (1) by finding critical points of the energy functional $J_b : W \to \mathbb{R}$ given by

$$\begin{aligned} J_\lambda(u) \quad : \quad &= \frac{a}{p}\Big(\int_\Omega |\nabla u|^p dx\Big) + \frac{b}{2p}\Big(\int_\Omega |\nabla u|^p dx\Big)^2 + \frac{\lambda}{1+q}\int_\Omega |u|^{1+q} dx \\ &-\frac{1}{1-\gamma}\int_\Omega m(x)|u|^{1-\gamma} dx, \end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$. By analyzing the associated minimization problems for the functional J_λ, one can study weak solutions for (1). As we known, the functional J_λ fails to be Fréchet differentiable because of the singular term, then we cannot apply the critical point theory to obtain the existence of solutions directly.

Throughout this paper, we make the following assumptions:

(H1) $0 < \gamma < 1$, $0 < q \leq p^* - 1$.

(H2) $m \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ with $m(x) > 0$ for almost every $x \in \Omega$.

LEMMA 2.1. The energy functional J_λ has a minimum c in W with $c < 0$.

PROOF. Since $0 < \gamma < 1$, $\lambda \geq 0$, by the Hölder inequality, we have

$$\begin{aligned}\int_\Omega m(x)|u|^{1-\gamma}dx &\leq \left(\int_\Omega |m(x)|^{\frac{p^*}{p^*-1+\gamma}}dx\right)^{\frac{p^*-1+\gamma}{p^*}}\left(\int_\Omega |u|^{(1-\gamma)(\frac{p^*}{1-\gamma})}dx\right)^{\frac{1-\gamma}{p^*}}\\ &= |m|_{\frac{p^*}{p^*-1+\gamma}}|u|_{p^*}^{1-\gamma}.\end{aligned}$$

Furthermore, by the Sobolev embedding theorem, we obtain that

$$\frac{1}{1-\gamma}|u|_{p^*}^{1-\gamma} \leq C\|u\|^{1-\gamma}.$$

Hence

$$\begin{aligned}J_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{\lambda}{1+q}\int_\Omega |u|^{1+q}dx - \frac{1}{1-\gamma}\int_\Omega m(x)|u|^{1-\gamma}dx\\ &\geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - C|m|_{\frac{p^*}{p^*-1+\gamma}}\|u\|^{1-\gamma},\end{aligned} \tag{3}$$

where $C > 0$ is a constant. This implies that I is coercive and bounded from below on W. Then $c = \inf_{u \in W} J_\lambda(u)$ is well defined. Moreover, since $0 < \gamma < 1$ and $m(x) > 0$ for almost every $x \in \Omega$, we have $J_\lambda(t\delta) < 0$ for all $\delta \neq 0$ and small $t > 0$. Thus, we obtain $c = \inf_{u \in W} J_\lambda(u) < 0$. The proof is complete.

Now, we can state our main result.

THEOREM 2.2. Assume that conditions (H1) and (H2) hold. Then problem (1) possesses a positive solution. Moreover, this solution is a global minimizer solution.

3 Proof of Theorem 2.2

The validity of the next lemma will be crucial in the sequel.

LEMMA 3.1. Assume that conditions (H1) and (H2) hold. Then J_λ attains the global minimizer in W, that is, there exists $u_* \in W$ such that $J_\lambda(u_*) = c < 0$.

PROOF. From Lemma 2.1, there exists a minimizing sequence $\{u_n\} \subset W$ such that $\lim_{n\to\infty} J_\lambda(u_n) = c < 0$. Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, we may assume that $u_n \geq 0$ for almost every $x \in \Omega$. By (3), the sequence $\{u_n\}$ is bounded in W. Since W is reflexive, we may extract a subsequence that for simplicity we call again $\{u_n\}$, there exists $u_* \geq 0$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u_*, & \text{strongly in } L^s(\Omega), \quad 1 \leq s < p^*, \\ u_n(x) \to u_*(x), & \text{a.e. in } \Omega, \end{cases} \tag{4}$$

as $n \to \infty$. As usual, letting $w_n = u_n - u_*$, we need to prove that $\|w_n\| \to 0$ as $n \to \infty$.

By Vitali's theorem [16], we find

$$\lim_{n\to\infty} \int_\Omega m(x)|u_n|^{1-\gamma}dx = \int_\Omega m(x)|u_*|^{1-\gamma}dx. \tag{5}$$

Moreover, by the weak convergence of $\{u_n\}$ in W and Brézis-Lieb's Lemma (see [6]), one obtains

$$\|u_n\|^p = \|w_n\|^p + \|u_*\|^p + o(1), \tag{6}$$

$$\|u_n\|^{2p} = \|w_n\|^{2p} + \|u_*\|^{2p} + 2\|w_n\|^p\|u_*\|^p + o(1), \tag{7}$$

and

$$\int_\Omega |u_n|^{p^*}dx = \int_\Omega |w_n|^{p^*}dx + \int_\Omega |u_*|^{p^*}dx + o(1), \tag{8}$$

where $o(1)$ is an infinitesimal as $n \to \infty$. Hence, in the case that $0 < q < p^* - 1$, from (4)–(7), we deduce that

$$\begin{aligned} c &= \lim_{n\to\infty} J_\lambda(u_n) \\ &= \lim_{n\to\infty} \Big(\frac{a}{p}\|u_n\|^p + \frac{b}{2p}\|u_n\|^{2p} + \frac{\lambda}{1+q}\int_\Omega |u_n|^{1+q}dx - \frac{1}{1-\gamma}\int_\Omega m(x)|u_n|^{1-\gamma}dx\Big) \\ &= \lim_{n\to\infty} \Big(\frac{a}{p}(\|w_n\|^p + \|u_*\|^p) + \frac{b}{2p}(\|w_n\|^{2p} + \|u_*\|^{2p} + 2\|w_n\|^p\|u_*\|^p) \\ &\quad + \frac{\lambda}{1+q}\int_\Omega |u_*|^{1+q}dx - \frac{1}{1-\gamma}\int_\Omega m(x)|u_*|^{1-\gamma}dx\Big) \\ &= \frac{a}{p}\|u_*\|^p + \frac{b}{2p}\|u_*\|^{2p} + \frac{\lambda}{1+q}\int_\Omega |u_*|^{1+q}dx - \frac{1}{1-\gamma}\int_\Omega m(x)|u_*|^{1-\gamma}dx \\ &\quad + \lim_{n\to\infty}\Big(\frac{a}{p}\|w_n\|^p + \frac{b}{2p}\|w_n\|^{2p} + \frac{b}{p}\|w_n\|^p\|u_*\|^p\Big) \\ &= J_\lambda(u_*) + \lim_{n\to\infty}\Big(\frac{a}{p}\|w_n\|^p + \frac{b}{2p}\|w_n\|^{2p} + \frac{b}{p}\|w_n\|^p\|u_*\|^p\Big) \\ &\geq J_\lambda(u_*) \geq \inf_{u_n \in W} J_\lambda(u_n) = c, \end{aligned}$$

which implies $J_\lambda(u_*) = c$. In the case that $q = p^* - 1$, it follows from (5)–(8) that

$$c = J_\lambda(u_*) + \lim_{n\to\infty}\Big(\frac{a}{p}\|w_n\|^p + \frac{b}{2p}\|w_n\|^{2p} + \frac{b}{p}\|w_n\|^p\|u_*\|^p + \frac{\lambda}{p^*}|w_n|_{p^*}^{p^*}\Big) \geq J_\lambda(u_*) \geq c,$$

which yields $J_\lambda(u_*) = c$. Thus $\inf_{u_n \in W} J_\lambda(u_n) = J_\lambda(u_*)$ and this completes the proof of Lemma 3.1. The proof is complete.

We are now in a position to prove Theorem 2.2.

PROOF OF THEOREM 2.2. We only need to prove that u_* is a weak solution of (1) and $u_* > 0$ in Ω. Firstly, we shaw that u_* is a weak solution of (1). From Lemma 2.1, we see that

$$\min J_\lambda(u_* + t\varphi) = J_\lambda(u_* + t\varphi)|_{t=0} = J_\lambda(u_*), \ \forall \varphi \in W.$$

This implies that

$$\Big(a + b\int_\Omega |\nabla u_*|^p dx\Big) \int_\Omega |\nabla u_*|^{p-2} \nabla u_* \nabla \varphi dx + \lambda \int_\Omega u_*^q \varphi dx - \int_\Omega m(x) u_*^{-\gamma} \varphi dx = 0, \quad (9)$$

for all $\varphi \in W$. Thus, u_* is a weak solution of (1).

Secondly, we prove that $u_* > 0$ for almost every $x \in \Omega$. Since $J_\lambda(u_*) = c < 0$, we obtain $u_* \geq 0$ and $u_* \not\equiv 0$. Then, $\forall \phi \in W$, $\phi \geq 0$ and $t > 0$, we have

$$\begin{aligned}
0 \ \leq \ & \frac{J_\lambda(u_* + t\phi) - J_\lambda(u_*)}{t} \\
= \ & \frac{a}{p}\Big[p\int_\Omega |\nabla u_*|^{p-2} \nabla u_* \nabla \phi \, dx + t(p-1)\int_\Omega |\nabla u_*|^{p-2} \, |\nabla \phi|^2 \, dx \\
& +\ldots + pt^{p-2}\int_\Omega \nabla u_* |\nabla \phi|^{p-1} \, dx\Big] + \frac{a}{p}\, t^{p-1}\int_\Omega |\nabla \phi|^p dx \\
& + \frac{b}{2p}(\int_\Omega |\nabla u_*|^p dx)\Big[2p(\int_\Omega |\nabla u_*|^{p-2} \nabla u_*, \nabla \phi \, dx) \\
& +2(p-1)t\int_\Omega |\nabla u_*|^{p-2}, |\nabla \phi|^2 \, dx + \ldots + 2pt^{p-2}\int_\Omega \nabla u_* |\nabla \phi|^{p-1} \, dx\Big] \\
& + \frac{b}{p} t^{p-1}(\int_\Omega |\nabla u_*|^p dx)(\int_\Omega |\nabla \phi|^p dx) + \frac{b}{2p}\,(\Big[pt\int_\Omega |\nabla u_*|^{p-2} \nabla u_* \nabla \phi \, dx \\
& +t^2(p-1)\int_\Omega |\nabla u_*|^{p-2} |\nabla \phi|^2 \, dx + \ldots + pt^{2p-3)}\int_\Omega \nabla u_* |\nabla \phi|^{p-1} \, dx\Big])^2 \\
& + \frac{b}{2p} t^{p-1}\int_\Omega |\nabla \phi|^p dx\,(\Big[2p\int_\Omega |\nabla u_*|^{p-2} \nabla u_* \nabla \phi \, dx \\
& +t2(p-1)\int_\Omega |\nabla u_*|^{p-2} |\nabla \phi|^2 \, dx + \ldots + 2pt^{p-2}\int_\Omega \nabla u_* |\nabla \phi|^{p-1} \, dx\Big]) \\
& +bt^{2p-1}(\int_\Omega |\nabla \phi|^p dx)^2 + \frac{\lambda}{1+q}\int_\Omega \frac{(u_* + t\phi)^{1+q} - u_*^{1+q}}{t} dx \\
& - \frac{1}{1-\gamma}\int_\Omega m(x) \frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t} dx. \qquad (10)
\end{aligned}$$

Using the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{t \to 0^+} \frac{1}{1+q}\int_\Omega \frac{(u_* + t\phi)^{1+q} - u_*^{1+q}}{t} dx = \int_\Omega u_*^q \phi dx. \quad (11)$$

For any $x \in \Omega$, we denote

$$g(t) = m(x)\frac{[u_*(x) + t\phi(x)]^{1-\gamma} - u_*^{1-\gamma}(x)}{(1-\gamma)t}.$$

Then

$$g'(t) = m(x)\frac{u_*(x)^{1-\gamma} - [\gamma t\phi(x) + u_*(x)]\,[u_*(x) + t\phi(x)]^{-\gamma}}{t^2(1-\gamma)} \leq 0,$$

which implies that $g(t)$ is non increasing for $t > 0$. Moreover, we have

$$\lim_{t\to 0^+} g(t) = ([u_*(x) + t\phi(x)]^{1-\gamma})' \mid_{t=0} = m(x)u_*^{-\gamma}(x)\phi(x),$$

for every $x \in \Omega$, which may be $+\infty$ when $u_*(x) = 0$ and $\phi(x) > 0$. Consequently, by the Monotone Convergence Theorem, we obtain

$$\lim_{t\to 0^+} \frac{1}{1-\gamma}\int_\Omega m(x)\frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t}dx = \int_\Omega m(x)u_*^{-\gamma}\phi dx,$$

which may equal to $+\infty$. Combining this with (11), let $t \to 0^+$, it follows from (10) that

$$\begin{aligned}0 &\leq a(\int_\Omega |\nabla u_*|^{p-2}\nabla u_*\nabla\phi dx) + b\Big((\int_\Omega (|\nabla u_*|^p dx)(\int_\Omega |\nabla u_*|^{p-2}\nabla u_*\nabla\phi dx)\Big)\\ &+ \lambda\int_\Omega u_*^q\phi dx - \int_\Omega m(x)u_*^{-\gamma}\phi dx.\end{aligned}$$

Then, we have

$$\int_\Omega m(x)u_*^{-\gamma}\phi dx \leq (a + b\|u_*\|^p(\int_\Omega |\nabla u_*|^{p-2}\nabla u_*\nabla\phi dx) + \lambda\int_\Omega u_*^q\phi dx,$$

for all $\phi \in W$ with $\phi > 0$. Let $e_1 \in W$ be the first eigenfunction of the operator $-\Delta_p$ with $e_1 > 0$ and $\|e_1\| = 1$. Particularly, taking $\phi = e_1$ in (9), one gets

$$\begin{aligned}\int_\Omega m(x)u_*^{-\gamma}e_1 dx &\leq (a + b\|u_*\|^p)(\int_\Omega |\nabla u_*|^{p-2}\nabla u_*\nabla e_1 dx) + \lambda\int_\Omega u_*^q e_1 dx\\ &\leq (a + b\|u_*\|^p)(\int_\Omega |\nabla u_*|^{p-1}\nabla e_1 dx) + \lambda\int_\Omega u_*^q e_1 dx\\ &\leq (a + b\|u_*\|^p)\Big[(\int_\Omega |\nabla u_*|^{(p-1)(\frac{p}{p-1})}dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla e_1|^p dx)^{\frac{1}{p}}\Big]\\ &+ \lambda(\int_\Omega |u_*|^{q(\frac{p}{p-1})}dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla e_1|^p dx)^{\frac{1}{p}}\\ &\leq (a + b\|u_*\|^p)(\|u_*\|^{p-1})(\|e_1\|) + \lambda|u_*|_{\frac{p}{p-1}}\|e_1\|\\ &< \infty,\end{aligned}$$

which implies that $u_* > 0$ for almost every $x \in \Omega$. Moreover, according to Lemma 2.1, we have $J_\lambda(u_*) = \inf_{u\in W} J_\lambda(u)$. Thus u_* is a global minimizer solution. That completes the proof of Theorem 2.1.

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