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On New Identities For Mersenne Numbers*

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Abstract

Some families of Toeplitz-Hessenberg determinants and permanents, the entries of which are the Mersenne numbers, are under consideration in this paper. In the course of these studies new identities for the Mersenne numbers have been discovered.

1 Introduction

A *Mersenne number*, denoted by M_n, is a number of the form $M_n = 2^n - 1$, where n is a nonnegative number. The *Mersenne sequence* $(M_n)_{n\geq 0}$ can be defined recursively as follows [3]:

$$M_0 = 0,\ M_1 = 1, \quad M_n = 3M_{n-1} - 2M_{n-2} \ \text{ for } n \geq 2. \tag{1}$$

The first few terms of the Mersenne sequence (sequence A000225 from [17]) are

$$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, \dots .$$

The Mersenne numbers are studied in a lot of courses devoted to the elementary number theory [1, 6, 8, 11, 16, 18].

A simple calculation shows that if M_n is a prime number, then n is a prime number, though not all M_n are prime. When M_n is a prime number it is called a *Mersenne prime*. The search for Mersenne primes is an active issue in the number theory, combinatorics, computer science, coding theory (see, for example, [4, 5, 14, 19], among others). Mersenne primes lead to prime repunits, meaning repeated units, i.e., numbers consisting exclusively of 1's in any given base system (note that the Mersenne primes are repunits in the binary number system). Mersenne primes are also noteworthy due to their connection with perfect numbers, i.e., the numbers that are equal to the sum of their proper divisors.

In the mathematical problem Tower of Hanoi, solving a puzzle with an n-disc tower requires taking M_n steps, assuming no mistakes have been made [9].

As examples of recent works involving the Mersenne numbers and its generalizations, see, for example, [2, 3, 7, 12, 15, 20]. For instance, Catarino et al. [3] established some new identities for the common factors of Mersenne numbers and Jacobsthal

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and Jacobsthal-Lucas numbers, and presented some results with matrices involving Mersenne numbers such as the generating matrix, tridiagonal matrices and circulant matrices. Koshy and Gao [12] investigated some divisibility properties of the Catalan numbers with Mersenne numbers M_n as their subscripts. Bravo and Gómes [2] found all k-Fibonacci numbers which are Mersenne numbers, i.e., k-Fibonacci numbers that are equal to 1 but less than a power of 2. For more information on classical and alternative approaches to the Mersenne numbers see [10].

2 Toeplitz-Hessenberg Matrices and Related Formulae

A *Toeplitz-Hessenberg matrix* is an $n \times n$ matrix of the form

$$A_n(a_0, a_1, \ldots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}, \tag{2}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

This class of matrices is encountered in various applications (see [13] and the references given there).

We expand the determinant $\det(A_n)$ and permanent $\operatorname{per}(A_n)$ according to the first row repeatedly. Then we obtain the following recurrent formulae for determinants and permanents of the Toeplitz-Hessenberg matrix (2):

$$\det(A_n) = \sum_{k=1}^{n} (-a_0)^{k-1} a_k \det(A_{n-k}), \tag{3}$$

$$\operatorname{per}(A_n) = \sum_{k=1}^{n} a_0^{k-1} a_k \operatorname{per}(A_{n-k}), \tag{4}$$

where, by definition, $\det(A_0) = 1$ and $\operatorname{per}(A_0) = 1$.

The following results are known as Trudi's formulae [13].

THEOREM 1. Let n be a positive integer and A_n be the matrix defined in (1). Then

$$\det(A_n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-a_0)^{n-(t_1+t_2+\cdots+t_n)} p_n(t)\, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}, \tag{5}$$

$$\operatorname{per}(A_n) = \sum_{t_1+2t_2+\cdots+nt_n=n} a_0^{n-(t_1+t_2+\cdots+t_n)} p_n(t)\, a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}, \tag{6}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + nt_n = n$, and

$$p_n(t) = \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1! t_2! \cdots t_n!}$$

is the multinomial coefficient.

3 Determinants and Permanents of the Toeplitz-Hessenberg Matrices Whose Entries Are Mersenne Numbers

Denote

$$\det(a_1, a_2, \ldots, a_n) = \det\left(A_n(1, a_1, a_2, \ldots, a_n)\right),$$

$$\operatorname{per}(a_1, a_2, \ldots, a_n) = \operatorname{per}\left(A_n(1, a_1, a_2, \ldots, a_n)\right).$$

PROPOSITION 1. Let M_n be the Mersenne number. For all $n \geq 1$, the following formulae hold:

$$\det(M_1, M_2, \ldots, M_n) = \frac{i}{2}\big((1+i)^n - (1-i)^n\big), \tag{7}$$

$$\operatorname{per}(M_1, M_2, \ldots, M_n) = \frac{\sqrt{2}}{4}\left((2+\sqrt{2})^n - (2-\sqrt{2})^n\right), \tag{8}$$

where $i = \sqrt{-1}$.

PROOF. We prove only formula (8), because the proof of (7) is similar. For simplicity of notation, we write P_n instead $\operatorname{per}(M_1, M_2, \ldots, M_n)$. Thus, we must prove

$$P_n = \operatorname{per}\begin{bmatrix} M_1 & 1 & \cdots & 0 & 0 \\ M_2 & M_1 & \cdots & 0 & 0 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ M_{n-1} & M_{n-2} & \cdots & M_1 & 1 \\ M_n & M_{n-1} & \cdots & M_2 & M_1 \end{bmatrix} = \frac{\sqrt{2}}{4}\left((2+\sqrt{2})^n - (2-\sqrt{2})^n\right).$$

We use induction on n. For $n = 1$ formula (8) holds. Suppose that assertion holds for all $k \leq n-1$ and proof its validity for n. From (4), using (1), we have

$$\begin{aligned} P_n &= \sum_{k=1}^{n} M_k P_{n-k} = M_1 P_{n-1} + \sum_{k=2}^{n} (3M_{k-1} - 2M_{k-2}) P_{n-k} \\ &= P_{n-1} + 3\sum_{k=2}^{n} M_{k-1} P_{n-k} - 2\sum_{k=3}^{n} M_{k-2} P_{n-k} \\ &= P_{n-1} + 3\sum_{k=1}^{n-1} M_k P_{n-k-1} - 2\sum_{k=1}^{n-2} M_k P_{n-k-2} \\ &= P_{n-1} + 3P_{n-1} - 2P_{n-2} = 4P_{n-1} - 2P_{n-2} \\ &= 4\cdot\frac{\sqrt{2}}{4}\left((2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}\right) - 2\cdot\frac{\sqrt{2}}{4}\left((2+\sqrt{2})^{n-2} - (2-\sqrt{2})^{n-2}\right) \\ &= \frac{\sqrt{2}}{4}\left((2+\sqrt{2})^n - (2-\sqrt{2})^n\right). \end{aligned}$$

Therefore, (8) holds for all positive integers.

We also derive several other formulae for determinants and permanents of Toeplitz-Hessenberg matrix whose entries are the sequential Mersenne numbers and Mersenne numbers with even or odd subscripts.

PROPOSITION 2. Let M_n be the Mersenne number. For all $n \geq 1$, the following formulae hold:

$$\det(M_2, M_4, \ldots, M_{2n}) = \frac{\sqrt{3}}{4}\left((\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}\right), \tag{9}$$

$$\operatorname{per}(M_2, M_4, \ldots, M_{2n}) = \begin{cases} 0 & \text{if } n = 3k, \\ -6 \cdot 8^{n-1} & \text{if } n = 3k-1, \\ \dfrac{3 \cdot 8^n + 5 \cdot \delta_{n1}}{8} & \text{if } n = 3k-2, \end{cases} \tag{10}$$

$$\det(M_3, M_5, \ldots, M_{2n+1}) = 9 \cdot 2^{n-1} - 2 \cdot \delta_{n1}, \tag{11}$$

$$\det(M_2, M_3, \ldots, M_{n+1}) = \begin{cases} 4 - n & \text{if } n = 1 \text{ and } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \tag{12}$$

where δ_{n1} is the Kronecker symbol.

PROOF. We prove only (11). About the other identities in this proposition, we have decided to omit its proof here because they can be easily proved. To simplify the notation, we write D_n instead $\det(M_3, M_5, \ldots, M_{2n+1})$. Thus, we must prove

$$D_n = \det \begin{bmatrix} M_3 & 1 & \cdots & 0 & 0 \\ M_5 & M_3 & \cdots & 0 & 0 \\ \ldots & \ldots & \ddots & \ldots & \ldots \\ M_{2n-1} & M_{2n-3} & \cdots & M_3 & 1 \\ M_{2n+1} & M_{2n-1} & \cdots & M_5 & M_3 \end{bmatrix} = \frac{9}{2} \cdot 2^n - 2 \cdot \delta_{n1}.$$

We use induction on n. For $n = 1$ the identity is trivial. Assuming (11) to hold for $k \leq n-1$, we proved it for n. Using (3) and (1), we get

$$\begin{aligned} D_n &= \sum_{k=1}^{n} (-1)^{k-1} M_{2k+1} D_{n-k} \\ &= \frac{9}{2} \cdot 2^n \left(2 \sum_{k=1}^{n-2} (-1)^{k-1} 2^k - \sum_{k=1}^{n-2} (-1)^{k-1} 2^{-k} \right) + \frac{3}{2}(-1)^n 2^{2n} - 6(-1)^n \\ &= \frac{9}{2} \cdot 2^n \left(4 \cdot \frac{1 - (-2)^{n-2}}{1+2} - \frac{1}{2} \cdot \frac{1 - \left(-\frac{1}{2}\right)^{n-2}}{1 + \frac{1}{2}} \right) + \frac{3}{2}(-1)^n 2^{2n} - 6(-1)^n \\ &= \frac{9}{2} \cdot 2^n \left(1 - \frac{1}{3}(-1)^n 2^n + \frac{4}{3}(-1)^n 2^{-n} \right) + \frac{3}{2}(-1)^n 2^{2n} - 6(-1)^n \\ &= \frac{9}{2} \cdot 2^n - \frac{3}{2}(-1)^n 2^{2n} + 6(-1)^n + \frac{3}{2}(-1)^n 2^{2n} - 6(-1)^n = 9 \cdot 2^{n-1}. \end{aligned}$$

Therefore, (11) holds for all positive integers.

4 Main Formulae

Using (5) for determinants in (7), (9), (11), (12), and using (6) for permanents in (8), (10), we obtain the following Mersenne identities.

PROPOSITION 3. Let M_n be the Mersenne number. For all $n \geq 1$, the following formulae hold:

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n-T} p_n(t) M_1^{t_1} M_2^{t_2} \cdots M_n^{t_n} = \frac{i}{2}\left((1+i)^n - (1-i)^n\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) M_1^{t_1} M_2^{t_2} \cdots M_n^{t_n} = \frac{\sqrt{2}}{4}\left((2+\sqrt{2})^n - (2-\sqrt{2})^n\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n-T} p_n(t) M_2^{t_1} M_4^{t_2} \cdots M_{2n}^{t_n} = \frac{\sqrt{3}}{4}\left((\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) M_2^{t_1} M_4^{t_2} \cdots M_{2n}^{t_n} = \begin{cases} 0 & \text{if } n = 3k, \\ -6 \cdot 8^{n-1} & \text{if } n = 3k-1, \\ \dfrac{3 \cdot 8^n + 5 \cdot \delta_{n1}}{8} & \text{if } n = 3k-2, \end{cases}$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n-T} p_n(t) M_3^{t_1} M_5^{t_2} \cdots M_{2n+1}^{t_n} = 9 \cdot 2^{n-1} - 2 \cdot \delta_{n1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) M_2^{t_1} M_3^{t_2} \cdots M_{n+1}^{t_n} = \begin{cases} 4-n & \text{if } n = 1 \text{ and } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases}$$

where the summation is over nonnegative integers satisfying

$$t_1 + 2t_2 + \cdots + nt_n = n,\ T = t_1 + t_2 + \cdots + t_n,\ p_n(t) = \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1! t_2! \cdots t_n!},$$

and δ_{n1} is the Kronecker symbol.

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