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A Fixed Point Approach To The Mittag-Leffler-Hyers-Ulam Stability Of Differential Equations $y'(x) = F(x, y(x))$*

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Abstract

Using a fixed point method, we prove the Mittag-Leffler-Hyers-Ulam stability for the differential equation of the form $y'(x) = F(x, y(x))$.

1 Introduction

Let Y be a normed space and let I be a bounded interval. Assume that for any function $f : I \longrightarrow Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + ... + a_1(x)y'(x) + a_0(x)y(x) + f(x) \right\| \leq \varepsilon$$

for all $x \in I$ and for some $\varepsilon \geq 0$. In this case there exists a solution $f_0 : I \longrightarrow Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + ... + a_1(x)y'(x) + a_0(x)y(x) + f_0(x) = 0$$

such that $\|f(x) - f_0(x)\| \leq K\varepsilon$ for any $x \in I$. Then , we say that the above differential equation has the Hyers-Ulam stability. For more study about functional equations the readers can see [9].

If the above statement is also true when we replace ε and $K\varepsilon$ by $\varphi(x)$ and $\Phi(x)$, where $\varphi, \Phi : I \longrightarrow [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

We may apply these terminologies for other differential equations. For more detailed definitions of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability, refer to [4, 5]. It seems that Obloza is the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [13, 14]. Here, we will introduce a result

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of Alsina and Ger (see [2]): If a differentiable function $f : I \longrightarrow \mathbb{R}$ is a solution of the differential inequality $|y'(x) - y(x)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \longrightarrow \mathbb{R}$ of the differential equation $y'(x) = y(x)$ such that $|f(x) - f_0(x)| \leq 3\varepsilon$ for any $x \in I$.

This result of Alsina and Ger has been generalized by Takahasi, Miura and Miyajima. They proved in [15] that the Hyers-Ulam stability holds for the differential equation $y'(x) = \lambda y(x)$ in Banach spaces (see also [10]).

Recently, Miura, Miyajima and Takahasi also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(x) + g(x)y(x) = 0$, where $g(x)$ is a continuous function, while the author proved the Hyers-Ulam stability of linear differential equations of other type (see [1, 6, 7, 8, 11, 12]).

In this paper, for a bounded and continuous function $F(x, y)$ we will adopt the idea of [3] and prove the Hyers-Ulam-Rassias stability as well as the Hyers-Ulam stability of the differential equations of the form

$$y'(x) = F(x, y(x)). \tag{1}$$

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \longrightarrow [0, +\infty]$ is called a generalized metric on X if and only if it satisfies

(A_1) $d(x, y) = 0$ if and only if $x = y$;

(A_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(A_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The above concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space.

We now introduce one of the fundamental results of Banach fixed point theorem in a generalized complete metric space.

THEOREM 1. Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \longrightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer K such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the following are true:

(a) The sequence $\Lambda^n x$ convergences to a fixed point x^* of Λ;

(b) x^* is the unique fixed point of Λ in

$$X^* = \{y \in X : d(\Lambda^k x, y) < \infty\}$$

(c) If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.

3 Mittag-Leffler-Hyers-Ulam Stability

Here, we prove the Mittag-Leffler-Hyers-Ulam Stability for the differential equation $y'(x) = F(x, y(x))$.

THEOREM 2. Given $c \in \mathbb{R}$ and $r > 0$. Let I denote a closed ball of radius r and center at c, that is,

$$I = \{x \in \mathbb{R} \mid c - r \leq x \leq c + r\}$$

and let $F : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function which satisfies a Lipschits condition

$$|F(x, y) - F(x, z)| \leq L|y - z| \tag{2}$$

for all $x \in I$ and $y, z \in \mathbb{R}$ where L is a constant with $0 < Lr < 1$. If a continuously differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies the differential inequality

$$|y'(x) - F(x, y(x))| \leq \varepsilon E_q(x^q) \tag{3}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, where E_q is a Mittag function, then there exists a unique continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x F(\tau, y_0(\tau))d\tau. \tag{4}$$

Furthermore, y_0 is a solution of (1) and

$$|y(x) - y_0(x)| \leq \frac{\varepsilon E_q(x^q)}{1 - Lr} \tag{5}$$

for any $x \in I$.

PROOF. First, we define a set X of all continuous functions $f : I \longrightarrow \mathbb{R}$ by

$$X = \{f : I \longrightarrow \mathbb{R} : f \text{ is a continuous function}\} \tag{6}$$

and introduce a generalized metric on X as follows:

$$d(f, g) = \inf\{c \in [0, \infty] : |f(x) - g(x)| \leq c \text{ for all } x \in I\}. \tag{7}$$

It is obvious that (X, d) is a generalized complete metric space. We define an operator $\Lambda : X \longrightarrow X$ by

$$(\Lambda f)(x) = y(c) + \int_c^x F(\tau, f(\tau))d\tau, \tag{8}$$

for any $x \in I$ and $f \in X$. (It is true that $\Lambda f \in X$, because Λf is continuously differentiable in view of the Fundamental Theorem of calculus. We now assert that Λ is strictly contractive on X.

For $f, g \in X$, let $c_{fg} \in [0, \infty]$ be an arbitrary constant with $d(f, g) \leq c_{fg}$, that is, we assume that

$$|f(x) - g(x)| \leq c_{fg} \tag{9}$$

for all $x \in I$. Moreover, it follows from (2), (8) and (9) that

$$\begin{aligned} |(\Lambda f)(x) - (\Lambda g)(x)| &= \left| \int_c^x \{F(\tau, f(\tau)) - F(\tau, g(\tau))\} d\tau \right| \\ &\leq \left| \int_c^x |F(\tau, f(\tau)) - F(\tau, g(\tau))| \, d\tau \right| \\ &\leq L \left| \int_c^x |f(\tau) - g(\tau)| d\tau \right| \leq Lrc_{fg} \end{aligned}$$

for each $x \in I$, that is, $d(\Lambda f, \Lambda g) \leq Lrc_{fg}$. Thus, it follows that $d(\Lambda f, \Lambda g) \leq Lrd(f, g)$ for all $f, g \in X$. We note that $0 < Lr < 1$. It follows from (6) and (8) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$|(\Lambda g_0)(x) - g_0(x)| = \left| y(c) + \int_c^x F(\tau, g_0(\tau)) d\tau - g_0(x) \right| \leq C$$

for all $x \in I$ since $F(x, g_0(x))$ and $g_0(x)$ are bounded on I. Thus (7) implies that $d(\Lambda g_0, g_0) < \infty$. Therefore, Theorem 1(a) implies that there exists a continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that $\Lambda^n g_0 \longrightarrow y_0$ in (X, d) as $n \longrightarrow \infty$ and $y_0 = \Lambda y_0$ where y_0 satisfies equation (8) for any $x \in I$. If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C > 0$ with $|g_0(x) - g(x)| \leq C$ for all $x \in I$.

This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently $\{g \in X : d(g_0, g) < \infty\} = X$. Therefore, according to Theorem 1(b), y_0 is a unique continuous function with the property (8). Furthermore, it follows from (3) that

$$-\varepsilon E_q(x^q) \leq y'(x) - F(x, y(x)) \leq \varepsilon E_q(x^q)$$

for all $x \in I$. If we integrate each term of the above inequality from c to x, then we have

$$\begin{aligned} |(\Lambda y)(x) - y(x)| &\leq \int_c^x \varepsilon E_q(\tau^q) d\tau = \int_c^x \varepsilon \sum_{k=o}^{\infty} \frac{\tau^{kq}}{\Gamma(kq+1)} d\tau \\ &= \varepsilon \sum_{k=o}^{\infty} \frac{1}{\Gamma(kq+1)} \int_c^x \tau^{kq} d\tau = \varepsilon \sum_{k=o}^{\infty} \frac{x^{kq+1} - C^{kq+1}}{\Gamma(kq+2)} \\ &\leq \varepsilon \sum_{k=o}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} = \varepsilon E_q(x^q) \end{aligned}$$

for any $x \in I$. That is, $d(\Lambda y, y) \leq \varepsilon E_q$ holds. It now follows from Theorem 1(c) that

$$d(y, y_0) \leq \frac{1}{1 - Lr} d(\Lambda y, y) \leq \frac{\varepsilon}{1 - Lr} E_q(x^q),$$

which implies the validity of (5) for each $x \in I$.

In the following Theorem we have used the Bielecki norm

$$\|x\|_B := \max_{t \in J} \left\{ |x(t)| e^{-\theta t}, : \theta > 0, : J \subset \mathbb{R}_+ \right\}$$

to derive the similar Theorem 2 for the fundamental equation (1) via the Bielecki norm.

THEOREM 3. Given $c \in \mathbb{R}$ and $r > 0$. Let I denote a closed ball of radius r and center at c, that is , $I = \{x \in \mathbb{R} \mid c - r \le x \le c + r\}$ and let $F : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function which satisfies a Lipschits condition

$$|F(x,y) - F(x,z)| \le L|y - z|$$

for all $x \in I$ and $y, z \in \mathbb{R}$ where L is a constant with $0 < L/\theta < 1$. If a continuously differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies the differential inequality

$$|y'(x) - F(x, y(x))| \le \varepsilon E_q(x^q)$$

for all $x \in I$ and some $\varepsilon \ge 0$, where E_q is a Mittag function, then the equation (1) is Mittag-Leffler-Hyers-Ulam stable via the Bielecki norm.

PROOF. Just like the discussion in Theorem 2, we only prove that Λ defined in (6) is a contraction mapping on X with respect to the Bielecki norm:

$$\begin{aligned} |(\Lambda f)(x) - (\Lambda g)(x)| &= \left| \int_c^x \{F(\tau, f(\tau)) - F(\tau, g(\tau))\} d\tau \right| \\ &\le \int_c^x |F(\tau, f(\tau)) - F(\tau, g(\tau))| d\tau \le L \int_c^x e^{\theta\tau} |f(\tau) - g(\tau)| e^{-\theta\tau} d\tau \\ &\le L\|f - g\|_B \int_c^x e^{\theta\tau} d\tau \le \frac{L}{\theta} \|f - g\|_B e^{\theta x}. \end{aligned}$$

Then

$$|(\Lambda f)(x) - (\Lambda g)(x)| e^{-\theta x} \le \frac{L}{\theta} \|f - g\|_B$$

for each $x \in I$, that is, $d(\Lambda f, \Lambda g) \le \frac{L}{\theta} \|f - g\|_B$. Hence we can conclude that $d(\Lambda f, \Lambda g) \le \frac{L}{\theta} d(f, g)$ for any $f, g \in X$. By letting $0 < L/\theta < 1$, the strictly continuous property is verified. Now by a similar process with Theorem 1, we have

$$d(y, y_0) \le \frac{1}{1 - \frac{L}{\theta}} d(\Lambda y, y) \le \frac{1}{1 - \frac{L}{\theta}} \varepsilon E_q(x^q),$$

which means that the equation (1) is Mittag-Leffler-Hyers-Ulam stable via the Bielecki norm.

4 Mittag-Leffler-Hyers-Ulam-Rassias Stability

In this section we prove the Mittag-Leffler-Hyers-Ulam-Rassias Stability for the equation (1).

THEOREM 4. For given real numbers a and b with $a < b$, let $I = [a, b]$ be a closed interval and choose a $c \in I$. Let K, M and L be positive constants with $0 < KL < 1$.

Assume that $F : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function which satisfies in a Lipschits condition

$$|F(x,y) - F(x,z)| \leq L|y-z| \tag{10}$$

for all $x \in I$ and $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies the differential inequality

$$|y'(x) - F(x, y(x))| \leq \varepsilon\varphi(x)E_q(x^q) \tag{11}$$

for all $x \in I$ and some $\varepsilon \geq 0$, where E_q is a Mittag function, and $\varphi : I \longrightarrow (0,\infty)$ is a continuous function with

$$\left|\int_0^x \varphi(\tau)d\tau\right| \leq K\varphi(x) \tag{12}$$

and

$$\left|(\int_0^x (\varphi(\tau))^{\frac{1}{1-p}} d\tau)^{1-p}\right| \leq M\varphi(x) \tag{13}$$

for each $x \in I$, then there exists a unique continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_0^x F(\tau, y_0(\tau))d\tau. \tag{14}$$

Furthermore, y_0 is a solution of (1) and

$$|y(x) - y_0(x)| \leq \frac{\varepsilon E_q(x^q)}{1 - LK} M\varphi(x) \tag{15}$$

for all $x \in I$.

PROOF. First, we define a set X of all continuous functions $f : I \longrightarrow \mathbb{R}$ by

$$X = \{f : I \longrightarrow \mathbb{R} : f \text{ is a continuous function}\}. \tag{16}$$

and introduce a generalized metric on X as follows:

$$d(f,g) = \inf\{c \in [0,\infty] : |f(x) - g(x)| \leq C\varphi(x) \text{ for all } x \in I\}. \tag{17}$$

It is obvious that (X,d) is a generalized complete metric space. We define an operator $\Lambda : X \longrightarrow X$ by

$$(\Lambda f)(x) = y(c) + \int_0^x F(\tau, f(\tau))d\tau, \tag{18}$$

for any $x \in I$ and $f \in X$. (It is true that $\Lambda f \in X$, because Λf is continuously differentiable in view of the Fundamental Theorem of calculus. We now assert that Λ is strictly contractive on X. For $f, g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \leq C_{fg}\varphi(x)$, that is, we assume that

$$|f(x) - g(x)| \leq C_{fg}\varphi(x) \tag{19}$$

for all $x \in I$. Moreover, it follows from (10), (17),(12) and (18) that

$$\begin{aligned} |(\Lambda f)(x) - (\Lambda g)(x)| &= \left| \int_0^x \{F(\tau, f(\tau)) - F(\tau, g(\tau))\} d\tau \right| \\ &\leq \left| \int_0^x |F(\tau, f(\tau)) - F(\tau, g(\tau))| d\tau \right| \leq L \left| \int_0^x |f(\tau) - g(\tau)| d\tau \right| \\ &\leq LC_{fg} \left| \int_0^x \varphi(\tau) d\tau \right| \leq LKC_{fg}\varphi(x) \end{aligned}$$

for each $x \in I$, that is, $d(\Lambda f, \Lambda g) \leq LKC_{fg}\varphi(x)$.

Thus, it follows that $d(\Lambda f, \Lambda g) \leq LKd(f, g)$ for all $f, g \in X$. We note that $0 < LK < 1$. It follows from (16) and (18) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < \infty$ with

$$|(\Lambda g_0)(x) - g_0(x)| = \left| y(c) + \int_c^x F(\tau, g_0(\tau)) d\tau - g_0(x) \right| \leq C\varphi(x)$$

for all $x \in I$ since $F(x, g_0(x))$ and $g_0(x)$ are bounded on I and $\min_{x \in I} \varphi(x) > 0$. Thus (17) implies that $d(\Lambda g_0, g_0) < \infty$. Therefore, Theorem 1(a) implies that there exists a continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that $\Lambda^n g_0 \longrightarrow y_0$ in (X, d) as $n \longrightarrow \infty$ and such that $y_0 = \Lambda y_0$ that is y_0 satisfies equation (14) for any $x \in I$. If $g \in X$, then g_0 and g are continuous functions defined on a compact interval I and $\min_{x \in I} \varphi(x) > 0$. Hence, there exists a constant $C_g > 0$ with $|g_0(x) - g(x)| \leq C_g\varphi(x)$ for all $x \in I$.

This implies that $d(g_0, g) < \infty$ for every $g \in X$, or equivalently

$$\{g \in X : d(g_0, g) < \infty\} = X.$$

Therefore, according to Theorem 1(b), y_0 is a unique continuous function with the property (14). Furthermore, it follows from (11) that

$$-\varepsilon\varphi(x)E_q(x^q) \leq y'(x) - F(x, y(x)) \leq \varepsilon\varphi(x)E_q(x^q)$$

for all $x \in I$. If we integrate each term of the above inequality from 0 to x, then we have

$$\begin{aligned} |(\Lambda y)(x) - y(x)| &\leq \int_0^x \varepsilon\varphi(\tau)E_q(\tau^q) d\tau \\ &= \int_0^x \varepsilon\varphi(\tau) \sum_{k=o}^{\infty} \frac{\tau^{kq}}{\Gamma(kq+1)} d\tau = \varepsilon \sum_{k=o}^{\infty} \frac{1}{\Gamma(kq+1)} \int_0^x \varphi(\tau)\tau^{kq} d\tau \\ &\leq \varepsilon \sum_{k=o}^{\infty} \frac{1}{\Gamma(kq+1)} \left(\int_0^x \varphi(\tau))^{\frac{1}{1-p}} d\tau \right)^{1-p} \left(\int_0^x (\tau^{kq})^{\frac{1}{p}} d\tau \right)^p \\ &\leq \varepsilon \sum_{k=o}^{\infty} \frac{1}{\Gamma(kq+1)} M\varphi(x) \left(\frac{x^{\frac{kq}{p}}}{\frac{kq+p}{p}} \right)^p \\ &\leq \varepsilon \sum_{k=o}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} M\varphi(x) \leq \varepsilon M\varphi(x)E_q(x^q) \end{aligned}$$

for any $x \in I$. That is, $d(\Lambda y, y) \leq \varepsilon M\varphi(x)E_q(x^q)$ holds. It now follows from Theorem 1(c) that

$$d(y, y_0) \leq \frac{1}{1-LK} d(\Lambda y, y) \leq \frac{\varepsilon}{1-LK} M\varphi(x)E_q(x^q),$$

which implies the validity of (14) for each $x \in I$.

EXAMPLE 1. We choose positives K and L with $KL < 1$. For a positive number $\varepsilon < 2K$, let $I = [0, 2K - \varepsilon]$ be a closed interval. Given a polynomial $p(x)$, we assume that a continuously differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies

$$|y'(x) - Ly(x) - p(x)| \leq (x + \varepsilon)E_q(x^q)$$

for all $x \in I$. If we set $F(x, y) = Ly + p(x)$ and $\varphi(x) = x + \varepsilon$, then the above inequality has the identical form with (11). Moreover, we obtain

$$\left| \int_0^x \varphi(\tau) d\tau \right| = \frac{1}{2} x^2 + x\varepsilon \leq K\varphi(x)E_q(x^q)$$

and

$$\left| \left(\int_0^x (\varphi(\tau))^{\frac{1}{1-p}} d\tau \right)^{1-p} \right| \leq M\varphi(x)$$

for each $x \in I$, since $K\varphi(x)E_q(x^q) - \frac{x^2}{2} - x\varepsilon \geq 0$. According to Theorem 4, there exists a unique continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that

$$y_0(x) = y(0) + \int_0^x \{Ly_0(\tau) + p(\tau)\} d\tau$$

and

$$|y(x) - y_0(x)| \leq \frac{M(x + \varepsilon)}{1 - LK} E_q(x^q)$$

for any $x \in I$.

EXAMPLE 2. Let r and L be positive constants with $0 < Lr < 1$ and define a closed interval $I = \{x \in \mathbb{R} : c - r \leq x \leq c + r\}$ for some real number c. Assume that a continuously differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies $|y'(x) - Ly(x) - p(x)| \leq \varepsilon E_q(x^q)$ for all $x \in I$ and some $\varepsilon \geq 0$, where $p(x)$ is a polynomial. If we set $F(x, y) = Ly(x) + p(x)$, then by Theorem 3.1 there exists a unique continuous function $y_0 : I \longrightarrow \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x \{Ly_0(\tau) + p(\tau)\} d\tau$$

and

$$|y(x) - y_0(x)| \leq \frac{\varepsilon}{1 - Lr} E_q(x^q)$$

for all $x \in I$.

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