

Existence And Uniqueness Of Solutions To Fractional Order Nonlinear Neutral Differential Equations*

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Abstract

The fractional order nonlinear neutral differential equation

$${}^c D_{0+}^\alpha (x(t) - g(t, x(t - \tau(t)))) = f(x(t), x(t - \tau(t))), \quad t \in [0, T],$$

is considered in this work. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, we establish some criteria for the existence and uniqueness of solutions to the fractional order neutral differential equation.

1 Introduction

During the last years, the use of fractional derivative has an important increase in many fields of science and engineering. In fact as the classical differential calculation provided powerful instruments for the explanation and modelization of a huge number of phenomena studied by applied mathematics, these instruments do not allow us to recognize the abnormal dynamics presented by some complex systems faced in nature or within social interactions, such as the diffusion of contamination in underground water, the relaxation of viscoelastic materials like the polymairies, the pollution propagation in the atmosphere. The diffusion of cells procedures, the signal transmission by magnetic fields, the network traffic, speculation effect on the store benefits in the financial markets, etc. [9, 11, 18].

In most of these cases, this kind of abnormal process has a complex microscopic and macroscopic behavior where the dynamic can't be characterized by the models based on the classical derivatives. It is proved by using the well known experimental results where many processes related to complex systems have a non-local dynamic leading to long term effects "long-memory" (or differentiated behavior). The operators of the fractional derivative and integration present similitude with certain characteristics that are more meaningful and adaptable to phenomena modeling.

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The concept of fractional derivatives or derivatives of fractional order is not recent. In 1695, Leibniz gave a signification to the derivative of order $1/2$. Later on a lot of mathematicians (Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Grunwald, and Letnikov) have contributed in the evolution of the fractional calculus theory which is a generalization of differential and integral calculus to a real or complex non-integer order. For more details, on related historic aspects we refer to [10, 12, 14, 15, 16, 17].

The fractional derivative generalizes the ordinary derivative notion within a certain measure. Hence this definition is not available except on a conceptual plan. To talk about generalization, a lot of progress should be done to establish the link between the two theories so that the ordinary derivatives can be interpreted as a sub-set of fractional derivatives. In fact, different approaches have been developed until now without being equivalent. This gives these derivatives power and limitation at the same time. We conclude that, the fractional derivative is a way of synthetics to describe intermediate behaviors between the classical derivatives which have a remarkable property to itself a non-local characteristic: the fractional derivatives contain some information on the function to interior points. So, they possess a memory effect that describe the past of the function [1, 2, 7, 19, 21].

In other words, the theory of fractional calculus provides many meanings potentially necessary for the resolution of the integral equations, differential and other problems using special functions in mathematical physics. In particular, the fractional differential equations like the important branch of fractional calculus researches keep a lot of attention. Theories on the local existence and uniqueness of the dependency in relation with the solution data of the fractional specific equations have been exploited during the recent years (see [5, 6]).

In mathematics delay differential equations are a type of differential equation in which the derivative of unknown function at a certain time is given in terms of the values of the function at previous times.

While physical events such as acceleration and deceleration take little time compared to the time needed to travel most distances. Time involved in biological processes such as gestation and maturation can be substantial when compared to the data-collection time in most population studies. Therefore, it is often imperative to explicitly incorporate these processes times into mathematical models of population dynamics. These process times are often called delay times and the models that incorporate such delay times are referred as delay differential equation models. [3, 4, 8, 13, 20].

In this article we study the existence of solutions of the fractional order nonlinear neutral differential equation with variable delay

$${}^c D_{0+}^{\alpha} (x(t) - g(t, x(t - \tau(t)))) = f(x(t), x(t - \tau(t))), \quad t \in [0, T], \quad (1)$$

with the initial condition

$$x(t) = \psi(t), \quad t \in [m_0, 0],$$

where ${}^c D_{0+}^{\alpha}$ is the standard Caputo fractional derivative of order $\alpha \in (0, 1)$, $m_0 = \inf_{t \in [0, T]} \{t - \tau(t)\}$, $\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $\psi : [m_0, 0] \rightarrow \mathbb{R}$ are continuous functions. The functions $g(t, x)$ and $f(x, y)$ are Lipschitz continuous in x and in x and y , respectively. That is, there are positive constants L_1, L_2, L_3 such that

$$|g(t, x) - g(t, y)| \leq L_1 |x - y|, \quad (2)$$

and

$$|f(x, y) - f(w, z)| \leq L_2 |x - w| + L_3 |y - z|. \quad (3)$$

To reach our desired end we have to transform (1) into an integral equation and then use Krasnoselskii's fixed point theorem to show the existence of solutions. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact.

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas, and state some preliminary results needed in later sections. Also, we present the inversion of (1) and state Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [22]. In Section 3, we present our main results on existence of solutions of (1).

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

DEFINITION 2.1. The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \rightarrow \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided the right side exists pointwise on \mathbb{R}^+ , where Γ is the gamma function.

For instance, $I^\alpha x$ exists for all $\alpha > 0$, when $x \in C(\mathbb{R}^+)$ then $I^\alpha x \in C(\mathbb{R}^+)$ and moreover $I^\alpha x(0) = 0$.

DEFINITION 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \rightarrow \mathbb{R}$ is given by

$${}^c D_{0+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds = I_{0+}^{n-\alpha} x^{(n)}(t),$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on \mathbb{R}^+ .

LEMMA 2.1. Suppose that $x \in C^{n-1}([0, \infty))$ and $x^{(n)}$ exists almost everywhere on any bounded interval of \mathbb{R}^+ . Then

$$(I_{0+}^\alpha {}^c D_{0+}^\alpha x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In particular, when $\alpha \in (0, 1)$, $(I_{0+}^\alpha {}^c D_{0+}^\alpha x)(t) = x(t) - x(0)$.

From Lemma 2.1, we deduce the following lemma.

LEMMA 2.2. Suppose that (3) holds. Let $x \in C([m_0, T])$, x' and $\frac{\partial g}{\partial t}$ exist. Then x is a solution of (1) if and only if

$$\begin{aligned} x(t) &= \psi(0) - g(0, \psi(-\tau(0))) + g(t, x(t - \tau(t))) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds, \end{aligned} \quad (4)$$

for $t \in [0, T]$ and $x(t) = \psi(t)$ for $t \in [m_0, 0]$.

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of solutions to (1). For its proof we refer the reader to [22].

THEOREM 2.1 (Krasnoselskii). Let B be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that F_1 and F_2 map B into X such that

- (i) for any $x, y \in B$, $F_1x + F_2y \in B$,
- (ii) F_1 is a contraction, and
- (iii) F_2 is continuous and $F_2(B)$ is contained in a compact set.

Then there exists $z \in B$ such that $z = F_1z + F_2z$.

3 Main Results

To apply Theorem 2.1, we need to define a Banach space X and a closed convex subset D of X , and to construct two mappings, one is a contraction and the other is compact. So, we let $(X, \|\cdot\|) = (BC([m_0, T], \mathbb{R}), \|\cdot\|)$ and $D = \{x \in BC([m_0, T], \mathbb{R}) : \|x\| \leq r\}$, where $BC([m_0, T], \mathbb{R})$ denotes the collection of all bounded and continuous functions from $[m_0, T]$ to \mathbb{R} and r is positive constant. We express equation (4) as

$$x(t) = (F_1x)(t) + (F_2x)(t) = (Fx)(t), \quad (5)$$

where $F_1, F_2 : D \rightarrow X$ are defined by

$$(F_1x)(t) = \psi(0) - g(0, \psi(-\tau(0))) + g(t, x(t - \tau(t))) \quad (6)$$

and

$$(F_2x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds. \quad (7)$$

LEMMA 3.1. Suppose that (2) holds. If F_1 is given by (6) with

$$L_1 < 1, \quad (8)$$

then $F_1 : D \rightarrow X$ is a contraction.

PROOF. Let F_1 be defined by (6). Obviously, F_1x is bounded and continuous. So, for $x, y \in D$, we have

$$\begin{aligned} |(F_1x)(t) - (F_1y)(t)| &= |g(t, x(t - \tau(t))) - g(t, y(t - \tau(t)))| \\ &\leq L_1 |x(t) - y(t)| \leq L_1 \|x - y\|. \end{aligned}$$

Then

$$\|F_1x - F_1y\| \leq L_1 \|x - y\|.$$

Thus $F_1 : D \rightarrow X$ is a contraction by (8).

LEMMA 3.2. Suppose that (3) holds. Then $F_2 : D \rightarrow X$, as defined by (7), is compact.

PROOF. Let F_2 be defined by (7). Clearly, F_2x is bounded and continuous. To prove the continuity of F_2 , we consider a sequence x_n converging to x . Taking the norm of $(F_2x_n)(t) - (F_2x)(t)$, we have

$$\begin{aligned} &|(F_2x_n)(t) - (F_2x)(t)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(x_n(s), x_n(s-\tau(s))) - f(x(s), x(s-\tau(s)))| ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (L_2 |x_n(s) - x(s)| + L_3 |x_n(s-\tau(s)) - x(s-\tau(s))|) ds \\ &\leq \frac{L_2 + L_3}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \|x_n - x\| \leq \frac{L_2 + L_3}{\Gamma(\alpha + 1)} T^\alpha \|x_n - x\|. \end{aligned}$$

From the above analysis we obtain

$$\|F_2x_n - F_2x\| \leq \frac{L_2 + L_3}{\Gamma(\alpha + 1)} T^\alpha \|x_n - x\|,$$

and hence whenever $x_n \rightarrow x$, $F_2x_n \rightarrow F_2x$. This proves the continuity of F_2 .

To show F_2 is compact. Observe that in view of (3) we arrive at

$$\begin{aligned} |f(x(t), y(t))| &= |f(x(t), y(t)) - f(0, 0) + f(0, 0)| \\ &\leq |f(x(t), y(t)) - f(0, 0)| + |f(0, 0)| \\ &\leq L_2 \|x\| + L_3 \|y\| + \delta_f, \end{aligned}$$

where $\delta_f = |f(0,0)|$. Now for $t_1 \leq t_2 \leq T$, we have

$$\begin{aligned}
& |(F_2x)(t_2) - (F_2x)(t_1)| \\
& \leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds \right. \\
& \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_2 - s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds \right. \\
& \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds \right. \\
& \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} f(x(s), x(s - \tau(s))) ds \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| |f(x(s), x(s - \tau(s)))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| (t_2 - s)^{\alpha-1} \right| |f(x(s), x(s - \tau(s)))| ds,
\end{aligned}$$

and then

$$\begin{aligned}
& |(F_2x)(t_2) - (F_2x)(t_1)| \\
& \leq \frac{(L_2 + L_3)r + \delta_f}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right| ds \\
& \quad + \frac{(L_2 + L_3)r + \delta_f}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| (t_2 - s)^{\alpha-1} \right| ds \\
& \leq \frac{(L_2 + L_3)r + \delta_f}{\Gamma(\alpha + 1)} |-2(t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha| \\
& \leq 2 \frac{(L_2 + L_3)r + \delta_f}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha.
\end{aligned}$$

The right-hand side of above expression does not depends on x . Thus we conclude that $F_2(D)$ is relatively compact and hence F_2 is compact by Arzela-Ascoli theorem.

THEOREM 3.1. Let $\delta_g = \max_{t \in [0, T]} \{|g(t, 0)|\}$ and $\delta_f = |f(0, 0)|$. Suppose (2), (3) and (8) hold. Suppose there is a positive constant r such that all solutions $x \in BC([m_0, T], \mathbb{R})$ of (1) satisfy $\|x\| \leq r$ and the inequality

$$|\psi(0) - g(0, \psi(-\tau(0)))| + L_1 r + \delta_g + \frac{(L_2 + L_3)r + \delta_f}{\Gamma(\alpha + 1)} T^\alpha \leq r$$

holds. Then equation (1) has a solution in $D = \{x \in BC([m_0, T], \mathbb{R}) : \|x\| \leq r\}$.

PROOF. By Lemma 3.1, the operator $F_1 : D \rightarrow X$ is a contraction. Also, from Lemma 3.2, the operator $F_2 : D \rightarrow X$ is compact and continuous. Moreover, if $x, y \in D$,

we see that

$$\begin{aligned}
& |(F_1x)(t) + (F_2y)(t)| \\
& \leq |\psi(0) - g(0, \psi(-\tau(0)))| + |g(t, x(t))| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(y(s), y(s-\tau(s)))| ds \\
& \leq |\psi(0) - g(0, \psi(-\tau(0)))| + L_1 \|x\| + \delta_g \\
& + \frac{(L_2 + L_3) \|y\| + \delta_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
& \leq |\psi(0) - g(0, \psi(-\tau(0)))| + L_1 r + \delta_g + \frac{(L_2 + L_3)r + \delta_f}{\Gamma(\alpha + 1)} T^\alpha \\
& \leq r.
\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in D$ such that $x = F_1x + F_2x$. By Lemma 2.2 this fixed point is a solution of (1) and the proof is complete.

THEOREM 3.2. Suppose that (2) and (3) hold. If

$$L_1 + \frac{L_2 + L_3}{\Gamma(\alpha + 1)} T^\alpha < 1,$$

then equation (1) has a unique solution.

PROOF. Let the mapping F be given by (5). For $x, y \in BC([m_0, T], \mathbb{R})$, in view of (4), we have

$$\begin{aligned}
& |(Fx)(t) - (Fy)(t)| \\
& \leq L_1 \|x - y\| + \frac{(L_2 + L_3) \|x - y\|}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} ds \\
& \leq \left(L_1 + \frac{L_2 + L_3}{\Gamma(\alpha + 1)} T^\alpha \right) \|x - y\|.
\end{aligned}$$

Then

$$\|Fx - Fy\| \leq \left(L_1 + \frac{L_2 + L_3}{\Gamma(\alpha + 1)} T^\alpha \right) \|x - y\|.$$

This completes the proof by invoking the contraction mapping principle.

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