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A Common Fixed Point Theorem For Generalized Almost Contractions In Metric-Like Spaces*

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Abstract

In this paper, we present a common fixed point theorem for two pairs of self mappings satisfying a generalized almost contractive condition in metric-like spaces. We provide two examples to illustrate our obtained results, also an application to study the existence of solution for a system of integral equations is given.

1 Introduction and Preliminaries

The idea of metric-like spaces (dislocated metric spaces) is initiated by Hitzler and Seda [20]. Later Amini-Herandi [3] discovered the metric like spaces are generalizations of metric spaces, and he obtained some fixed point results in such spaces. In last years, many authors established some fixed point or common fixed point theorems in metric-like spaces, see for instance [5, 6, 7, 8, 9, 12, 21, 24].

On the other hand, Berinde [13] defined the weak contraction (contraction of Berinde-type) and obtained some results under some contractive conditions. He also introduced the concept of almost contraction in [14]. Recently, Babu et al. [10] introduced a new type of contractive condition, which is called "condition (B)", they proved the existence of a fixed point for this class of mappings, later Abbas et al. [2] generalized the last concept to "generalized condition (B)". Quite recently, Ćirić et al.[16] introduced the concept of almost generalized contractive condition and they obtained some fixed point results in order metric spaces.

Firstly, we recall some basic definitions and properties of partial metric spaces and metric-like spaces.

DEFINITION 1 ([25]). Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for all x, y and z in X, the following conditions hold:

(P1) $p(x, x) = p(y, t) = p(x, y)$ if and only if $x = y$,

(P2) $p(x, x) \leq p(x, y)$,

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(P3) $p(x,y) = p(y,x)$,

(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The space (X,p) is called a partial metric space.

Clearly if $p(x,y) = 0$, then (P1) and (P2) imply $x = y$. If p is a partial metric on X, then the function $p^s : X \times X \to R_+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

defines a metric on X.

Every partial metric p on X generates a topology τ_p on X, which has as a base the family of open p-balls $\{B_p(x,\varepsilon); x \in X, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \{y \in X, |p(x,y) - p(x,x)| < \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

DEFINITION 2 ([25]). Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ with respect to τ_p if and only if
$$\lim_{n\to\infty} p(x,x_n) = p(x,x).$$

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy if $\lim_{n\to\infty} p(x_n,x_m)$ exists and is finite.

(iii) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is convergent with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.

In this case, we say that the partial metric p is complete.

DEFINITION 3 ([3]). Let X be a nonempty set. A function: $\sigma : X \times X \to \mathbb{R}_+$ is said to be a metric-like on X if the following conditions hold:

(i) $\sigma(x,y) = 0$ implies that $x = y$,

(ii) $\sigma(x,y) = \sigma(y,x)$,

(iii) $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

The space (X,σ) is said to be a metric-like space.

Every partial metric is a metric-like.

EXAMPLE 1 ([3, 4]). Let $X = \mathbb{R}$ and define σ for all $x,y \in X$ by:

$$\sigma(x,y) = \frac{|x-y| + |x| + |y|}{2}.$$

If $X = (0, \infty)$, we see that $\sigma(x, y) = \max\{x, y\}$. In this case σ is also a partial metric.

EXAMPLE 2 ([3]). Let $X = \{0, 1\}$ and define $\sigma : X \times X \to \mathbb{R}_+$ as follows:

$$\sigma(x, y) = \begin{cases} 2 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then σ is a metric-like on X. Since $\sigma(0, 0) > \sigma(0, 1)$, σ is not a partial metric.

Each metric-like σ on X generates a topology τ_σ on X, which has as a base the family of open σ-balls $\{B_\sigma(x, \varepsilon); x \in X, \varepsilon > 0\}$, where

$$B_\sigma(x, \varepsilon) = \{y \in X, |\sigma(x, y) - \sigma(x, x)| < \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$.

DEFINITION 4 ([3]). Let (X, σ) be a metric-like space.

1. A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if, and only if

$$\lim_{n\to\infty} \sigma(x, x_n) = \sigma(x, x).$$

2. A sequence $\{x_n\}$ in X is said to be a σ-Cauchy sequence if $\lim_{n\to\infty} \sigma(x_n, x_m)$ exists and is finite.
3. (X, σ) is said to be σ-complete if every σ-Cauchy sequence $\{x_n\}$ in X is convergent to a point $x \in X$ such that $\lim_{n\to\infty} \sigma(x, x_n) = \sigma(x, x)$.

DEFINITION 5 ([26, 19]). Let (X, σ) be a metric-like space.

1. A sequence $\{x_n\}$ in X is said to be a $0-\sigma$-Cauchy sequence if $\lim_{n\to\infty} \sigma(x_n, x_m) = 0$.
2. (X, σ) is said to be $0 - \sigma$-complete if every $0 - \sigma$-Cauchy sequence $\{x_n\}$ in X is convergent to a point $x \in X$ such that $\sigma(x, x)$.

REMARK 1.

1. Every $0 - \sigma$-Cauchy sequence is a σ-Cauchy sequence.
2. Every $0 - \sigma$-complete metric-like space is a $\sigma-$ complete metric-like space.

Abbas et al. [1] introduced the concept of almost contraction property for two self mappings on a metric space which generalizes that given by Berinde [13, 15].

DEFINITION 6 ([1]). Let f, g be two self mappings on a metric space (X, d). The mapping g is said to be an almost contraction with respect to f, if there exist $\delta \in [0, 1)$ and $L > 0$ such that

$$d(gx, gy) \leq \delta d(fx, fy) + Ld(fy, gx).$$

Babu et al. [10] defined that a self mapping g on metric space (X, d) is said to satisfy the condition (B) if there exist $\delta \geq 0$ and $L \geq 0$ such that $\delta + L < 1$ and for all $x, y \in X$ we have

$$d(gx, gy) \leq \delta d(x, y) + L \min(d(x, gx), d(y, gy), d(x, gy), d(y, gx)).$$

Abbas et al. [2] generalized the last definition for two self mappings, called generalized condition (B).

DEFINITION 7 ([2]). Let (X, d) be a metric space and let there be two self mappings $f, g : X \to X$. The mapping g satisfies the generalized condition (B) associated with f, if there exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(gx, gy) \leq \delta M(x, y) + L \min\{d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\},$$

where

$$M(x, y) = \max\left\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx, gy) + d(fy, gx)}{2}\right\}.$$

We find the same definition in paper [2], called g generalized almost f-contraction. In our framework, we will need the following definition. Jungck and Rhoades [22] defined weak compatibility of two self mappings as follows.

DEFINITION 8 ([22]). Let X be nonempty set. Two self mappings $f, S : X \to X$ are said to be weakly compatible if they commute at their coincidence points; i.e., if $fu = Su$ for some $u \in X$, then $fSu = Sfu$.

2 Main Results

DEFINITION 9. Let (X, σ) be a metric-like space and let $A, S : X \to X$. The mapping S satisfies the generalized condition (B) associated with A, if there exists $\delta \in (0, 1)$ and $L \geq 0$ such that

$$\sigma(Sx, Sy) \leq \delta M(x, y) + L \min\{\sigma(Ax, Sx), \sigma(Ay, Sy), \sigma(Ax, Sy), \sigma(Ay, Sx)\}, \tag{1}$$

where

$$M(x, y) = \max\left\{\sigma(Ax, Ay), \sigma(Ax, Sx), \sigma(Ay, Sy), \frac{\sigma(Ax, Sy) + \sigma(Ay, Sx)}{4}\right\}.$$

If $A = id_X$, then S satisfies the generalized condition (B).

EXAMPLE 4. Let $X = \{0, 1, 2\}$ and define σ as follows: $\sigma(0, 0) = 0$,

$$\sigma(1, 1) = 2, \quad \sigma(2, 2) = 3, \quad \sigma(0, 1) = \sigma(1, 0) = 1,$$

$$\sigma(0, 2) = \sigma(2, 0) = 2 \text{ and } \sigma(1, 2) = \sigma(2, 1) = 4.$$

Let A and S be two self mappings such that

$$Sx = \begin{cases} 2 & x \in \{0, 1\}, \\ 0 & x = 2, \end{cases} \quad \text{and} \quad Ax = \begin{cases} 2 & x \in \{0, 1\}, \\ 1 & x = 2. \end{cases}$$

We show the inequality (1) is satisfied, with $\delta = \frac{4}{5}$ and $L = 0$.

1. For $x = y = 0$ or $x = y = 1$, we have

$$\sigma(S0, S0) = \sigma(1, 1) = 2 \leq \frac{12}{5} = \frac{4}{5}\sigma(2, 2) = \frac{4}{5}\sigma(A0, A0) = \frac{4}{5}\sigma(A1, A).$$

2. For $x = y = 2$, we have

$$\sigma(S2, S2) = \sigma(0, 0) = 0 \leq \frac{4}{5}\sigma(A2, A2).$$

3. For $x, y \in \{0.2\}$ with $x \neq y$, we have

$$\sigma(S0, S2) = \sigma(0, 1) = 1 \leq \frac{16}{5} = \frac{4}{5}\sigma(2, 1) = \frac{1}{3}\sigma(A(0), A(2)).$$

4. For $x = y \in \{0, 1\}$ with $x \neq y$, we have

$$\sigma(S0, S1) = \sigma(1, 1) = 2 \leq \frac{12}{5} = \frac{4}{5}\sigma(A0, A1).$$

5. For $x = y \in \{1, 2\}$ with $x \neq y$, we have

$$\sigma(S1, S2) = \sigma(1, 0) = 1 \leq \frac{16}{3} = \frac{4}{5}\sigma(2, 1) = \frac{4}{5}\sigma(A1, A2).$$

Consequently, S satisfies the generalized condition (B) associated with A.

In our work, we will apply this condition for two pairs of self mappings to prove the existence of common fixed points in a metric like-space.

Let (X, σ) be a metric-like space and let A, B, S and T be self mappings on X satisfying (1) and

$$T(X) \subset A(X) \text{ and } S(X) \subset B(X). \tag{2}$$

Let $x_0 \in X$. Since $S(X) \subset B(X)$ there is a point $x_1 \in X$ such that $y_0 = Bx_1 = Sx_0$, for this point y_0 there exists a point $y_1 = Tx_1$, and since $T(X) \subseteq A(X)$ there is $x_2 \in X$ such that $y_1 = Ax_2 = Tx_1$, so by continuing in this manner, we construct a sequence $\{y_n\}$ in X as follows.

$$\begin{cases} y_{2n+1} = Bx_{2n+1} = Sx_{2n}, \\ y_{2n+2} = Ax_{2n+2} = Tx_{2n+1}. \end{cases} \tag{3}$$

LEMMA 1. The sequence $\{y_n\}$ which is defined by (3) is a σ-Cauchy sequence in (X, σ).

PROOF. First we prove

$$\lim_{n\to\infty} \sigma(y_n, y_{n+1}) = 0.$$

We have

$$\begin{aligned} & M(x_{2n}, x_{2n+1}) \\ = \ & \max\Big\{\sigma(Ax_{2n}, Bx_{2n+1}), \sigma(Ax_{2n}, Sx_{2n}), \sigma(Bx_{2n+1}, Tx_{2n+1}), \\ & \frac{\sigma(Ax_{2n}, Tx_{2n+1}) + \sigma(Bx_{2n+1}, Sx_{2n})}{4})\Big\} \\ \leq \ & \max\Big\{\sigma(y_{2n}, y_{2n+1}), \sigma(y_{2n}, y_{2n+1}), \sigma(y_{2n+1}, y_{2n+2}, \\ & \frac{\sigma(y_{2n}, y_{2n+2} + \sigma(y_{2n+1}, y_{2n+1})}{4}\Big\} \\ \leq \ & \max\{\sigma(y_{2n}, y_{2n+1}), \sigma(y_{2n+1}, y_{2n+2})\} \end{aligned}$$

and

$$\begin{aligned} & \min\{\sigma(Ax_{2n}, Sx_{2n}), \sigma(Bx_{2n+1}, Tx_{2n+1}), \sigma(Ax_{2n}, Tx_{2n+1}), \sigma(Bx_{2n+1}, Sx_{2n})\} \\ = \ & \min\{\sigma(y_{2n}, y_{2n+1}), \sigma(y_{2n}, y_{2n+1}), \sigma(y_{2n+1}, y_{2n+2}), \sigma(y_{2n}, _{2n+2}), \sigma(y_{2n+1}, y_{2n+1})\} \\ = \ & \min\{\sigma(y_{2n}, y_{2n+1}), \sigma(y_{2n}, y_{2n+2}), \sigma(y_{2n+1}, y_{2n+1})\}. \end{aligned}$$

If $\sigma(y_{2n}, y_{2n+1}) \leq \sigma(y_{2n+1}, y_{2n+2})$, by (1) we get

$$\begin{aligned} \sigma(y_{2n+1}, y_{2n+2}) &= \sigma(Sx_{2n}, Tx_{2n+1}) \leq \delta\sigma(y_{2n+1}, y_{2n+2}) + L\sigma(y_{2n+1}, y_{2n+2}) \\ &= (\delta + L)\sigma(y_{2n+1}, y_{2n+2}) < \sigma(y_{2n+1}, y_{2n+2}), \end{aligned}$$

which is a contradiction. Then

$$\sigma(y_{2n+1}, y_{2n+2}) \leq (\delta + L)\sigma(y_{2n}, y_{2n+1}).$$

Since $\delta + L < 1$, we put $\lambda = (\delta + L) < 1$, and by induction we obtain

$$\sigma(y_{n+1}, y_{n+2}) \leq \lambda^n \sigma(y_0, y_1).$$

Hence $\{\sigma(y_n, y_{n+1}\}$ is convergent to 0. For all $n, m \in \mathbb{N}$ such that $m > n$ we have

$$\begin{aligned} d_\sigma(y_n, y_m) &= 2\sigma(y_n, y_m) - \sigma(y_n, y_n) - \sigma(y_m, y_m) \\ &\leq 2\big(\sigma(y_n, y_{n+1}) + \sigma(y_{n+1}, y_{n+2}) + ... + \sigma(y_{m-1}, y_m)\big) \\ &\leq 2\lambda^n\Big(\sigma(y_0, y_1) + \lambda\sigma(y_0, y_1) + ... + \lambda^{m-n-1}\sigma(y_0, y_1)\Big) \\ &\leq 2\lambda^n \frac{1-\lambda^{m-n}}{1-\lambda}\sigma(y_0, y_1) \quad \to 0 \;\text{ as }\; n, m \to \infty. \end{aligned}$$

This yields that $\{y_n\}$ is a Cauchy sequence in (X, d_σ), so it is a σ-Cauchy sequence in (X, σ). Since (X, σ) is σ-complete, $\{y_n\}$ is convergent to $z \in X$. Moreover, we have

$$\lim_{n\to\infty} \sigma(y_n, z) = \lim_{n\to\infty} \sigma(y_n, y_m) = \sigma(z, z) = 0.$$

THEOREM 1. Let (X, σ) be a complete metric-like space, and let A, B, S and T be self mappings on X satisfying (1) and (2). If $A(X)$ or $B(X)$ is closed, then A, B, S and T have a unique common fixed point in X.

PROOF. From Lemma 1 the sequence $\{y_n\}$ is a σ-Cauchy sequence, and X is σ-complete, so it converges to $z \in X$. Also, the subsequence $\{y_{2n+2}\} = \{Ax_{2n+2}\}$ is convergent to z. Suppose AX is closed, then $z \in AX$ and there exists $u \in X$ such that $z = Au$. We claim $z = Au = Su$, if not by using (1) we get

$$\begin{aligned} \sigma(Su, Tx_{2n+}) \;&\leq\; \delta M(u, x_{2n+1}) + L\min\Big\{\sigma(Au, Bx_{2n+1}), \sigma(Au, Su), \\ &\quad \sigma(Bx_{2n+1}, Tx_{2n+1}), \sigma(Az, Tx_{2n+1}), \sigma(Bx_{2n+1}, Sz)\Big\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\sigma(Su, Au) \leq \delta\sigma(Au, Su) + L\sigma(Au, Su) < \sigma(Au, Su),$$

which is a contradiction, and so u is a coincidence point for A and S. Since $S(X) \subseteq B(X)$, there is $v \in X$ such that $z = Su = Bv$. We show $Bv = Tv$, if not by using (1) we get

$$\begin{aligned} \sigma(Bv, Tv) \;&=\; \sigma(Su, Tv) \leq \delta M(u, v) + L\min\Big\{\sigma(fu, Su), \sigma(Bv, Tv), \\ &\quad \sigma(fu, Tv), \sigma(Bv, Su))\Big\} \\ &\leq\; \delta\max\{\sigma(z, z), \sigma(Bv, Tv)\} + L\min\{\sigma(z, z), \sigma(Bv, Tv)\}. \end{aligned}$$

If $\sigma(z, Tv) \leq \sigma(z, z)$, we get

$$\sigma(z, Tv) \leq (\delta + L)\sigma(z, z) < \sigma(z, Tv),$$

which is a contradiction. Then $z = Bv = Tv$ and the pair (B, T) is weakly compatible imply that $Bz = Tz$. Now we prove $Az = Bz$, if not by using (1) we get

$$\begin{aligned}
\sigma(Az, Bz) &= \sigma(Sz, Tz) \\
&\leq \delta \max\left\{\sigma(Az, Bz), \sigma(Az, Sz), \sigma(Bz, Tz), \frac{1}{4}(\sigma(Az, Tz) + \sigma(Bz, Sz)\right\} \\
&\quad + L \min\{\sigma(Az, Sz), \sigma(Bz, Tz), \sigma(Az, Tz), \sigma(Bz, Sz)\} \\
&\leq (\delta + L)\sigma(Az, Bz) < \sigma(Az, Bz),
\end{aligned}$$

which is a contradiction. Then $Az = Bz$. Next, we prove $z = Az$, if not by using (1) we get

$$\sigma(Sz, Tx_n) \leq \delta M(z, x_n) + L \min\{\sigma(Az, Sz), \sigma(Bx_n, Tx_n), \sigma(Az, Tx_n), \sigma(Bx_n, Sz)\}.$$

Letting $n \to \infty$, we get

$$\sigma(Az, z) = \sigma(Sz, z) \leq (\delta + L)\sigma(Az, z) < \sigma(Az, z),$$

which is a contradiction. Then z is a common fixed point for A, B, S and T. For uniqueness, suppose there are two common fixed points z and w, by using (1) we get

$$\begin{aligned}
\sigma(z, w) &= \sigma(Sz, Tw) \\
&\leq \delta M(z, w) + L \min\{\sigma(Az, Sz), \sigma(Bx_n, Tx_n), \sigma(Az, Tx_n), \sigma(Bx_n, Sz)\} \\
&\leq (\delta + L)\sigma(z, w) < \sigma(z, w),
\end{aligned}$$

which is a contradiction. Then $z = w$.

Theorem 1 extends Theorem 2.1 in [23] to the setting of metric-like spaces.

COROLLARY 1. Let (X, σ) be a complete metric-like space and let A and S be two self mappings on X satisfying (2). If S satisfies generalized condition (B) associated with A and the pair (A, S) is weakly compatible, then A and S have a unique common fixed point in X.

If $A = B = id_X$, we get the following corollary.

COROLLARY 2. Let (X, σ) be a complete metric-like space and let S and T be two self mappings on X such that

$$\sigma(Sx, Ty) \leq \delta N(x, y) + L \min\{\sigma(x, Sx), \sigma(y, Ty), \sigma(x, Ty), \sigma(y, Sx)\},$$

where

$$N(x, y) = \min\left\{\sigma(x, y), \sigma(x, Sx), \sigma(y, Ty), \frac{\sigma(x, Ty) + \sigma(y, Sx)}{4}\right\}.$$

Then A and S have a unique common fixed point in X.

In order to support our results, we give the following illustrative examples.

EXAMPLE 5. Let $X = \{0, 1, 2\}$. Define σ as follows:

$$\sigma(0,0) = 0, \sigma(1,1) = 2, \sigma(2,2) = 3,$$
$$\sigma(0,1) = \sigma(1,0) = 1,$$
$$\sigma(0,2) = \sigma(2,0) = 2,$$
$$\sigma(1,2) = \sigma(2,1) = 4.$$

Let A and S be two self mappings such that $A0 = 0, A1 = A2 = 2$ and $S0 = S1 = 0, S2 = 1$. We have also $SX = \{0, 1\} \subset X = AX$. For the inequality (1), we discuss the following cases.

1. For $x = y$ and $x, y \in \{0, 1\}$, we have $\sigma(Sx, Sy) = \sigma(0, 0) = 0$, obviously the inequality (1) holds.
2. For $x = y = 2$, there exists $\delta = \frac{4}{5}$ such that
$$\sigma(S2, S2) = \sigma(1, 1) = 2 \leq \frac{12}{5} = \frac{4}{5}\sigma(A2, A2).$$
3. For $x, y \in \{0.1\}$ and $x \neq y$, obviously the inequality (1) holds.
4. For $x, y \in \{0.2\}$ and $x \neq y$, there exists $\delta = \frac{4}{5}$ such that
$$\sigma(S0, S2) = \sigma(0, 1) = 1 \leq \frac{8}{5} = \frac{4}{5}\sigma(A0, A2).$$
5. For $x, y \in \{1, 2\}$ and $x \neq y$, there exists $\delta = \frac{4}{5}$ such that
$$\sigma(S1, S2) = \sigma(0, 1) = 1 \leq \frac{12}{5} = \frac{4}{5}\sigma(A1, A2).$$

Consequently, all hypotheses of Corollary 2 are satisfied (with $L = 0$), the point 0 is the unique fixed point for A and S.

EXAMPLE 6. Let $X = [0, \infty)$ be endowed with a metric-like: $\sigma(x, y) = \max\{x, y\}$ (it is a partial metric) and let A, B, S and T be four mappings defined by:

$$Ax = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 4, & x > 1 \end{cases}, \quad Bx = \begin{cases} \frac{3}{2}x, & 0 \leq x \leq 1 \\ 2, & x > 1 \end{cases}$$
$$Sx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ \frac{1}{4}, & x > 1 \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0, & 0 \leq x \leq 1 \\ \frac{1}{2}, & x > 1. \end{cases}$$

In this example we will utilize Theorem 1 with $L = 0$. Firstly we have
$$SX = [0, \frac{1}{2}] \subset [0, \frac{3}{2}] \cup \{2\} = BX$$
and
$$TX = \{0, \frac{1}{4}\} \subset [0, 2] \cup \{4\} = AX.$$
For the inequality (1), there exists $\delta = \frac{2}{3}$ such that

1. For $x, y \in [0,1]$, we have

$$\sigma(Sx,Ty) = \frac{x}{2} \leq \frac{4}{3}x = \frac{2}{3}\sigma(Ax,Sx).$$

2. For $x \in [0,1]$ and $y > 1$, we have

$$\sigma(Sx,Ty) = \frac{1}{2} \leq \frac{8}{3} = \frac{2}{3}\sigma(Ax,By).$$

3. For $x > 1$ and $y \in [0,1]$, we have

$$\sigma(Sx,Ty) = \frac{1}{4} \leq \frac{8}{3} = \frac{2}{3}\sigma(Ax,Sx).$$

4. For $x, y \in (1,\infty)$, we have

$$\sigma(Sx,Ty) = \frac{1}{2} \leq \frac{4}{3} = \frac{2}{3}\sigma(By,Ty).$$

Consequently all the conditions of Theorem 1 are satisfied. Moreover the point 0 is the unique common fixed point for A, B, S and T.

3 Applications

In this section, we will apply our results of Corollary 2 to prove the existence of solution for the following system of Fredholm integral equations:

$$\begin{cases} x(t) = \int_0^1 K_1(t,s,x(s))ds, \\ x(t) = \int_0^1 K_2(t,s,x(s))ds, \end{cases} \tag{4}$$

where $K_i : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. For $x, y \in X$ define a metric-like σ as follows:

$$\sigma(x,y) = \|x-y\|_\infty + \|x\|_\infty + \|y\|_\infty,$$

where $\|x(t)\|_\infty = \max_{0 \leq t \leq 1} |x|$. Since (X, d_σ) is a complete metric space, where $d_\sigma(x,y) = 2\|x-y\|_\infty$, so (X,σ) is a σ-complete metric-like space.

THEOREM 2. Assume that:

1. There exists a function $\theta : [0,1] \times [0,1] \to \mathbb{R}_+$ such that

$$\Big| \int_0^1 K_1(t,s,x(s)) - K_2(t,s,y(s))ds \Big| \leq \theta(t,s)|x(t) - y(t)|.$$

2. There exists a function $\eta : [0,1] \times [0,1] \to \mathbb{R}_+$ such that

$$|f(t) + \int_0^1 K_i(t,s,x(s)ds| \leq \eta(t,s)|x(t)|, i = 1,2.$$

3. $$\sup_{t\in[0,1]} \theta(t,s) = \delta_1, \ \sup_{t\in[0,1]} \eta(t,s) = \delta_2 \text{ and } \ \delta = \max\{\delta_1, \delta_2\} < \frac{1}{3}.$$

Then the system (4) has a solution in X.

PROOF. Define

$$Sx(t) = \int_0^1 K_1(t,s,x(s))ds \text{ and } Tx(t) = \int_0^1 K_2(t,s,x(s))ds.$$

The system (4) has a solution, if and only if the two self mappings S and T have a common fixed point in X. Since f and K_i are continuous, so S and T are two self mappings from X into itself. We have also

$$\begin{aligned} |Sx(t) - Ty(t)| &= \left| \int_0^1 K_1(t,s,x(s) - K_2(t,s,y(s)ds \right| \leq \theta(t,s)\, |x(t) - y(t)| \\ &\leq \theta(t,s) \max\{|x(t)-y(t)|, |Sx(t)-x(t)|, |y(t)-Ty(t)|, |x(t), Ty(t)| + |y(t) - S_1|x(t)|\} \\ &\leq \delta_1 \|x(t) - y(t)\|_\infty. \end{aligned}$$

On the other hand, we have

$$|Sx(t)| \leq \eta(t,s)|x|, |Ty(t)| \leq \eta(t,s)|y|,$$

so

$$\|Sx(t)\|_\infty \leq \delta_2, \quad \|Ty(t)\|_\infty \leq \delta_2,$$

which implies that

$$\begin{aligned} \sigma(Sx(t), Ty(t)) &= \|x(t) - y(t)\|_\infty + \|Sx(t)\|_\infty + \|Ty(t)\|_\infty \leq \delta_1 + 2\delta_2 \\ &\leq 3\delta < 1. \end{aligned}$$

Consequently, all hypotheses of Corollary 2 are satisfied (with $L = 0$), then the system (4) has a unique solution.

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