

# Dominating Chromatic Weights Of Graphs\*

Johan Kok<sup>†</sup>, Sudev Naduvath<sup>‡</sup>, Muhammad Kamran Jamil<sup>§</sup>

Received 16 February 2017

## Abstract

Let  $X$  be a minimum dominating set of a graph  $G$ . The application of primary data streaming from an external system into a graph  $G$  can be efficient if the primary streaming is limited to the vertices  $v \in X$  only, then followed by a secondary data streaming from each neighbourhood head  $v$  to  $N(v)$ . If certain weighted concepts  $c_i$ ,  $1 \leq i \leq \ell$  have to find residence with the vertices of  $X$  under minimum or maximum constraints the concepts of minimin, maximin, minimax and maximax sums of weights can be considered. This paper reports on some interesting introductory results.

## 1 Introduction

For general notation and concepts in graphs and digraphs see [1, 2, 6, 11]. Unless mentioned otherwise, all graphs are simple, finite, connected and undirected graphs. Recall that the minimum and maximum vertex degrees of a graph  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$  respectively. Also the order and size of  $G$  are denoted  $\nu(G)$  and  $\varepsilon(G)$ , respectively.

We recall that the domination number of a graph  $G$ , denoted by  $\gamma(G)$ , is the number of vertices in a minimum dominating set say,  $X \subseteq V(G)$ . A vertex  $v \in X$  can be viewed as a neighbourhood head of the closed neighbourhood  $N[v]$ . It is possible to have  $v \neq u$  and  $v \in N[u] \Leftrightarrow u \in N[v]$ . It is also important to note that either  $N(v) \cap N(u) = \emptyset$  or  $N(v) \cap N(u) \neq \emptyset$ . Also,  $\bigcup_{v \in X} N[v] = V(G)$ . Finding a minimum dominating set of a graph  $G$  in general is hard. A few linear algorithms such as the algorithms for finding a minimum dominating set for trees, cactus graphs and parallel-series graphs are known (see [4, 7, 8]). The inherent limitation of these algorithms is that they converge to a minimum dominating set but not all minimum dominating sets are obtained. This study requires all minimum dominating sets of a graph  $G$  to determine the range of the sum of weights.

The application of primary data streaming from an external system into a graph  $G$  can be efficient if the primary streaming is limited to the vertices  $v \in X$  only, then

---

\*Mathematics Subject Classifications: 05C15, 05C38, 05C75, 05C85.

<sup>†</sup>Tshwane Metropolitan Police Department, City of Tshwane, South Africa

<sup>‡</sup>Centre for Studies in Discrete Mathematics, Vidya Academy of Science & Technology, Thrissur, India

<sup>§</sup>Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore, Pakistan

followed by a secondary data streaming protocol from each neighbourhood head  $v$  to  $N(v)$ .

For a graph  $G$ , consider the set of weighted concepts  $\mathcal{W} = \{c_i : 1 \leq i \leq \ell, \ell \geq \chi(G)\}$ . Weighted concepts could be budget allocation, further ICT types or other response units or other. If all vertices  $v \in X$  have to be residence to some and hence not necessary all, of the different weighted concepts  $c_i$ , with the proviso that the number of distinct  $c_i$  allocated must be minimum, without two or more similar weighted concepts being adjacent, the initial allocation is a minimum proper colouring (chromatic colouring) problem in respect of  $G$ . Thereafter, all colouring of vertices  $V(G) - X$  is cancelled.

## 2 Preliminary Results for Certain Graphs

Without loss of generality, consider the weighted concepts,  $c_1 \leq c_2 \leq c_3 \leq \dots \leq c_\ell$ . Further criteria are to minimize the sum of weights or to maximise the sum of weights. Henceforth a weighted concept  $c_i$  will be called the colour  $c_i$ . In all colouring, we follow the *Rainbow Colouring Convention* which is described as follows: Let  $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_{\chi(G)}\}$  be a minimal colouring of  $G$ . Colour the maximum number of vertices with the colour  $c_1$ , then colour maximum possible number of remaining uncoloured vertices with colour  $c_2$  and until all vertices are coloured. Some very good algorithms to find the chromatic number of a graph have been described in [3, 5]. In particular, the algorithm described in [3] provides a chromatic colouring in accordance with the Rainbow Colouring Convention and is called an *optimal independent colouring*. In our application, the colouring will be in the order of increasing consecutive subscripts.

The chromatic number  $\chi(G) \geq 1$  of a graph  $G$  is the minimum number of distinct colours that allow a proper colouring of  $G$ . Such colouring is called a chromatic colouring. In [10], the non-zero number of times a colour  $c_j$  have been allocated is defined as the colour weight,  $\theta(c_j) \geq 1$ . Hence, the colours can be partitioned into non-empty colour classes  $\mathcal{C}_i$ ,  $1 \leq i \leq \chi(G)$  with each colour class say,  $\mathcal{C}_j = \underbrace{(c_j, c_j, c_j, \dots, c_j)}_{\theta(c_j)}$ .

Similar to set theory notation, let  $\mathbb{C} = \bigcup_{1 \leq i \leq \chi(G)} \mathcal{C}_i$ , and be called a colour cluster. It

is easy to see that for  $\chi(G) \geq 2$  a graph  $G^{\mathbb{C}}$  with  $\max \varepsilon(G^{\mathbb{C}})$  that requires the colour cluster  $\mathbb{C}$  to allow exactly a chromatic colouring (minimum proper colouring) is the complete  $\chi(G)$ -partite graph  $K_{\theta(c_1), \theta(c_2), \dots, \theta(c_{\chi(G)})}$  or in other words, the complete  $\ell$ -partite graph with vertex partitioning  $V_i(G^{\mathbb{C}})$ ,  $1 \leq i \leq \ell$  such that  $|V_i(G^{\mathbb{C}})| = |\mathcal{C}_i|$  and  $\bigcup_{1 \leq i \leq \ell} V_i(G^{\mathbb{C}}) = V(G^{\mathbb{C}})$ .

Let  $\chi(G) = t$  and consider any  $t$ -subset of  $\mathcal{W}$ . Clearly, allocating all the permissible chromatic colourings using  $\{c_1, c_2, c_3, \dots, c_t\}$  will provide a range of minimum sum of weights and some chromatic colourings will provide *minimin sum* of weights whilst some others will provide *maximin sum* of weights. Similarly, allocating all the permissible chromatic colourings from  $\{c_{t-1}, c_t, c_{t+1}, \dots, c_\ell\}$  will provide a range of maximum sum of weights, some of which will provide *minimax sum* and *maximax sum* of weights respectively. These extremal values are called *dominating chromatic weights* and will

be denoted by  $\Phi^{(-,-)}(G)$ ,  $\Phi^{(+,-)}(G)$ ,  $\Phi^{(-,+)}(G)$  and  $\Phi^{(+,+)}(G)$  respectively.

To illustrate the principles we consider the path  $P_5 = v_1v_2v_3v_4v_5$  with minimum dominating sets  $\{v_1, v_4\}$ ,  $\{v_2, v_4\}$ ,  $\{v_2, v_5\}$ . By using the reference chromatic colourings  $c(v_1) = c_1$ ,  $c(v_2) = c_2$ ,  $c(v_3) = c_1$ ,  $c(v_4) = c_2$ ,  $c(v_5) = c_1$  and  $c(v_1) = c_2$ ,  $c(v_2) = c_1$ ,  $c(v_3) = c_2$ ,  $c(v_4) = c_1$ ,  $c(v_5) = c_2$ , we obtain the minimum sum of weights to be  $c_1 + c_2$ ,  $2c_2$ ,  $c_2 + c_1$ ,  $c_2 + c_1$ ,  $2c_1$ ,  $c_1 + c_2$ . Therefore,  $\Phi^{(-,-)}(P_5) = 2c_1$  and  $\Phi^{(+,-)}(P_5) = 2c_2$ . Obviously, it follows that  $\Phi^{(-,+)}(P_5) = 2c_{\ell-1}$  and  $\Phi^{(+,+)}(P_5) = 2c_\ell$ .

**THEOREM 2.1.** For all complete graphs  $K_n$  and  $W = \{c_i : 1 \leq i \leq \ell, \ell \geq n\}$ , we have  $\Phi^{(-,-)}(K_n) = c_1$ ,  $\Phi^{(+,-)}(K_n) = c_n$ ,  $\Phi^{(-,+)}(K_n) = c_{\ell-1}$  and  $\Phi^{(+,+)}(K_n) = c_\ell$ .

**PROOF.** For  $\forall n \in \mathbb{N}$ , we have  $\gamma(K_n) = 1$ . Hence, the results follows immediately.

We now present the results for certain well-known graph classes.

**PROPOSITION 2.2.** For all paths  $P_n$ ,  $n \in \mathbb{N}$  and  $W = \{c_i : 1 \leq i \leq \ell, \ell \geq 2\}$ , we have

- (i)  $\Phi^{(-,-)}(P_1) = \Phi^{(+,-)}(P_1) = c_1$  and  $\Phi^{(-,+)}(P_1) = \Phi^{(+,+)}(P_1) = c_\ell$ .
- (ii) For  $n = 2, 3$ ,  $\Phi^{(-,-)}(P_n) = c_1$ ;  $\Phi^{(+,-)}(P_n) = c_2$ ;  $\Phi^{(-,+)}(P_n) = c_{\ell-1}$ ;  $\Phi^{(+,+)}(P_n) = c_\ell$ .
- (iii) For  $n = 4, 5$ ,  $\Phi^{(-,-)}(P_n) = 2c_1$ ;  $\Phi^{(+,-)}(P_n) = 2c_2$ ;  $\Phi^{(-,+)}(P_n) = 2c_{\ell-1}$ ;  $\Phi^{(+,+)}(P_n) = 2c_\ell$ .
- (iv) Let  $n \geq 6$ ,  $n \equiv 0 \pmod{3}$ . Then
  - (a) If  $n$  is even, then

$$\begin{aligned}\Phi^{(-,-)}(P_n) &= \Phi^{(+,-)}(P_n) = \frac{n}{2}(c_1 + c_2), \\ \Phi^{(-,+)}(P_n) &= \Phi^{(+,+)}(P_n) = \frac{n}{2}(c_{\ell-1} + c_\ell).\end{aligned}$$

- (b) If  $n$  is odd, then

$$\begin{aligned}\Phi^{(-,-)}(P_n) &= \lceil \frac{n}{2} \rceil c_1 + \lfloor \frac{n}{2} \rfloor c_2, \\ \Phi^{(+,-)}(P_n) &= \lfloor \frac{n}{2} \rfloor c_1 + \lceil \frac{n}{2} \rceil c_2, \\ \Phi^{(-,+)}(P_n) &= \lceil \frac{n}{2} \rceil c_{\ell-1} + \lfloor \frac{n}{2} \rfloor c_\ell, \\ \Phi^{(+,+)}(P_n) &= \lfloor \frac{n}{2} \rfloor c_{\ell-1} + \lceil \frac{n}{2} \rceil c_\ell.\end{aligned}$$

(v) If  $n \geq 6$  and  $n \equiv 1, 2 \pmod{3}$  and  $\gamma(P_n) = t$ , then

$$\begin{aligned}\Phi^{(-,-)}(P_n) &= \lceil \frac{t}{2} \rceil c_1 + \lfloor \frac{t}{2} \rfloor c_2 \\ \Phi^{(+,-)}(P_n) &= \lfloor \frac{t}{2} \rfloor c_1 + \lceil \frac{t}{2} \rceil c_2, \\ \Phi^{(-,+)}(P_n) &= \lceil \frac{t}{2} \rceil c_{\ell-1} + \lfloor \frac{t}{2} \rfloor c_\ell, \\ \Phi^{(+,+)}(P_n) &= \lfloor \frac{t}{2} \rfloor c_{\ell-1} + \lceil \frac{t}{2} \rceil c_\ell.\end{aligned}$$

PROOF.

- (i) This part of the result is straight forward from the fact that  $\gamma(P_1) = 1$  and  $\chi(P_1) = 1$ .
- (ii) This part immediately follows from the fact that  $\gamma(P_2) = \gamma(P_3) = 1$  and  $\chi(P_2) = \chi(P_3) = 2$ .
- (iii) This part of the proposition follows as a direct consequence of the facts that  $\gamma(P_2) = \gamma(P_3) = 2$  and  $\chi(P_2) = \chi(P_3) = 2$ .
- (iv) Let  $t = \ell - (i - 1)$ . Note that the rotation colour mapping  $c_1 \mapsto c_i, c_2 \mapsto c_{i+1}, c_3 \mapsto c_{i+2}, \dots, c_t \mapsto c_\ell, c_{t+1} \mapsto c_1, c_{t+2} \mapsto c_2, \dots, c_\ell \mapsto c_{i-1}, 1 \leq i \leq \ell$  always results in a chromatic colouring which provides the full range of minimum sum of weights. Therefore,  $\Phi^{(-,-)}(P_n)$  and  $\Phi^{(+,-)}(P_n)$  follow easily by colouring with  $(c_1, c_2)$ . Finally, by colouring with  $(c_{\ell-1}, c_\ell)$  and utilising the rotation colour mapping the values,  $\Phi^{(-,+)}(P_n)$  and  $\Phi^{(+,+)}(P_n)$  follow similarly.
- (v) The proof is exactly the same as that of Part (iv) written above.

PROPOSITION 2.3. For all cycles  $C_n$ ;  $n \geq 3$  and  $W = \{c_i : 1 \leq i \leq \ell\}$ , we have

- (i)  $\Phi^{(-,-)}(C_3) = c_1, \Phi^{(+,-)}(C_3) = c_3, \Phi^{(-,+)}(C_3) = c_{\ell-1}$  and  $\Phi^{(+,+)}(C_3) = c_\ell$ .
- (ii)  $\Phi^{(-,-)}(C_4) = 2c_1; \Phi^{(+,-)}(C_4) = 2c_2; \Phi^{(-,+)}(C_4) = 2c_{\ell-1}$  and  $\Phi^{(+,+)}(C_4) = 2c_\ell$
- (iii)  $\Phi^{(-,-)}(C_5) = 2c_1; \Phi^{(+,-)}(C_5) = 2c_3; \Phi^{(-,+)}(C_5) = 2c_{\ell-1}$  and  $\Phi^{(+,+)}(C_5) = 2c_\ell$ .
- (iv) For  $n \geq 6, n \equiv \pmod{0}$  and

(a) if  $n$  is even, then

$$\begin{aligned}\Phi^{(-,-)}(C_n) &= \Phi^{(+,-)}(C_{3n}) = \frac{n}{2}(c_1 + c_2), \\ \Phi^{(-,+)}(C_n) &= \Phi^{(+,+)}(C_{3n}) = \frac{n}{2}(c_{\ell-1} + c_\ell).\end{aligned}$$

(b) if  $n$  is odd, then

$$\begin{aligned}\Phi^{(-,-)}(C_n) &= \lceil \frac{n}{2} \rceil c_1 + \lfloor \frac{n}{2} \rfloor c_2, \\ \Phi^{(+,-)}(C_n) &= \lfloor \frac{n}{2} \rfloor c_2 + \lceil \frac{n}{2} \rceil c_3, \\ \Phi^{(-,+)}(C_n) &= \lceil \frac{n}{2} \rceil c_{\ell-2} + \lfloor \frac{n}{2} \rfloor c_{\ell-1}, \\ \Phi^{(+,+)}(C_n) &= \lfloor \frac{n}{2} \rfloor c_{\ell-1} + \lceil \frac{n}{2} \rceil c_{\ell}.\end{aligned}$$

(v) For  $n \geq 6$ ,  $n \equiv (\text{mod } 0)$  and

(a) if  $n$  is even and if  $\gamma(C_n) = t$ , then

$$\begin{aligned}\Phi^{(-,-)}(C_n) &= \lceil \frac{t}{2} \rceil c_1 + \lfloor \frac{t}{2} \rfloor c_2, \\ \Phi^{(+,-)}(C_n) &= \lfloor \frac{t}{2} \rfloor c_1 + \lceil \frac{t}{2} \rceil c_2, \\ \Phi^{(-,+)}(C_n) &= \lceil \frac{t}{2} \rceil c_{\ell-1} + \lfloor \frac{t}{2} \rfloor c_{\ell}, \\ \Phi^{(+,+)}(C_n) &= \lfloor \frac{t}{2} \rfloor c_{\ell-1} + \lceil \frac{t}{2} \rceil c_{\ell}.\end{aligned}$$

(b) if  $n$  is odd and if  $\gamma(C_n) = t$ , then

$$\begin{aligned}\Phi^{(-,-)}(C_n) &= \lceil \frac{t}{2} \rceil c_1 + \lfloor \frac{t}{2} \rfloor c_2, \\ \Phi^{(+,-)}(C_n) &= \lfloor \frac{t}{2} \rfloor c_2 + \lceil \frac{t}{2} \rceil c_3, \\ \Phi^{(-,+)}(C_n) &= \lceil \frac{t}{2} \rceil c_{\ell-2} + \lfloor \frac{t}{2} \rfloor c_{\ell-1}, \\ \Phi^{(+,+)}(C_n) &= \lfloor \frac{t}{2} \rfloor c_{\ell-1} + \lceil \frac{t}{2} \rceil c_{\ell}.\end{aligned}$$

PROOF. As  $C_3 = K_3$ , this part of the result follows by Theorem 2.1. We have  $\gamma(P_n) = \gamma(C_n)$  and the addition of edge  $v_1v_n$  results in  $\chi(C_n) = \chi(P_n) + 1$  if  $n$  is odd and  $\chi(C_n) = \chi(P_n)$  if  $n$  is even. Then, the results follow directly from the corresponding results of Proposition 2.2., by substituting the colour  $c_3$  and  $c_{\ell-2}$  accordingly. The rotation colour mapping allows this substitution.

THEOREM 2.4. For any tree  $T$  of order  $n \geq 1$  we have

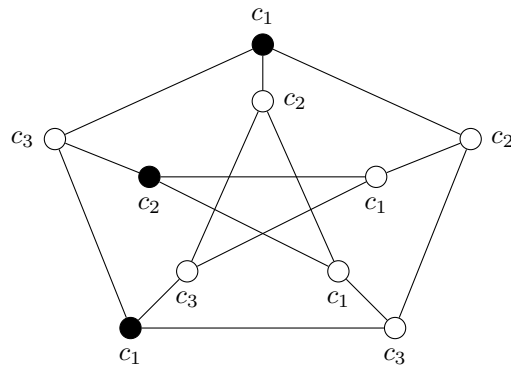
$$\begin{aligned}c_1 &\leq \Phi^{(-,-)}(T) \leq \Phi^{(-,-)}(P_n), \\ c_2 &\leq \Phi^{(+,-)}(T) \leq \Phi^{(+,-)}(P_n), \\ c_{\ell-1} &\leq \Phi^{(-,+)}(T) \leq \Phi^{(-,+)}(P_n), \\ c_{\ell} &\leq \Phi^{(+,+)}(T) \leq \Phi^{(+,+)}(P_n).\end{aligned}$$

PROOF. Note that  $K_{1,0} = K_1$ . Then the result follows directly from the fact that  $\gamma(K_{1,(n-1)}) \leq \gamma(T) \leq \gamma(P_n)$ .

PROPOSITION 2.5. For the Petersen graph  $PG$  and  $W = \{c_i : 1 \leq i \leq \ell, \ell \geq 3\}$ , we have

$$\begin{aligned}\Phi^{(-,-)}(PG) &= 2c_1 + c_2, \\ \Phi^{(+,-)}(PG) &= c_2 + 2c_3, \\ \Phi^{(-,+)}(PG) &= 2c_{\ell-2} + c_{\ell-1}, \\ \Phi^{(+,+)}(PG) &= c_{\ell-1} + 2c_\ell.\end{aligned}$$

PROOF. Consider the Petersen graph given below which depicts both a chromatic colouring and a minimum dominating set (see solid dots in the following figure).



By symmetry and up to isomorphism the minimum dominating set depicted, results in  $\Phi^{(-,-)}(PG) = 2c_1 + c_2$ . Utilising rotation colour mapping, all other results follow.

### 3 Finite Linear Jaco Graphs

For the terms and definitions on the family of *trivial* finite linear Jaco Graphs, see [9]. The linear Jaco graphs and directed graphs are derived from the infinite linear Jaco graph called, the  $x$ -root digraph. Note that the underlying graph will be denoted  $J_n^*(x)$  and if the context is clear, both the directed and undirected graph are referred to as a *linear Jaco graph*. Similarly, the difference between *arc* and *edge* will be understood.

The following are some of the definitions provided in [9], which are relevant in our present study.

DEFINITION 3.1 ([9]). For  $x \in \mathbb{N}$ , the *infinite linear Jaco Graph*, denoted by  $J_\infty(x)$ , is the graph with vertex set is  $V(J_\infty(x)) = \{v_i : i \in \mathbb{N}\}$  such that two vertices  $v_i$  and  $v_j$  are adjacent in  $J_n(x)$  if and only if  $2i - d^-(v_i) \geq j$ . (That is, the edge set (arc set) of  $J_\infty(x)$  denoted by  $A(J_\infty(x)) \subseteq \{(v_i, v_j) : 2i - d^-(v_i) \geq j, i < j\}$ ).

DEFINITION 3.2 ([9]). The family of finite linear Jaco Graphs is defined by  $\{J_n(x) \subseteq J_\infty(x) : n, x \in \mathbb{N}\}$ . A member of the family is referred to as the Jaco Graph,  $J_n(x)$ .

Although linear Jaco graphs are complex in respect of many graph parameters, the family has surprisingly simple results for domination chromatic weights. Note that vertex labeling  $v_1, v_2, v_3, \dots, v_n$  is well defined. For ease of reference we recall three more important definitions.

DEFINITION 3.3 ([9]). A vertex of the linear Jaco graph which attains the maximum degree  $\Delta(J_n(x))$  is called a *Jaconian vertex* of the graph  $J_n(x)$ . The set of all Jaconian vertices of a linear Jaco graph  $J_n(x)$  is called the *Jaconian set* of  $J_n(x)$ . The Jaconian set of  $J_n(x)$  is denoted by  $\mathbb{J}(J_n(x))$  or  $\mathbb{J}_n(x)$  for brevity.

DEFINITION 3.4 ([9]). The lowest numbered (subscripted) Jaconian vertex is called the *prime Jaconian vertex* of a linear Jaco Graph.

DEFINITION 3.5 ([9]). If  $v_i$  is the prime Jaconian vertex of a linear Jaco Graph  $J_n(x)$ , the complete subgraph on vertices  $v_{i+1}, v_{i+2}, \dots, v_n$  is called the *Hope subgraph*<sup>1</sup> or *Hope graph* of a linear Jaco graph and is denoted by  $\mathbb{H}(J_n(x))$  or  $\mathbb{H}_n(x)$  for brevity.

We now state an important lemma to assist in the proof of the main result of this section.

LEMMA 3.1 The vertices of a linear Jaco graph  $J_n^*(x)$  can be coloured such that all vertices in a minimum dominating set  $X$  have identical colours.

PROOF. The basis of the proof through induction is the fact that linear Jaco graphs are well defined in respect of vertex labeling, number of vertices, existence of arcs and the orientation of arcs. It means that  $J_{n+i}(x)$  can be obtained recursively from  $J_n(x)$ ,  $n \in \mathbb{N}$ ,  $i = 1, 2, 3, \dots$ . Therefore,  $J_n^*(x)$  is well defined.

Without loss of generality, let the identical colour to be considered be,  $c_1$ . Clearly for  $J_{1 \leq n \leq 4}^*(x)$  the linear Jaco graphs are paths so the result is trivial. For  $J_5^*(x)$  the set  $\{v_1, v_3\}$  is a minimum dominating set and both vertices may be coloured  $c_1$  since edge  $v_1 v_3$  does not exist. Assume the result holds for  $J_t^*(x)$ ,  $1 \leq t \leq k$ . Now, consider the linear Jaco graph  $J_{k+1}^*(x)$ . Let the highest subscript of a vertex in  $X$  be  $s$ .

If the edge  $v_s v_{k+1}$  exists in  $J_{k+1}(x)$  then  $X$  is a minimum dominating set for  $J_{k+1}(x)$  as well and the result holds. If not, colour  $c(v_{k+1}) = c_1$  and note that no conflict of chromatic colouring arises because of the existence of a Hope graph. Also note that  $X \cup \{v_{k+1}\}$  is a minimum dominating set of  $J_{k+1}(x)$ . Therefore, the result for  $J_n(x)$ ,  $n \in \mathbb{N}$ .

THEOREM 3.2. For a linear Jaco graph  $J_n(x)$ ,  $n \in \mathbb{N}$ , we have

---

<sup>1</sup>Named after the first author's loving mommy, Hope Kok.

$$\begin{aligned}
\text{(i)} \quad \Phi^{(-,-)}(J_n^*(x)) &= \begin{cases} [(n-i)+1]c_1, & \text{if and only if the edge } v_i v_n \text{ exists,} \\ (n-i)c_1, & \text{otherwise.} \end{cases} \\
\text{(ii)} \quad \Phi^{(+,-)}(J_n^*(x)) &= \begin{cases} [(n-i)+1]c_{[(n-i)+1]}, & \text{if and only if the edge } v_i v_n \text{ exists,} \\ (n-i)c_{n-i}, & \text{otherwise.} \end{cases} \\
\text{(iii)} \quad \Phi^{(-,+)}(J_n^*(x)) &= \begin{cases} [(n-i)+1]c_{\ell-(n-i)}, & \text{if and only if the edge } v_i v_n \text{ exists,} \\ (n-i)c_{\ell-(n-i-1)}, & \text{otherwise.} \end{cases} \\
\text{(iv)} \quad \Phi^{(+,+)}(J_n^*(x)) &= \begin{cases} [(n-i)+1]c_\ell, & \text{if and only if the edge } v_i v_n \text{ exists,} \\ (n-i)c_\ell, & \text{otherwise.} \end{cases}
\end{aligned}$$

PROOF. It is known that:

$$\chi(J_n^*(x)) = \begin{cases} (n-i)+1, & \text{if and only if the edge } v_i v_n \text{ exists,} \\ n-i, & \text{otherwise.} \end{cases}$$

Therefore, by Lemma 3.1, colour all vertices in the minimum dominating set  $X$  either  $c_1$  or  $c_{n-i}$  or  $c_{[(n-i)+1]}$  or  $c_{\ell-(n-i)}$  or  $c_{\ell-(n-i-1)}$  or  $c_\ell$ , respectively in accordance to each subcase.

The observation made for linear Jaco graphs leads to the next general result. The result may be found to be very useful for further research.

### 3.1 Further Observations

It is important to note that by convention  $c_i \geq c_j \Leftrightarrow i \geq j$ . However, there is no relationship defined between  $c_i + c_j$  and  $c_{i+j}$ .

**THEOREM 3.3.** For any connected graph  $G$  and any minimum dominating set  $X$  of  $G$ , we have that  $1 \leq \chi(\langle X \rangle) \leq \gamma(G)$ .

PROOF. The lower bound is obvious. Clearly, if  $\chi(\langle X \rangle) = 1$  then  $\langle X \rangle$  is a null graph (edgeless). Also since  $\chi(G) \leq \nu(G)$ , it follows that  $\chi(\langle X \rangle) \leq |X| = \gamma(G)$ .

**THEOREM 3.4.** Consider a connected graph  $G$  with  $\gamma(G) = t$ ,  $\chi(G) = q$  and  $W = \{c_i : 1 \leq i \leq \ell, \ell \geq q\}$ . Then,

$$\begin{aligned}
\Phi^{(-,-)}(G) &\geq tc_1, \\
\Phi^{(+,-)}(G) &\leq tc_q, \\
\Phi^{(-,+)}(G) &\geq tc_{\ell-(q-1)}, \\
\Phi^{(+,+)}(G) &\leq tc_\ell.
\end{aligned}$$

PROOF. All bounds follow from the fact that if there exists a minimum dominating set  $X$  for  $G$  for which  $\langle X \rangle$  is a null graph, then all vertices  $v \in X$  may possibly (not necessarily for all  $G$ ) be coloured the same colour.



Since it is possible to find a chromatic colouring for a connected graph  $G$  such that  $\theta(c_1) \geq \theta(c_2) \geq \theta(c_3) \geq \dots \geq \theta(c_q)$  and  $\chi(G) = q$ , the following proposition discusses dominating chromatic weights for the corona of graphs  $G$  and  $H$ .

**THEOREM 3.5** If a connected graph  $G$  has  $\nu(G) = p$ ,  $\chi(G) = q$  and  $\theta(c_1) \geq \theta(c_2) \geq \theta(c_3) \geq \dots \geq \theta(c_q)$  and  $W = \{c_i : 1 \leq i \leq \ell, \ell \geq q\}$  and any other graph  $H$ , then we have

$$\begin{aligned}\Phi^{(-,-)}(G \circ H) &= \sum_{i=1}^q \theta(c_i) c_i, \\ \Phi^{(+,-)}(G \circ H) &= \sum_{i=1}^q \theta(c_i) c_{q-(i-1)}, \\ \Phi^{(-,+)}(G \circ H) &= \sum_{i=1}^q \theta(c_{q-(i-1)}) c_{\ell-(i-1)}, \\ \Phi^{(+,+)}(G \circ H) &= \sum_{i=1}^q \theta(c_i) c_{\ell-(i-1)}.\end{aligned}$$

**PROOF.** In the corona  $G \circ H$  the set  $V(G)$  is a minimum dominating set. Therefore  $\gamma(G \circ H) = p$ . Since  $G$  is not necessary complete,  $\theta(c_i) \geq 1, \forall i$ . Hence, the results follows directly from the definitions of the dominating chromatic weights.

The following result discusses dominating chromatic weights of the join of two graphs.

**PROPOSITION 3.6.** If a graph  $G$  has  $\chi(G) = q_1$  and  $\theta(c_1) \geq \theta(c_2) \geq \theta(c_3) \geq \dots \geq \theta(c_{q_1})$  and a graph  $H$  has  $\chi(H) = q_2$  and  $\theta'(c_1) \geq \theta'(c_2) \geq \theta'(c_3) \geq \dots \geq \theta'(c_{q_2})$  and  $W = \{c_i : 1 \leq i \leq \ell, \ell \geq q_1 + q_2\}$ , then for the join  $G + H$ , we have

(i) if at least  $\chi(G) = 1$ , then

$$\begin{aligned}\Phi^{(-,-)}(G + H) &= c_1, \\ \Phi^{(+,-)}(G + H) &= c_{q_1+q_2}, \\ \Phi^{(-,+)}(G + H) &= c_{\ell-(q_1+q_2-1)}, \\ \Phi^{(+,+)}(G + H) &= c_\ell.\end{aligned}$$

(ii) if without loss of generality,  $\chi(G) \geq \chi(H) \geq 2$ , then

$$\begin{aligned}\Phi^{(-,-)}(G + H) &= c_1 + c_2, \\ \Phi^{(+,-)}(G + H) &= c_{q_1+q_2-1} + c_{q_1+q_2}, \\ \Phi^{(-,+)}(G + H) &= c_{\ell-(q_1+q_2-1)} + c_{\ell-(q_1+q_2-2)}, \\ \Phi^{(+,+)}(G + H) &= c_{\ell-1} + c_\ell.\end{aligned}$$

PROOF. **Case 1:** If at least say,  $\chi(G) = 1$  then  $\gamma(G + H) = 1$  and the results are immediate. **Case 2:** If without loss of generality,  $\chi(G) \geq \chi(H) \geq 2$ , then  $\gamma(G + H) = 2$ . Because all minimum dominating sets have two adjacent vertices the minimum dominating chromatic weight must be  $c_1 + c_2$ . Also since smallest  $\mathcal{W}$  is the set  $\{c_1, c_2, c_3, \dots, c_{q_1+q_2}\}$ , the maximum dominating chromatic weight must be  $c_{q_1+q_2-1} + c_{q_1+q_2}$ . The last two results follow by similar reasoning for the colours  $c_{\ell-(q_1+q_2-1)}$ ,  $c_{\ell-(q_1+q_2)+2}$ ,  $c_{\ell-1}$  and  $c_\ell$ .

## 4 Conclusion

The paper is an introduction to dominating chromatic weight(s) of a graph. Clearly, there is a wide scope for further research in that many families of graphs remain open. Graph operations should be studied as well as other known variations of domination in graphs.

An interesting observation is that for the path  $P_3$ , we have  $\gamma(P_3) = 1$  and  $\chi(P_3) = 2$  while for the power graph  $P_3^2$  we have  $\gamma(P_3^2) = 1$  and  $\chi(P_3^2) = 3$ . Generally, it is expected that the chromatic number increases for higher powers of  $G$  whereas the domination number decreases. This could be an interesting study in respect of dominating chromatic weights.

Perhaps, the greatest challenge is to efficiently find all distinct minimum dominating sets of a graph  $G$ . Let these minimum dominating sets be  $X_1, X_2, \dots, X_s$ . Then, finding chromatic colourings for the induced subgraphs  $\langle X_1 \rangle, \langle X_2 \rangle, \dots, \langle X_s \rangle$  will provide the result  $\Phi^{(-,-)}(G) = \min\{\Phi^{(-,-)}(\langle X_i \rangle)\}$ ,  $1 \leq i \leq s$ . Similarly, the results  $\Phi^{(+,-)}(G) = \max\{\Phi^{(+,-)}(\langle X_i \rangle)\}$ ,  $\Phi^{(-,+)}(G) = \min\{\Phi^{(-,+)}(\langle X_i \rangle)\}$  and  $\Phi^{(+,+)}(G) = \max\{\Phi^{(+,+)}(\langle X_i \rangle)\}$  for  $1 \leq i \leq s$  will follow through rotation colour mapping.

All the above mentioned facts highlight that there is a wide scope for further investigations in this area.

## References

- [1] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.
- [2] G. Chartrand and L. Lesniak, Graphs and Digraphs, CRC Press, Boca Raton, 2000.
- [3] N. Christofides, An algorithm for the chromatic number of a graph, Comput. J., 14(1)(1971), 39–39.
- [4] E. J. Cockayne, S. Goodman and S.T. Hedetniemi, A linear algorithm for the domination number of a tree, Inform. Process. Lett., 4(1975), 41–44.
- [5] D. G. Cornell and B. Graham, An algorithm for determining the chromatic number of a graph, SIAM J. Comput., 2(4)(1973), 311–318.
- [6] F. Harary, Graph Theory, New Age International, New Delhi, 2001

- [7] S.T. Hedetniemi, R. Laskar and J. Pfaff, A linear algorithm for finding the domination number of a series-parallel graph, *Discrete Appl. Math.*, 13, (1986), 287–292.
- [8] T. Kikuno, N. Yoshida and Y. Kakuda, A linear algorithm for finding a minimum dominating set in a cactus, *Discrete Appl. Math.*, 5(1983), 299–311.
- [9] J. Kok, Linear Jaco graphs: A critical review, *J. Inform. Math. Sci.* 8(2)(2016), 67–103.
- [10] J. Kok, N.K. Sudev and K. P. Chithra, General colouring sums of graphs, *Cogent Math.*, 3(2016), 1–11.
- [11] D. B. West, *Introduction to Graph Theory*, Pearson Education Inc., Delhi, 2001.