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# Some Indices Of Edge Corona Of Two Graphs*

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**Abstract**

The edge corona of two graphs $G$ and $H$, denoted by $G \diamond H$, is obtained by taking one copy of $G$ and $|E(G)|$ copies of $H$ and joining each end vertices of $i$-th edge of $G$ to every vertex in the $i$-th copy of $H$. In this paper, the Hyper Zagreb index of edge corona of two graphs is presented. Also, the vertex Padmakar Ivan (vertex-PI) and the Szeged indices of edge corona of two trees are computed. Moreover, the upper bounds on the multiplicative Zagreb indices and the multiplicative sum Zagreb index of edge corona of two graphs are obtained. As applications, we use these results to compute the Hyper Zagreb index, the vertex-PI index and the Szeged index of certain classes of graphs.

## 1 Introduction

Throughout this paper we consider only simple connected graphs. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the shortest distance between two vertices $u$ and $v$ in $G$ by $d_G(u, v)$ and the degree of a vertex $v$ in $G$ by $d_G(v)$. A topological index is a real number derived from the structure of a graph, which is invariant under graph isomorphism. Many topological indices are closely correlated with some physico-chemical characteristics of the underlying compounds. The first and second Zagreb indices of a graph denoted by $M_1(G)$ and $M_2(G)$, respectively, are degree based topological indices introduced more than thirty years ago by I. Gutman and N. Trinajstic [7]. They are defined as

$$M_1(G) = \sum_{e=uv\in E(G)} (d_G(u) + d_G(v)) = \sum_{v\in V(G)} d_G^2(v),$$

$$M_2(G) = \sum_{e=uv\in E(G)} d_G(u)d_G(v).$$

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These indices were introduced to study the structure-dependency of the total $\pi$-electron energy ($\varepsilon$). It was found that the $\varepsilon$ depends on $M_1(G)$ and thus provides a measure of carbon skeleton of the underlying molecules.

The Hyper Zagreb index of a graph denoted by $HM(G)$, was introduced by Shirdel et al., as a new version of Zagreb index [14]. Aftar that, Basavanagoud et al. found this index for some graph operations [1]. For a graph $G$, the Hyper Zagreb index is defined as follows:

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2.$$

For an edge $e = uv \in E(G)$, the number of vertices of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$ in $G$ is denoted by $n_G(u|e)$; analogously, $n_G(v|e)$ is the number of vertices of $G$ whose distance to the vertex $v$ in $G$ is smaller than the distance to the vertex $u$. Notice that vertices equidistant from $u$ and $v$ are not taken into account. The vertex-PI index of $G$, denoted by $PI_v(G)$, is defined as follows:

$$PI_v(G) = \sum_{e=uv\in E(G)} (n_G(u|e) + n_G(v|e)).$$

We encourage the reader to consult [9, 18] for the chemical applications of the vertex-PI index. The Szeged index is another topological index introduced by Gutman [6]. It has been found useful in modeling various biological activities viz. antihypertensive, antimalarial, antituberculotic, anti HIV, CA inhibitory antagonists, lipoxygenase inhibitory activity, lipophilicity etc. [10]

$$Sz_v(G) = \sum_{e=uv\in E(G)} n_G(u|e)n_G(v|e).$$

Todeschini et al. [16] proposed that multiplicative variants of molecular structure descriptors be considered. When this idea is applied to Zagreb indices, one arrives at their multiplicative versions $\Pi_1(G)$ and $\Pi_2(G)$, defined as

$$\Pi_1(G) = \prod_{v\in V(G)} d^2(v) \quad \text{and} \quad \Pi_2(G) = \prod_{uv\in E(G)} d(v)d(u).$$

Finally, Eliasi et al. [5] defined a multiplicative version of $M_1$ as

$$\Pi_1^*(G) = \prod_{uv\in E(G)} (d(u) + d(v)),$$

and is called as the multiplicative sum Zagreb index by Xu and Das [17].

The edge corona of two graphs $G$ and $H$ denoted by $G \diamond H$ is obtained by taking one copy of $G$ and $|E(G)|$ copies of $H$ and joining each end vertices of $i$-th edge of $G$ to every vertex in the $i$-th copy of $H$ [8, 12, 13]. As an example, we may see Figure 1.

Following Yan et al. [19], the graph $R(G)$ is obtained from $G$ by adding a new vertex corresponding to each edge of $G$, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of $G$ by a triangle. It is clear that if $G$ is a graph and $H$ is a trivial graph, then $G \diamond H \cong R(G)$. In

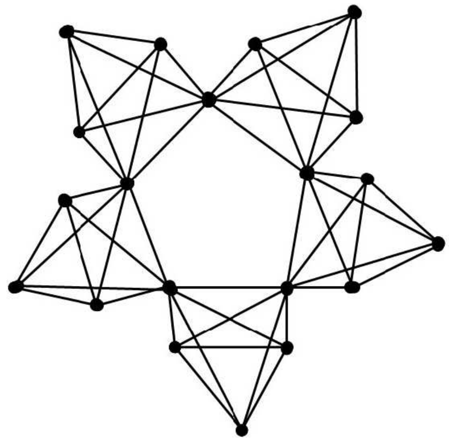

Figure 1: $C_5 \diamond K_3$.

what follows, some indices of the edge corona of two graphs are presented. We denote the path and cycle graphs of order $n$ by $P_n$ and $C_n$, respectively.

Throughout this paper our notation is standard and taken mainly from [2, 4].

## 2 Vertex-PI Index of $T_1 \diamond T_2$

In this section, we compute the vertex-PI index of the edge corona of two trees.

THEOREM 1. If $T_1$ and $T_2$ are trees of orders $n_1$ and $n_2$, respectively, then

$$PI_v(T_1 \diamond T_2) = (n_2+1)^2 PI_v(T_1) + (n_1-1)M_1(T_2) + 2(1-n_1)(3n_2-2).$$

PROOF. We have that

$$PI_v(T_1 \diamond T_2) = \sum_{e=uv\in E(T_1\diamond T_2)} \Big(n_{T_1\diamond T_2}(u|e) + n_{T_1\diamond T_2}(v|e)\Big) = A_1 + A_2 + A_3,$$

where

$$A_1 = \sum_{e=uv\in E_1=E(T_1)} \Big(n_{T_1\diamond T_2}(u|e) + n_{T_1\diamond T_2}(v|e)\Big),$$

$$A_2 = \sum_{e=uv\in E_2=E(T_2)} \Big(n_{T_1\diamond T_2}(u|e) + n_{T_1\diamond T_2}(v|e)\Big),$$

$$A_3 = \sum_{e=uv\in E_3=\{e=uv|u\in V(T_1), v\in V(T_2)\}} \Big(n_{T_1\diamond T_2}(u|e) + n_{T_1\diamond T_2}(v|e)\Big).$$

We have three following cases:

1) If $e = uv \in E_1$, then

$$n_{T_1 \diamond T_2}(u|e) = n_{T_1}(u|e)(1 + n_2) - n_2,$$
$$n_{T_1 \diamond T_2}(v|e) = n_{T_1}(v|e)(1 + n_2) - n_2.$$

So,

$$\begin{aligned} A_1 &= \sum_{e=uv \in E(T_1)} \Big( (n_{T_1}(u|e) + n_{T_1}(v|e))(1 + n_2) - 2n_2 \Big) \\ &= (1 + n_2)PI_v(T_1) - 2n_2(n_1 - 1). \end{aligned}$$

2) Let $e = uv \in E_2$ and $T_{2_i}$, $1 \leq i \leq n_1 - 1$, be a copy of $T_2$ corresponding to the edge $e_i \in E_1$. If $uv \in E(T_{2_i})$, then for every vertex $w$ in $V(T_1 \diamond T_2) \backslash V(T_{2_i})$, we have $d(u, w) = d(v, w)$. Therefore,

$$n_{T_1 \diamond T_2}(u|e) = d_{T_{2_i}}(u) \quad \text{and} \quad n_{T_1 \diamond T_2}(v|e) = d_{T_{2_i}}(v).$$

Hence,

$$A_2 = (n_1 - 1) \sum_{e=uv \in E(T_2)} \Big( d_{T_2}(u) + d_{T_2}(v) \Big) = (n_1 - 1)M_1(T_2).$$

3) Let $e = uv \in E_3$ and $T_{2_i}$ be a copy of $T_2$ corresponding the edge $e_i = ux$ of $E_1$. We have, $n_{T_1 \diamond T_2}(u|e) = B_1 + B_2$, where $B_1$ is the number of all vertices of $(T_1 \diamond T_2) \cap T_{2_i}$ which are closer to $u$ than $v$ in $T_1 \diamond T_2$, and $B_2$ is the number of other vertices of $T_1 \diamond T_2$ which are closer to $u$ than $v$. Clearly, $B_1 = n_2 - d_{T_{2_i}}(v) - 1$, and $B_2 = (n_2 + 1)n_{T_1}(u|e_i) - n_2$. So,

$$n_{T_1 \diamond T_2}(u|e) = B_1 + B_2 = (n_2 + 1)n_{T_1}(u|e_i) - d_{T_{2_i}}(v) - 1.$$

We have also, $n_{T_1 \diamond T_2}(v|e) = 1$, Hence, for an edge $uv \in E_3$,

$$n_{T_1 \diamond T_2}(u|e) + n_{T_1 \diamond T_2}(v|e) = (n_2 + 1)n_{T_1}(u|e_i) - d_{T_{2_i}}(v).$$

Note that, for each edge $e_i = ux \in E_1$, there are $n_2$ edges $uv \in E_3$ and $n_2$ edges $xv \in E_3$. Therefore,

$$\begin{aligned} A_3 &= \sum_{uv \in E_3} (n_{T_1 \diamond T_2}(u|e) + n_{T_1 \diamond T_2}(v|e)) \\ &= n_2(n_2 + 1) \sum_{i=1}^{n_1 - 1} (n_{T_1}(u|e_i) + n_{T_1}(x|e_i)) - 2 \sum_{i=1}^{n_1 - 1} \sum_{v \in V(T_{2_i})} d_{T_{2_i}}(v) \\ &= n_2(n_2 + 1)PI_v(T_1) - 4(n_1 - 1)(n_2 - 1). \end{aligned}$$

Now, by summation of $A_1$, $A_2$ and $A_3$, we obtain the result.

Since $PI_v(P_n) = n(n-1)$ and $M_1(P_n) = 4n - 6$ $(n \geq 2)$, we have the following result.

COROLLARY 1. For $P_n$ and $P_m$,

(i) $PI_v(P_n \diamond P_m) = (m+1)(n-1)(mn+n-2)$.

(ii) $PI_v(P_n \diamond P_n) = (n-1)^2(n+1)(n+2)$.

# 3 Szeged Index of $T_1 \diamond T_2$

By a similar argument applied in the proof of Theorem 1, we have the next theorem.

THEOREM 2. Let $T_1$ and $T_2$ be trees of orders $n_1$ and $n_2$, respectively. Then

$$Sz_v(T_1 \diamond T_2) = (n_2+1)^2 Sz(T_1) + (n_1 - 1)M_2(T_2) + n_2^2 - 6n_1n_2 + 6n_2 + 4n_1 - 4.$$

Since $Sz(P_n) = \frac{1}{6}n(n^2-1)$ and $M_2(P_n) = 4(n-2)$, we have the following result.

COROLLARY 2. For $P_n$ and $P_m$,

(i) $Sz_v(P_n \diamond P_m) = \frac{1}{6}(m^2n^3 - m^2n + 6m^2 + 2mn^3 + 58mn + 12m + n^3 - 25n + 24)$.

(ii) $Sz_v(P_n \diamond P_n) = \frac{1}{6}(n^5 + 2n^4 + 64n^2 - 13n + 24)$.

# 4 Hyper Zagreb Index of $G_1 \diamond G_2$

Pattabiraman and Vijayaragavan have recently calculated the Hyper Zagreb index of edge corona, [11]. In this section, we improve their results. We begin with a crucial lemma, that we will use later on.

LEMMA 1. Let $G_1$ and $G_2$ be two graphs. Then

$$d_{G_1 \diamond G_2}(v) = \begin{cases} (n_2+1)d_{G_1}(v) & v \in V(G_1), \\ d_{G_2}(v) + 2 & v \in V(G_2). \end{cases}$$

Let $G_1$ and $G_2$ be two graphs with vertex sets $V(G_1) = \{v_1, v_2, ..., v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, ..., u_{n_2}\}$ and edge sets $E(G_1) = \{e_1, e_2, ..., e_{m_1}\}$ and $E(G_2) = \{e'_1, e'_2, ..., e'_{m_2}\}$, respectively. Let the vertex set of $i$-th copy of $G_2$ be denoted by $V_i(G_2) = \{u_{i1}, u_{i2}, ..., u_{in_2}\}$.

THEOREM 3. The Hyper Zagreb of $G_1 \diamond G_2$ is given by

$$\begin{aligned} HM(G_1 \diamond G_2) &= (n_2+1)^3 HM(G_1) + m_1 HM(G_2) + 10m_1M_1(G_2) \\ &+ 4(n_2+1)(n_2+m_2)M_1(G_1) - 2n_2(n_2+1)^2 M_2(G_1) \\ &+ 8m_1(n_2 + 4m_2). \end{aligned}$$

PROOF. We have

$$HM(G_1 \diamond G_2) = \sum_{e=uv\in E(G_1\diamond G_2)} (d_{G_1\diamond G_2}(u) + d_{G_1\diamond G_2}(v))^2 = A_1 + A_2 + A_3,$$

where

$$A_1 = \sum_{e=v_iv_j\in E(G_1)} (d_{G_1\diamond G_2}(v_i) + d_{G_1\diamond G_2}(v_j))^2,$$

$$A_2 = \sum_{e_i\in E(G_1)} \sum_{e=u_{ij}u_{ik}\in E_i(G_2)} (d_{G_1\diamond G_2}(u_{ij}) + d_{G_1\diamond G_2}(u_{ik}))^2,$$

$$A_3 = \sum_{e_i=v_lv_m\in E(G_1)} \sum_{u_{ij}\in V_i(G_2)} \Big((d_{G_1\diamond G_2}(u_{ij}) + d_{G_1\diamond G_2}(v_l))^2 + (d_{G_1\diamond G_2}(u_{ij}) + d_{G_1\diamond G_2}(v_m))^2\Big).$$

We now compute $A_i$ for $i = 1, 2, 3$. By Lemma 1, we see that

$$A_1 = (n_2+1)^2 \sum_{e=v_iv_j\in E(G_1)} (d_{G_1}(v_i) + d_{G_1}(v_j))^2 = (n_2+1)^2 HM(G_1),$$

$$\begin{aligned} A_2 &= \sum_{e_i\in E(G_1)} \sum_{e=u_{ij}u_{ik}\in E_i(G_2)} (d_{G_2}(u_{ij}) + d_{G_2}(u_{ik}) + 4)^2 \\ &= \sum_{e_i\in E(G_1)} (HM(G_2) + 16m_2 + 8M_1(G_2)) \\ &= m_1 HM(G_2) + 16m_1m_2 + 8m_1M_1(G_2), \end{aligned}$$

$$\begin{aligned} A_3 &= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{u_{ij}\in V_i(G_2)} \Big(2d^2_{G_1\diamond G_2}(u_{ij}) + d^2_{G_1\diamond G_2}(v_l) + d^2_{G_1\diamond G_2}(v_m) \\ &\quad +2d_{G_1\diamond G_2}(u_{ij})d_{G_1\diamond G_2}(v_l) + 2d_{G_1\diamond G_2}(u_{ij})d_{G_1\diamond G_2}(v_m)\Big) \\ &= \sum_{e_i=v_lv_m\in E(G_1)} \sum_{u_{ij}\in V_i(G_2)} \Big(2(d_{G_2}(u_{ij})+2)^2 + (n_2+1)^2(d^2_{G_1}(v_l) + d^2_{G_1}(v_m)) \\ &\quad +2(d_{G_2}(u_{ij})+2)(n_2+1)(d_{G_1}(v_l) + d_{G_1}(v_m))\Big) \\ &= 2m_1M_1(G_2) + 8n_2m_1 + 16m_2m_1 + (n_2+1)^2 n_2 HM(G_1) \\ &\quad -2(n_2+1)^2 n_2 M_2(G_1) + 4(n_2+1)(m_2+n_2)M_1(G_1). \end{aligned}$$

Note that, $A_3$ is a positive value since $HM(G_1) > 2M_2(G_1)$. So, the summation of $A_1$, $A_2$ and $A_3$ completes the proof.

Using the facts that $M_1(P_n) = 4n - 6$ $(n \geq 2)$, $M_2(P_n) = 4n - 8 (n \geq 3)$, $M_1(C_n) = M_2(C_n) = 4n$, $M_1(K_n) = n(n-1)^2, M_2(K_n) = n(n-1)^3/2$,

$HM(P_n) = 16n - 30$ $(n \geq 3)$, $HM(C_n) = 16n$, $HM(K_n) = 2n(n-1)^3$, [14], and Theorem 3, we obtain the following result.

COROLLARY 3.

1. $HM(P_n \diamond P_m) = 2(4m^3n - 7m^3 + 32m^2n - 53m^2 + 76mn - 97m - 61n + 58)$,
2. $HM(C_n \diamond C_m) = 8n(m^3 + 8m^2 + 21m + 2)$,
3. $HM(P_n \diamond C_m) = 2(4nm^3 + 32nm^2 + 84nm + 8n - 7m^3 - 53m^2 - 109m - 15)$,
4. $HM(C_n \diamond P_m) = 2n(4m^3 + 32m^2 + 76m - 61)$,
5. $HM(K_n \diamond K_m) = n(n-1)(m^3n^2 + 4n^2m^2 + m^4 + 5mn^2 - 4nm^2 + m^3 + 2n^2 - 8nm + m^2 - 4n + 3m + 2)$,
6. $HM(K_n \diamond P_m) = n(n-1)(n^2m^3 + 4n^2m^2 - 2m^3n + 5mn^2 + m^3 + 2n^2 - 6nm - 4m^2 - 8n + 49m - 55)$,
7. $HM(P_n \diamond K_m) = 2(m^4n - m^4 + 10m^3n - 15m^3 + 25m^2n - 42m^2 + 24mn - 43m + 8n - 15)$,
8. $HM(K_n \diamond C_m) = n(n-1)(m^3n^2 + 4m^2n^2 - 2m^3n + 5mn^2 + m^3 + 2n^2 - 2nm - 4m^2 - 4n + 45m + 2)$,
9. $HM(C_n \diamond K_m) = 2n(m^4 + 10m^3 + 25m^2 + 24m + 8)$.

## 5 Multiplicative Zagreb Indices of $G_1 \diamond G_2$

We begin this section with the following standard inequality.

LEMMA 2 (AM-GM inequality). Let $x_1, x_2, ..., x_n$ be nonnegative numbers. Then

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 + x_2 + ... + x_n}$$

holds with equality if and only if all the $x_k$'s are equal.

THEOREM 4. Let $G_1$ and $G_2$ be two graphs. Then

$$\Pi_1(G_1 \diamond G_2) \leq (n_2+1)^{2n_1} \Big[\frac{M_1(G_2) + 4n_2 + 8m_2}{n_2}\Big]^{m_1 n_2} \Pi_1(G_1),$$

$$\begin{aligned}\Pi_2(G_1 \diamond G_2) \quad &\leq \quad (n_2+1)^{2m_1 + 2n_2m_1} \Big[\frac{M_1(G_2) + 4n_2 + 8m_2}{n_2}\Big]^{m_1 n_2} \\ &\times \Big[\frac{M_2(G_2) + 2M_1(G_2) + 4m_2}{m_2}\Big]^{m_1 m_2} (\Pi_2(G_1))^2.\end{aligned}$$

PROOF. By definition of $\Pi_1$, Lemmas 1 and 2, we have

$$\begin{aligned}
\Pi_1(G_1 \diamond G_2) &= \prod_{v\in V(G_1\diamond G_2)} d^2_{G_1\diamond G_2}(v) \\
&= \prod_{v_i\in V(G_1)} d^2_{G_1\diamond G_2}(v_i) \prod_{e_i\in E(G_1)} \prod_{v=u_{ij}\in V_i(G_2)} d^2_{G_1\diamond G_2}(u_{ij}) \\
&= \prod_{v_i\in V(G_1)} ((n_2+1)d_{G_1}(v_i))^2 \prod_{e_i\in E(G_1)} \prod_{v=u_{ij}\in V_i(G_2)} (d_{G_2}(u_{ij})+2)^2 \\
&= (n_2+1)^{2n_1}\Pi_1(G_1) \prod_{e_i\in E(G_1)} \prod_{v=u_{ij}\in V_i(G_2)} (d^2_{G_2}(u_{ij})+4+4d_{G_2}(u_{ij})) \\
&\leq (n_2+1)^{2n_1}\Pi_1(G_1) \prod_{e_i\in E(G_1)} \Big[\frac{\sum_{v=u_{ij}\in V_i(G_2)}(d^2_{G_2}(u_{ij})+4+4d_{G_2}(u_{ij}))}{n_2}\Big]^{n_2} \\
&= (n_2+1)^{2n_1}\Big[\frac{M_1(G_2)+4n_2+8m_2}{n_2}\Big]^{m_1n_2}\Pi_1(G_1).
\end{aligned}$$

On the other hand, by definition of $\Pi_2$, Lemmas 1 and 2,

$$\Pi_2(G_1 \diamond G_2) = \prod_{e=uv\in E(G_1\diamond G_2)} d_{G_1\diamond G_2}(u)d_{G_1\diamond G_2}(v) = A_1A_2A_3$$

where

$$\begin{aligned}
A_1 &= \prod_{e=v_iv_j\in E(G_1)} d_{G_1\diamond G_2}(v_i)d_{G_1\diamond G_2}(v_j), \\
A_2 &= \prod_{e_i\in E(G_1)} \prod_{e=u_{ij}u_{ik}\in E_i(G_2)} d_{G_1\diamond G_2}(u_{ij})d_{G_1\diamond G_2}(u_{ik}), \\
A_3 &= \prod_{e_i=v_lv_m\in E(G_1)} \prod_{u_{ij}\in V_i(G_2)} d^2_{G_1\diamond G_2}(u_{ij})d_{G_1\diamond G_2}(v_l)d_{G_1\diamond G_2}(v_m).
\end{aligned}$$

Now, we compute $A_i$ for $i = 1, 2, 3$. By Lemmas 1 and 2,

$$A_1 = \prod_{e=v_iv_j\in E(G_1)} \big[(n^2+1)^2 d_{G_1}(v_i)d_{G_1}(v_j)\big] = (n_2+1)^{2m_1}\Pi_2(G_1),$$

$$\begin{aligned}
A_2 &= \prod_{e_i\in E(G_1)} \prod_{e=u_{ij}u_{ik}\in E_i(G_2)} (d_{G_2}(u_{ij})+2)(d_{G_2}(u_{ik})+2) \\
&\leq \prod_{e_i\in E(G_1)} \Big[\frac{\sum_{e=u_{ij}u_{ik}\in E_i(G_2)}(d_{G_2}(u_{ij})d_{G_2}(u_{ik})+2(d_{G_2}(u_{ij})+d_{G_2}(u_{ik}))+4)}{m_2}\Big]^{m_2} \\
&= \Big[\frac{M_2(G_2)+2M_1(G_2)+4m_2}{m_2}\Big]^{m_1m_2},
\end{aligned}$$

$$\begin{aligned}
A_3 &= \prod_{e_i=v_lv_m\in E(G_1)}\prod_{u_{ij}\in V_i(G_2)}(d_{G_2}(u_{ij})+2)^2(n_2+1)^2 d_{G_1}(v_l)d_{G_1}(v_m)\\
&\le \prod_{e_i=v_lv_m\in E(G_1)}(n_2+1)^{2n_2}\Big[\frac{\sum_{u_{ij}\in V_i(G_2)}\big(d^2_{G_2}(u_{ij})+4+4d_{G_2}(u_{ij})\big)d_{G_1}(v_l)d_{G_1}(v_m)}{n_2}\Big]^{n_2}\\
&= \prod_{e_i=v_lv_m\in E(G_1)}(n_2+1)^{2n_2}\Big[\frac{M_1(G_2)+4n_2+8m_2}{n_2}\Big]^{n_2} d^{n_2}_{G_1}(v_l)d^{n_2}_{G_1}(v_m)\\
&= (n_2+1)^{2n_2m_1}\Big[\frac{M_1(G_2)+4n_2+8m_2}{n_2}\Big]^{n_2m_1}\Pi_2(G_1).
\end{aligned}$$

By multiplication of $A_1$, $A_2$ and $A_3$ we obtain the result.

THEOREM 5. Let $G_1$ and $G_2$ be two graphs. Then

$$\begin{aligned}
\Pi_1^*(G_1\diamond G_2) \le &\Big[\frac{(m_1+2(n_2+1)(m_2+n_2))M_1(G_1)}{m_1}\\
&+\frac{4m_1(n_2+2m_2)+n_2(n_2+1)^2M_2(G_1)}{m_1}\Big]^{m_1n_2}\\
&\times\Big[\frac{M_1(G_2)+4m_2}{m_2}\Big]^{m_1m_2}\frac{(n_2+1)^{m_1}}{n_2^{n_2m_1}}\Pi_1^*(G_1).
\end{aligned}$$

PROOF. By definition of the multiplicative sum Zagreb index, Lemmas 1 and 2, we have

$$\Pi_1^*(G_1\diamond G_2) = \prod_{e=uv\in E(G_1\diamond G_2)}(d_{G_1\diamond G_2}(u)+d_{G_1\diamond G_2}(v)) = A_1A_2A_3,$$

where

$$A_1 = \prod_{e=v_iv_j\in E(G_1)}(d_{G_1\diamond G_2}(v_i)+d_{G_1\diamond G_2}(v_j)),$$

$$A_2 = \prod_{e_i\in E(G_1)}\prod_{e=u_{ij}u_{ik}\in E_i(G_2)}(d_{G_1\diamond G_2}(u_{ij})+d_{G_1\diamond G_2}(u_{ik})),$$

$$A_3 = \prod_{e_i=v_lv_m\in E(G_1)}\prod_{u_{ij}\in V_i(G_2)}(d_{G_1\diamond G_2}(u_{ij})+d_{G_1\diamond G_2}(v_l))(d_{G_1\diamond G_2}(u_{ij})+d_{G_1\diamond G_2}(v_m)).$$

Now, we compute $A_i$ for $i=1,2,3$.

$$A_1 = \prod_{e=v_iv_j\in E(G_1)}[(n_2+1)(d_{G_1}(v_i)+d_{G_1}(v_j))] = (n_2+1)^{m_1}\Pi_1^*(G_1),$$

$$\begin{aligned}
A_2 &= \prod_{e_i\in E(G_1)}\prod_{e=u_{ij}u_{ik}\in E_i(G_2)}(d_{G_2}(u_{ij})+d_{G_2}(u_{ik})+4)\\
&\le \prod_{e_i\in E(G_1)}\Big[\frac{\sum_{e=u_{ij}u_{ik}\in E_i(G_2)}(d_{G_2}(u_{ij})+d_{G_2}(u_{ik})+4)}{m_2}\Big]^{m_2}\\
&= \Big[\frac{M_1(G_2)+4m_2}{m_2}\Big]^{m_1m_2},
\end{aligned}$$

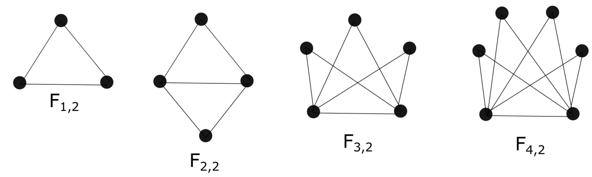


Figure 2: Some examples of fan graphs.

$$\begin{aligned}
A_3 &= \prod_{e_i=v_lv_m\in E(G_1)}\prod_{u_{ij}\in V_i(G_2)}\big[(d_{G_2}(u_{ij})+2)^2+(d_{G_2}(u_{ij})+2)(n_2+1)d_{G_1}(v_m)\\
&\quad +(d_{G_2}(u_{ij})+2)(n_2+1)d_{G_1}(v_l)+(n_2+1)^2(d_{G_1}(v_l)d_{G_1}(v_m))\big]\\
&\leq \frac{1}{n_2^{n_2m_1}}\prod_{e_i=v_lv_m\in E(G_1)}\Big[M_1(G_2)+4n_2+8m_2+2(m_2+n_2)(n_2+1)\\
&\quad \times(d_{G_1}(v_l)+d_{G_1}(v_m))+n_2(n_2+1)^2(d_{G_1}(v_l)d_{G_1}(v_m))\Big]^{n_2}\\
&\leq \frac{1}{n_2^{n_2m_1}}\big[\frac{m_1M_1(G_2)+2(n_2+1)(m_2+n_2)M_1(G_1)}{m_1}\\
&\quad +\frac{4m_1(n_2+2m_2)+n_2(n_2+1)^2M_2(G_1)}{m_1}\big]^{m_1n_2}.
\end{aligned}$$

It is easy to see that the multiplication of $A_1$, $A_2$ and $A_3$ completes the proof.

# 6 Application

Reducing the problem of computing indices of edge corona of two graphs, to the problem of computing some parameters of the factor graphs will be desirable. For example, The sun graph, $S_n$, is an important class of graphs is defined in terms of edge corona product. There are several differing definitions of the sun graph. In [3, 15], defines a sun graph $S_n$ as a graph obtained by replacing every edge of a cycle $C_n$ by a triangle $C_3$. A sun graph also has $2n$ vertices. It is clear that if $G \cong C_n$ and $H \cong K_1$, then $G \diamond H \cong S_n$. Hence by Corollary 3 and Theorem 4, we can establish the following result.

COROLLARY 4. For a sun graph $S_n$, we have

1. $HM(S_n) = 136n$,
2. $\Pi_1(S_n) \leq 64^n$, actually $\Pi_1(S_n) = 64^n$ and it shows that the bound in Theorem 4 is sharp.

Another example is fan graph $F_{r,2}$, which is isomorphic to the $K_2 \diamond \bar{K}_r$, Figure 2 shows some fan graphs, Hence by Corollary 3 and Theorem 4, we have the following result.

COROLLARY 5. For a fan graph $F_{r,2}$, we have

1. $HM(F_{r,2}) = 2r^3 + 16r^2 + 26r + 4$,
2. $\Pi_1(F_{r,2}) \leq 4^r(r+1)^4$, this bound is sharp.

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