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Travelling Profile Solutions For Nonlinear Degenerate Parabolic Equation And Contour Enhancement In Image Processing

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Abstract

We propose in this work to find explicit exact solutions called travelling profile solutions to a nonlinear diffusion equation that occurs in image processing. Some of these explicit solutions are related with the phenomenon of contour enhancement in image processing. We present a generalization of the results obtained by Barenblatt to study the contour enhancement in image processing for exponent range of parameter enhancement.

1 Introduction

Many models which use nonlinear diffusion equations are proposed in image processing and contour enhancement. The technique of using PDEs in the image edge has firstly been introduced by Perona and Malik [6]. They have proved that image intensity flux can lead to an enhancement of image edge if the flux is directed opposite to the image intensity gradient. Different approaches to the edge enhancement problem were proposed by several authors [1]. In particular, Malladi and Sethian [7] proposed a model based on the differential-geometric approach, this model leads (after proper scaling) to the following equation for image intensity:

$$u_t(x,t) = \frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{(1+u_x^2+u_y^2)^{1+\gamma}}, \tag{1}$$

where $u(x,y,t)$ is the image intensity flux and $\gamma \geq 0$ is a positive constant. This equation represents a general movement by curvature flow.

The asymptotic treatment of these models shows the enhancement of the intensity contrasts by formation of regions of large intensity gradients.

In order to focus on the boundary layer where large gradients concentrate, further simplifications of this model were proposed in [2], and lead to unidimensional version of (1):

$$\frac{\partial u}{\partial t} = u_x^{-2(1+\gamma)} u_{xx}. \tag{2}$$

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This equation governing the evolution of the image intensity in the boundary layer. In the range $\gamma > 0$, equation (2) falls into the class of degenerate parabolic equations with degeneracy at $u_x = \infty$.

In this paper, we first propose to find exact solutions to equation (2) by introducing the "travelling profile solutions" which have been proposed by Benhamidouche et al. in [4, 5], in seeking exact solutions for some nonlinear partial differential equations. We seek solutions in the following form:

$$u(x,t) = c(t) f\left(\frac{x-b(t)}{a(t)}\right), \tag{3}$$

where the coefficients $a(t)$, $b(t)$, $c(t)$ are functions which depend on time t and f is the basic profile, these functions are to be determined.

On the other hand, we want to generalize the results proposed in [2], and we show that such a behavior can be observed in the larger exponent range $\gamma > \frac{-1}{2}$, directly by using solutions (3). We note that this problem has been investigated by Vasquez and Barenblatt in [3] but with complicated conjugate formulation.

2 Exact Solutions to Nonlinear Diffusion Equation

We now construct exact solutions of equation (2) and give a prescription for properties such as the asymptotic behavior, blow up, etc., for some exact solutions. For that, we use the travelling profile solution as a special form of exact solutions to nonlinear partial differential equations. Travelling profile solutions of equation (2) are represented in the following form:

$$u(x,t) = c(t) f(\xi) \quad \text{with } \xi = \frac{x-b(t)}{a(t)} \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{4}$$

where the parameters $a(t)$, $c(t)$, $b(t)$ and the profile f should be determined. This form permits us reducing the PDE (2) to an ODE.

Substituting the form of solutions (4) into equation (2), we obtain:

$$\frac{\dot{c}}{c} f - \frac{\dot{a}}{a}\xi f_\xi - \frac{\dot{b}}{a} f_\xi = \frac{c^{-2-2\gamma}}{a^{-2\gamma}} f_\xi^{-2(1+\gamma)} f_{\xi\xi}. \tag{5}$$

A simple separation of variables argument implies that the following conditions must hold:

$$\begin{cases} \frac{\dot{c}}{c} = \alpha \frac{c^{-2-2\gamma}}{a^{-2\gamma}}, \\ \frac{\dot{a}}{a} = -\beta \frac{c^{-2-2\gamma}}{a^{-2\gamma}}, \\ \frac{\dot{b}}{a} = -\lambda \frac{c^{-2-2\gamma}}{a^{-2\gamma}}, \end{cases} \tag{6}$$

with parameters $\alpha, \beta, \lambda \in \mathbb{R}$, and the basic profile f must satisfy the following ordinary differential equation in ξ:

$$f_\xi^{-2(1+\gamma)} f_{\xi\xi} = \alpha f + \beta \xi f_\xi + \lambda f_\xi. \tag{7}$$

We have to solve system (6) to find the coefficients $c(t), a(t)$ and $b(t)$. Indeed, from (6) we have:

$$\begin{cases} c(t) = K_0 a(t)^{\frac{-\alpha}{\beta}}, \\ b(t) = \frac{\lambda}{\beta} a(t) + A_1, \end{cases} \tag{8}$$

where $K_0 > 0, A_1$ are integration constants. Inserting (8) into (6), we get:

(i) **In the first case**, we have two time behaviors of the coefficients $c(t), a(t)$ and $b(t)$. They are given by:

$$\begin{cases} a(t) = \left[K_0^{-2-2\gamma} S\beta t + A_0\right]^{\frac{-1}{S}}, \\ c(t) = K_0 \left[K_0^{-2-2\gamma} S\beta t + A_0\right]^{\frac{\alpha}{\beta S}}, \\ b(t) = \frac{\lambda}{\beta} \left[K_0^{-2-2\gamma} S\beta t + A_0\right]^{\frac{-1}{S}} + A_1, \end{cases} \quad \text{for } 0 < t < T \text{ and } S\beta < 0, \tag{9}$$

with

$$T = -\frac{A_0 K_0^{2+2\gamma}}{S\beta}.$$

And

$$\begin{cases} a(t) = \left[K_0^{-2-2\gamma} S\beta t + A_0\right]^{\frac{-1}{S}}, \\ c(t) = K_0 \left[K_0^{-2-2\gamma} S\beta t + A_0\right]^{\frac{\alpha}{\beta S}}, \\ b(t) = \frac{\lambda}{\beta} \left[K_0^{-2-2\gamma} S\beta t + A_0\right]^{\frac{-1}{S}} + A_1, \end{cases} \quad \text{for } 0 < t < \infty \text{ and } S\beta > 0, \tag{10}$$

with

$$S = \frac{2(1+\gamma)\alpha + 2\gamma\beta}{\beta} \quad \text{and} \quad A_0 > 0,\ K_0 > 0,\ A_1 \text{ are constants.}$$

(ii) **In the second case** (for $S = 0$), we have:

$$\begin{cases} a(t) = A_0 \exp(-K_0^{-2-2\gamma} \beta t), \\ c(t) = K_0 A_0^{-\frac{\alpha}{\beta}} \exp(K_0^{-2-2\gamma} \alpha t), \\ b(t) = \frac{\lambda}{\beta} A_0 \exp(-K_0^{-2-2\gamma} \beta t) + A_1, \end{cases} \quad \text{for } 0 < t < \infty, \tag{11}$$

with $A_0 > 0$, $K_0 > 0$, A_1 are constants.

We have three time behaviors of coefficients $c(t)$, $a(t)$ and $b(t)$; these behaviors depend on parameters of similarity α and β.

3 Some Explicit Exact Solutions of Nonlinear Diffusion Equation

We consider now two interesting particular cases according to the values of α, β and λ, and we seek new exact solutions of (2), we show also the asymptotic behaviors of the solutions obtained.

3.1 Case1: $\alpha = 0,\ \beta \in \mathbb{R}_+^*,\ \lambda \in \mathbb{R}$

A particular case in the discussion now is the case where $\alpha = 0$, $\beta \in \mathbb{R}_+^*$, $\lambda \in \mathbb{R}$. This corresponds to the following form of solutions:

$$u(x,t) = c_0 f(\xi) \quad \text{with } \xi = \frac{x - b(t)}{a(t)}, \tag{12}$$

with interesting application of these solutions in image processing generalizing the results proposed in [2].

To analyze the process of contour enhancement in image processing and resolving the free boundary problem formulated with equation (2), Barenblatt [2] has introduced the "intermediate asymptotic solution" :

$$u(x,t) = c_0 f(\xi) \quad \text{with } \xi = \frac{x - x_0}{(t + t_0)^{-\frac{1}{2\gamma}}},$$

for a constant $c_0 > 0$, and x_0, t_0 fixed, the profile f is an increasing function to be determined. He proves that the rate of enhancement depends on the hypotheses concerning the image intensity flow (the parameter γ).

The mathematical problem was investigated by Barenblatt [2] and consists in solving this equation with suitable boundary data, namely, $u = 0$ on the left-hand side of the contour and $u = 1$ on the right-hand side, and initial conditions $u(x,0) = u_0(x)$, satisfying $0 < u_0 < 1$ and $u_0' > 0$ in an interval $I = (a,b)$ and constant values otherwise, *zero* to the left, 1 to the right, joining the levels $f = 0$ at a finite distance $\xi = -c < 0$ to the level $f = 1$ at $\xi = c$. At these levels, when taken at a finite distance $\xi = \pm c$, the gradients are infinite.

In this model, the phenomenon of gradient enhancement takes place for all $\gamma > 0$: The spatial gradient of the solutions, u_x, increases with time, and its support shrinks [2, 3]. Indeed, the scaling implies that:

$$u_x(x,t) = c_0 (t + t_0)^{1/2\gamma} f'(\xi) \quad \text{with } \xi = \frac{x - x_0}{(t + t_0)^{-\frac{1}{2\gamma}}},$$

which shows that the solution is concentrated in an increasingly narrower strip $S = \{(x,t) : |x - x_0| \leq c(t + t_0)^{-1/2\gamma}\}$ with gradients that diverge like $t^{1/2\gamma}$ as $t \to \infty$.

With the form of solutions (12), we can generalize these results. Indeed, if we replace this form of solutions in (2), we obtain:

$$-\frac{\dot{a}}{a}\xi f_\xi' - \frac{\dot{b}}{a} f_\xi' = \frac{c_0^{-2-2\gamma}}{a^{-2\gamma}} f_\xi^{-2(1+\gamma)} f_{\xi\xi}. \tag{13}$$

A separation of variables argument implies that:

$$\begin{cases} \frac{\dot{a}}{a} = -\beta \frac{c_0^{-2-2\gamma}}{a^{-2\gamma}}, \\ \frac{\dot{b}}{a} = -\lambda \frac{c_0^{-2-2\gamma}}{a^{-2\gamma}}, \end{cases} \tag{14}$$

where β, λ are arbitrary constants and $\gamma > \frac{-1}{2}$.

The equation for the profile f (13) becomes:

$$(\beta\xi+\lambda)\frac{df}{d\xi}=\left(\frac{df}{d\xi}\right)^{-2(1+\gamma)}\frac{d^2f}{d\xi^2}. \tag{15}$$

According to different value of parameter γ, we discuss the solutions of (14) and (15).

The case: $\gamma\in\left]\frac{-1}{2};\infty\right[-\{0\}$:

The resolution of system (14) is not difficult, indeed from (14) we have:

$$b(t)=\frac{\lambda}{\beta}a(t)+A_1. \tag{16}$$

If we replace (16) in (14), we get

$$\begin{cases} a(t)=\left(\frac{2\gamma\beta}{c_0^{2(1+\gamma)}}t+A_0\right)^{\frac{-1}{2\gamma}}, \\ b(t)=\frac{\lambda}{\beta}\left(\frac{2\gamma\beta}{c_0^{2(1+\gamma)}}t+A_0\right)^{\frac{-1}{2\gamma}}+A_1, \end{cases} \quad \text{for } 0<t<\infty, \tag{17}$$

and

$$\begin{cases} a(t)=\left(\frac{2\gamma\beta}{c_0^{2(1+\gamma)}}t+A_0\right)^{\frac{-1}{2\gamma}}, \\ b(t)=\frac{\lambda}{\beta}\left(\frac{2\gamma\beta}{c_0^{2(1+\gamma)}}t+A_0\right)^{\frac{-1}{2\gamma}}+A_1, \end{cases} \quad \text{for } 0<t<T=-\frac{A_0c_0^{2(1+\gamma)}}{2\gamma\beta}, \tag{18}$$

where $\beta>0$, and $A_0>0$, A_1 are constants.

The equation for the profile f (15) can be written as:

$$\left(\frac{df}{d\xi}\right)^{-1-2(1+\gamma)}\frac{d^2f}{d\xi^2}=(\beta\xi+\lambda)\quad\text{where } \beta>0 \text{ and } \lambda\in\mathbb{R},$$

after integration, we obtain

$$\frac{-1}{2(1+\gamma)}\left(\frac{df}{d\xi}\right)^{-2(1+\gamma)}=\beta\frac{\xi^2}{2}+\lambda\xi+k,\quad\text{with } k \text{ constant},$$

and another integration gives

$$\frac{df}{d\xi}=\left[\frac{1}{1+\gamma}\right]^{\frac{1}{2(1+\gamma)}}C^{\frac{-1}{1+\gamma}}\left[1-\left(\frac{\beta\xi+\lambda}{C\sqrt{\beta}}\right)^2\right]^{\frac{-1}{2(1+\gamma)}}, \tag{19}$$

for

$$-\frac{\lambda+C\sqrt{\beta}}{\beta}\leq\xi\leq-\frac{\lambda-C\sqrt{\beta}}{\beta}.$$

For $\xi=\frac{x-b(t)}{a(t)}$, this relation suggests a free-boundary problem for determination of the image intensity evolution in the boundary layer $l(t)$ and $r(t)$ such that

$$l(t)=b(t)-\frac{\lambda+C\sqrt{\beta}}{\beta}a(t)\leq x\leq b(t)-\frac{\lambda-C\sqrt{\beta}}{\beta}a(t)=r(t),$$

where $a(t)$, $b(t)$ are given by (17) and (18). If we replace $b(t)$ "given by (16)" we obtain

$$l(t) = A_1 - \frac{C}{\sqrt{\beta}} a(t) \leq x \leq A_1 + \frac{C}{\sqrt{\beta}} a(t) = r(t).$$

Here, C is an integration constant. Further integration and using the boundary conditions $f\left(-\frac{\lambda + C\sqrt{\beta}}{\beta}\right) = 0$, $f\left(-\frac{\lambda - C\sqrt{\beta}}{\beta}\right) = 1$, then we obtain

$$f(\xi) = \frac{1}{\sqrt{\beta}} \left[\frac{1}{1+\gamma}\right]^{\frac{1}{2(1+\gamma)}} C^{\frac{\gamma}{1+\gamma}} \int_{-1}^{\frac{\beta\xi+\lambda}{C\sqrt{\beta}}} \frac{d\eta}{\left[1-\eta^2\right]^{\frac{1}{2(1+\gamma)}}}$$

and

$$C = \left[\frac{2}{\sqrt{\beta}} \left[\frac{1}{1+\gamma}\right]^{\frac{1}{2(1+\gamma)}} \int_{0}^{1} \frac{d\eta}{\left[1-\eta^2\right]^{\frac{1}{2(1+\gamma)}}}\right]^{-\frac{\gamma+1}{\gamma}}.$$

Thus, the travelling profile solutions assumes the form:

$$u(x,t) = c_0 \left[\frac{1}{\sqrt{\beta}} \left[\frac{1}{1+\gamma}\right]^{\frac{1}{2(1+\gamma)}} C^{\frac{\gamma}{1+\gamma}} \int_{-1}^{\sqrt{\beta}\frac{x-A_1}{Ca(t)}} \frac{d\eta}{\left[1-\eta^2\right]^{\frac{1}{2(1+\gamma)}}}\right].$$

The gradients are given by

$$u_x(x,t) = \frac{c_0}{a(t)} f'(\xi).$$

So by (19), we have

$$u_x(x,t) = \frac{c_0}{a(t)} \left[\frac{1}{1+\gamma}\right]^{\frac{1}{2(1+\gamma)}} C^{\frac{-1}{1+\gamma}} \left[1 - \left(\sqrt{\beta}\frac{x-A_1}{Ca(t)}\right)^2\right]^{\frac{-1}{2(1+\gamma)}}.$$

Therefore, the validity of the asymptotic equation (2) improves with time. Thus, the gradients blow up like "$O(t^{\frac{1}{2\gamma}})$ " as $t \to \infty$, and like "$O((T-t)^{\frac{1}{2\gamma}})$" as $t \to T$. The width of the transition region $r(t) - l(t)$ is equal to

$$2\frac{C}{\sqrt{\beta}} a(t) = 2\frac{C}{\sqrt{\beta}} \left(\frac{2\gamma\beta}{c_0^{2(1+\gamma)}} t + A_0\right)^{\frac{-1}{2\gamma}},$$

which decreases with time.

In this case the phenomenon of gradient enhancement takes place: The spatial gradient of the solutions u_x increases with time, and its support shrinks, see figures 1 and 2.

The case: $\gamma = 0$:

The equation (2) becomes:

$$u_t = u_x^{-2} u_{xx}. \tag{20}$$

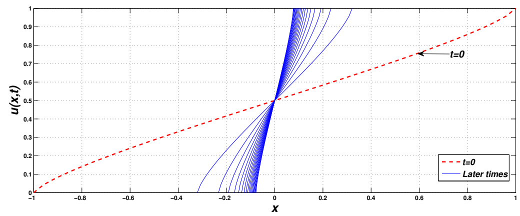

Figure 1: The evolution of the image intensity distribution $u(x,t)$ for $\gamma = 1$, "the Beltrami flow".

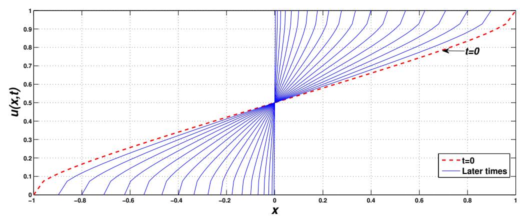

Figure 2: The evolution of the image intensity distribution $u(x,t)$ for $\gamma = -1/4$.

If we replace the solution (12) in equation (20), we obtain

$$-\frac{\dot{a}}{a}\xi f'_\xi - \frac{\dot{b}}{a} f'_\xi = c_0^{-2} f_\xi^{-2} f_{\xi\xi}. \tag{21}$$

A separation of variables argument implies that the following conditions must hold:

$$\begin{cases} \frac{\dot{a}}{a} = -\beta c_0^{-2}, \\ \frac{\dot{b}}{a} = -\lambda c_0^{-2}, \end{cases} \tag{22}$$

where β, λ are arbitrary constants.

The resolution of system (22) gives:

$$\begin{cases} a(t) = A_0 e^{-\beta c_0^{-2} t}, \\ b(t) = \frac{\lambda}{\beta} A_0 e^{-\beta c_0^{-2} t} + A_1, \end{cases} \quad \text{for } 0 < t < \infty. \tag{23}$$

The equation for the profile f becomes:

$$(\beta\xi + \lambda)\frac{df}{d\xi} = \left(\frac{df}{d\xi}\right)^{-2} \frac{d^2 f}{d\xi^2}, \tag{24}$$

which can be written as

$$\left(\frac{df}{d\xi}\right)^{-3} \frac{d^2 f}{d\xi^2} = (\beta\xi + \lambda) \ \text{ where } \beta \in \mathbb{R}_+^* \text{ and } \lambda \in \mathbb{R},$$

after integration, we obtain

$$\frac{df}{d\xi} = \frac{1}{C}\left[1 - \left(\frac{\beta\xi + \lambda}{C\sqrt{\beta}}\right)^2\right]^{\frac{-1}{2}},$$

where $C > 0$ is an integration constant. By using the boundary conditions $f\left(-\frac{\lambda + C\sqrt{\beta}}{\beta}\right) = 0$ and $f\left(-\frac{\lambda - C\sqrt{\beta}}{\beta}\right) = 1$, we obtain, after integration:

$$f(\xi) = \frac{1}{\pi}\arcsin\left(\frac{\pi^2\xi + \lambda}{\pi C}\right) + \frac{1}{2}; \quad \beta = \pi^2,$$

for

$$\frac{-C\pi - \lambda}{\pi^2} \le \xi \le \frac{C\pi - \lambda}{\pi^2}.$$

Thus, the travelling profiles solutions assumes the form

$$u(x,t) = c_0\left[\frac{1}{\pi}\arcsin\left(\pi\frac{x - A_1}{Ca(t)}\right) + \frac{1}{2}\right]$$

for

$$l(t) = b(t) - \frac{\lambda + C\pi}{\pi^2}a(t) \le x \le b(t) - \frac{\lambda - C\pi}{\pi^2}a(t) = r(t),$$

where $a(t)$ and $b(t)$ are given by (23). We can write:

$$l(t) = A_1 - \frac{C}{\pi}a(t) \le x \le A_1 + \frac{C}{\pi}a(t) = r(t).$$

The gradients are given by

$$u_x(x,t) = \frac{c_0}{a(t)}f'(\xi) = \frac{c_0}{Ca(t)}\left[1 - \left(\pi\frac{x - A_1}{Ca(t)}\right)^2\right]^{\frac{-1}{2}}.$$

The gradients blow up exponentially as $t \to \infty$. The width of the transition region $r(t) - l(t)$ is given by $2\frac{C}{\pi}A_0 e^{-\left(\frac{\pi}{c_0}\right)^2 t}$ which decreases with time. In this case the phenomenon of edge enhancement takes place for this class of equations ($\gamma = 0$), see figure 3.

3.2 Case2: $\alpha = \beta \in \mathbb{R},\ \lambda \in \mathbb{R}$

Now, we discuss another case so we assume that

$$\int_{\mathbb{R}} u(s,t)\,ds = const.$$

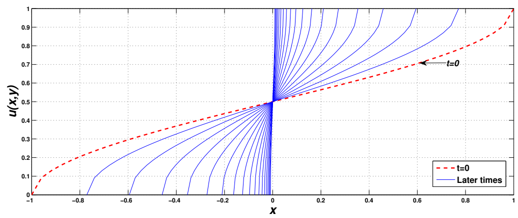

Figure 3: The evolution of the image intensity distribution $u(x,t)$ for $\gamma = 0$, "the mean curvature flow".

We can see that from (4):

$$\int_{\mathbb{R}} u(s,t)\,ds = \int_{\mathbb{R}} c(t)\, f\left(\frac{s-b(t)}{a(t)}\right) ds = a(t)\,c(t)\int_{\mathbb{R}} f(\xi)\,d\xi = const,$$

this implies

$$c(t) = \frac{const}{a(t)}, \tag{25}$$

from relation (8) and (25), we have $\alpha = \beta$, this case corresponds to physical law of conservation of mass.

In this case, new explicit solutions are obtained to equation (2), which depends on the parameter γ.

The case: $\gamma \in \left]\frac{-1}{2};\infty\right[- \{0\}$:

The equation of profile f (7) becomes:

$$f_{\xi}^{-2(1+\gamma)} f_{\xi\xi} = \left[(\beta\xi + \lambda) f\right]_{\xi} \quad \text{where } \alpha, \lambda \in \mathbb{R}. \tag{26}$$

After integration, we obtain:

$$\frac{-1}{2\gamma+1} f_{\xi}^{-2\gamma-1} = (\beta\xi + \lambda) f + k \quad \text{with } k \text{ is constant}.$$

If we put for example $f(0) = f_{\xi}(0) = 0$, then $k = 0$ and we obtain:

$$f_{\xi}^{-2\gamma-1} = -(2\gamma+1)(\beta\xi+\lambda) f,$$

which implies that

$$f^{\frac{1}{2\gamma+1}}\,df = \left((2\gamma+1)(-\beta\xi-\lambda)\right)^{\frac{-1}{2\gamma+1}} d\xi.$$

Integrating once more, we get

$$\frac{2\gamma+1}{2\gamma+2} f^{\frac{2\gamma+2}{2\gamma+1}} = \frac{(2\gamma+1)^{\frac{2\gamma}{2\gamma+1}}}{-2\gamma\beta}(-\beta\xi-\lambda)^{\frac{2\gamma}{2\gamma+1}} + K, \quad \text{for } \gamma \in \left]\frac{-1}{2};\infty\right[- \{0\},$$

where K is an integration constant.

Then, the solution of (26) is written under the form:

$$f(\xi)=\left[(\gamma+1)(2\gamma+1)^{\frac{-1}{2\gamma+1}}\right]^{\frac{2\gamma+1}{2\gamma+2}}\left[\left(\frac{-1}{\gamma\beta}\left[(-\beta\xi-\lambda)_+\right]^{\frac{2\gamma}{2\gamma+1}}+K\right)_+\right]^{\frac{2\gamma+1}{2\gamma+2}}$$

for $\gamma\in\left]\frac{-1}{2};\infty\right[-\{0\}$. Finally, an explicit exact solution to the nonlinear degenerate parabolic equation (2) is given as follows:

$$u(x,t)=c(t)\left[(\gamma+1)(2\gamma+1)^{\frac{-1}{2\gamma+1}}\right]^{\frac{2\gamma+1}{2\gamma+2}}\left[\left(\frac{-1}{\gamma\beta}\left[(-\beta\frac{x-b(t)}{a(t)}-\lambda)_+\right]^{\frac{2\gamma}{2\gamma+1}}+K\right)_+\right]^{\frac{2\gamma+1}{2\gamma+2}} \tag{27}$$

for $\gamma\in\left]\frac{-1}{2};\infty\right[-\{0\}$. The coefficients $c(t)$, $a(t)$ and $b(t)$ are given, explicitly, by:

$$\begin{cases} a(t)=\left[K_0^{-2-2\gamma}S\beta t+A_0\right]^{\frac{-1}{S}}, \\ c(t)=K_0\left[K_0^{-2-2\gamma}S\beta t+A_0\right]^{\frac{1}{S}}, \\ b(t)=\frac{\lambda}{\alpha}\left[K_0^{-2-2\gamma}S\beta t+A_0\right]^{\frac{-1}{S}}+A_1, \end{cases} \quad \text{for } 0<t<T \text{ and } \beta<0, \tag{28}$$

with

$$T=\frac{-A_0K_0^{2+2\gamma}}{S\beta}.$$

And

$$\begin{cases} a(t)=\left[K_0^{-2-2\gamma}S\beta t+A_0\right]^{\frac{-1}{S}}, \\ c(t)=K_0\left[K_0^{-2-2\gamma}S\beta t+A_0\right]^{\frac{1}{S}}, \\ b(t)=\frac{\lambda}{\alpha}\left[K_0^{-2-2\gamma}S\beta t+A_0\right]^{\frac{-1}{S}}+A_1, \end{cases} \quad \text{for } 0<t<\infty \text{ and } \beta>0, \tag{29}$$

where

$$S=2(1+2\gamma)\ >0\ \text{ for } \gamma\in\left]\frac{-1}{2};\infty\right[-\{0\}\ \text{ and } A_0,K_0>0,\ A_1 \text{ are constants.}$$

The case: $\gamma=0$:

Clearly, if we set $\gamma=0$, in (9) and (10), the coefficients $c(t)$, $a(t)$ and $b(t)$ are given by:

$$\begin{cases} a(t)=\left[K_0^{-2}2\alpha t+A_0\right]^{\frac{-1}{2}}, \\ c(t)=K_0\left[K_0^{-2}2\alpha t+A_0\right]^{\frac{1}{2}}, \\ b(t)=\frac{\lambda}{\alpha}\left[K_0^{-2}2\alpha t+A_0\right]^{\frac{-1}{2}}+A_1, \end{cases} \quad \text{for } 0<t<T \text{ and } \beta=\alpha<0, \tag{30}$$

with

$$T = \frac{-A_0 K_0^2}{2\alpha}.$$

And

$$\begin{cases} a(t) = \left[K_0^{-2} 2\alpha t + A_0\right]^{\frac{-1}{2}}, \\ c(t) = K_0 \left[K_0^{-2} 2\alpha t + A_0\right]^{\frac{1}{2}}, \\ b(t) = \frac{\lambda}{\alpha} \left[K_0^{-2} 2\alpha t + A_0\right]^{\frac{-1}{2}} + A_1, \end{cases} \quad \text{for } 0 < t < \infty \text{ and } \beta = \alpha > 0, \tag{31}$$

where $A_0, K_0 > 0$, A_1 are constants.

The equation of profile f (7) becomes:

$$f_\xi^{-2} f_{\xi\xi} = \left[(\alpha\xi + \lambda) f\right]_\xi \quad \text{where } \alpha, \lambda \in \mathbb{R}. \tag{32}$$

Integration of (32) and putting $f(0) = f_\xi(0) = 0$ and after a routine computation yield

$$f_\xi f = \frac{-1}{\alpha\xi + \lambda}.$$

Integrating once more, we obtain:

$$\frac{1}{2} f^2 = \frac{-1}{\alpha} \ln|\alpha\xi + \lambda| + K,$$

where K is an integration constant.

Therefore, the solution of (32) is written under the form:

$$f(\xi) = \left[\left(\frac{-2}{\alpha} \ln|\alpha\xi + \lambda| + K\right)_+\right]^{\frac{1}{2}}.$$

Returning to the variables x and t we obtain:

$$u(x,t) = c(t) \left[\left(\frac{-2}{\alpha} \ln\left|\alpha \frac{x - b(t)}{a(t)} + \lambda\right| + K\right)_+\right]^{\frac{1}{2}}, \tag{33}$$

where $c(t)$, $a(t)$ and $b(t)$ are given by (30) and (31).

4 Conclusion

In this work we have presented some explicit exact solutions to a nonlinear degenerate parabolic equation that occurs in image processing. We have introduced a general form of self similar solutions, called "travelling profiles solutions". We have also given general results obtained for a free boundary problem investigated by Barenblatt [2], to study the contour enhancement in image processing. We have obtained new results in the larger exponent range of parameter enhancement $\gamma > \frac{-1}{2}$.

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