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An Elementary Upper Bound For The Number Of Generic Quadrisecants Of Polygonal Knots*

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Abstract

Let K be a polygonal knot in general position with vertex set V. A *generic quadrisecant* of K is a line that is disjoint from the set V and intersects K in exactly four distinct points. We give an upper bound for the number of generic quadrisecants of a polygonal knot K in general position. This upper bound is in terms of the number of edges of K.

1 Introduction

In this article, we study polygonal knots in three dimensional space that are in general position. Given such a knot K, we define a *quadrisecant* of K as an unoriented line that intersects K in exactly four distinct points. We require that these points are not vertices of the knot, in which case we say that the quadrisecant is *generic*.

Using geometric and combinatorial arguments, we give an upper bound for the number of generic quadrisecants of a polygonal knot K in general position. This bound is in terms of the number $n \geq 3$ of edges of K. More precisely, we prove the following.

THEOREM 1. Let K be a polygonal knot in general position, with exactly n edges. Then K has at most $U_n = \frac{n}{12}(n-3)(n-4)(n-5)$ generic quadrisecants.

Applying Theorem 1 to polygonal knots with few edges, we obtain the following.

1. If $n \leq 5$, then K has no generic quadrisecant.

2. If $n = 6$, then K has at most three generic quadrisecants.

3. If $n = 7$, then K has at most 14 generic quadrisecants.

Using a result of G. Jin and S. Park ([3]), we can prove that the above bound is sharp for $n = 6$. In other words, a hexagonal trefoil knot has exactly three quadrisecants, all of which are generic.

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Quadrisecants of polygonal knots in $\mathbb{R}^3$ have been studied by many people, such as E. Pannwitz, H. Morton, D. Mond, G. Kuperberg and E. Denne. The study of quadrisecants started in 1933 with E. Pannwitz's doctoral dissertation ([7]). There, she found a lower bound for the number of quadrisecants of non-trivial generic polygonal knots. This bound is in terms of the minimal number of boundary singularities for a disk bounded by K. Later, H. Morton and D. Mond ([5]) proved that every non-trivial generic knot has a quadrisecant, and G. Kuperberg extended their result to non-trivial tame knots and links ([4]). More recently, E. Denne ([1]) proved that essential alternating quadrisecants exist for all non-trivial tame knots.

Notation

Unless otherwise stated, all polygonal knots studied in this article are embedded in the three-dimensional Euclidean space $\mathbb{R}^3$. Such a knot will be denoted by K. The cardinality of a set A is denoted by $|A|$. Given a set A, with $|A| = n$, $\binom{n}{k}$ denotes the number of subsets of A of cardinality k. The symbol $\sqcup$ denotes the disjoint union of sets.

2 Preliminaries

It is well-known that a triple of pairwise skew lines E_1, E_2, E_3 determines a unique quadric. This quadric is a doubly-ruled surface S that is either a hyperbolic paraboloid, if the three lines are parallel to one plane, or a hyperboloid of one sheet, otherwise (see for example [2]). The lines E_1, E_2, E_3 belong to one of the rulings of S, and every line intersecting all those three lines belongs to the other ruling of S. Further, every point in S lies on a unique line from each ruling (see [1], [6] and [8]).

We now define the type of polygonal knots that we will consider in this article.

DEFINITION 1. We say that the polygonal knot K in $\mathbb{R}^3$ is in *general position* if the following conditions are satisfied:

(i) No four vertices of K are coplanar.

(ii) Given three edges e_1, e_2, e_3 of K that are pairwise skew, no other edge of K is contained in the quadric generated by e_1, e_2, e_3.

The quadrisecants of knots that we will study are defined as follows.

DEFINITION 2. Let K be a polygonal knot in general position with vertex set V. A *generic quadrisecant* of K is an unoriented line that is disjoint from the set V and intersects K in exactly four distinct points.

In this paper we are interested in giving an upper bound for the number of generic quadrisecants of a polygonal knot K in general position. This upper bound is in terms of the number of edges of K. We start by estimating the number of generic quadrisecants that intersect a given collection of four edges of K that are pairwise skew.

PROPOSITION 1. Let K be a knot in general position. Let $\mathcal{E}_4$ be a collection of four distinct edges of K that are pairwise skew. Then there are at most two generic quadrisecants of K that intersect all edges in $\mathcal{E}_4$.

PROOF. Let e_1, e_2, e_3, e_4 be the four edges in the collection $\mathcal{E}_4$. Each edge e_i generates a line E_i $(i = 1, 2, 3, 4)$. Let S be the doubly-ruled quadric generated by E_1, E_2, E_3. Since K is in general position, the edge e_4 is not contained in S. Therefore, e_4 intersects the quadric S in at most two points.

Let $\mathcal{Q}_{\mathcal{E}_4}$ be the set of all generic quadrisecants of K that intersect all edges in $\mathcal{E}_4$. For $l \in \mathcal{Q}_{\mathcal{E}_4}$, we define the point p_l as the point of intersection between the edge e_4 and the line l. Since l intersects all lines E_1, E_2, E_3, then it belongs to a ruling $\mathcal{R}$ of S. Also, $p_l \in e_4 \cap S$, and so the cardinality of the set $\{p_l \colon l \in \mathcal{Q}_{\mathcal{E}_4}\}$ is at most two. To complete the proof, we show that the function $l \mapsto p_l$ is one-to-one. Suppose that $p_l = p_{l'}$, where $l \in \mathcal{Q}_{\mathcal{E}_4}$ and $l' \in \mathcal{Q}_{\mathcal{E}_4}$. Then the point $p_l = p_{l'}$ lies in two lines, l and l', that belong to the ruling $\mathcal{R}$ of S. Since every point in S lies on a unique line from $\mathcal{R}$, then $l = l'$.

Our next result complements Proposition 1.

PROPOSITION 2. Let K be a knot in general position. Let $\mathcal{E}_4$ be a collection of four distinct edges of K, two of which are coplanar. Then there is at most one generic quadrisecant of K that intersects all edges in $\mathcal{E}_4$.

PROOF. Let e_1, e_2, e_3, e_4 be the four edges in the collection $\mathcal{E}_4$, and suppose that e_1 and e_2 lie in a plane P. By general position, e_1 and e_2 are adjacent edges. Arguing toward a contradiction, suppose that l_1 and l_2 are two distinct generic quadrisecants of K that intersect all edges in $\mathcal{E}_4$.

Since e_1 and e_2 lie in P, then the same is true for l_1 and l_2. Since both l_1 and l_2 intersect the edge e_i, then so does P $(i = 3, 4)$. By general position, the edge e_i intersects P in a single point p_i, which is a point of intersection between the lines l_1 and l_2 $(i = 3, 4)$. Thus, $p_3 = p_4$, and so the edge e_3 intersects the edge e_4. This means that the point $p_3 = p_4$ is a vertex of both e_3 and e_4, and this vertex is different from those of edges e_1 and e_2 (because K is a knot). This contradicts general position.

3 Quadrisecants Intersecting Consecutive Edges of the Knot

To prove some of the results in the next section, we will need to analyze collections of edges of a polygonal knot that have the property defined below.

DEFINITION 3. Let $\mathcal{E}'$ be a collection of distinct edges of a polygonal knot K. We will say that the edges in $\mathcal{E}'$ are *consecutive* if their union (with the subspace topology induced from K) is connected.

Since two consecutive edges of a polygonal knot are always coplanar, then Proposition 2 implies the following.

PROPOSITION 3. Let K be a knot in general position. Let $\mathcal{E}_4$ be a collection of four distinct edges of K that contains a pair of consecutive edges. Then there is at most one generic quadrisecant of K that intersects all edges in $\mathcal{E}_4$.

We now investigate the existence of generic quadrisecants intersecting two or three consecutive edges of a polygonal knot.

PROPOSITION 4. There are no generic quadrisecants of K intersecting three distinct consecutive edges of K.

PROOF. Let n be the number of edges of K. If $n = 3$, then the result is clear. Suppose that $n > 3$ and that l is a generic quadrisecant that intersects three distinct consecutive edges of K. Then the plane P that contains l and one of the three consecutive edges also contains the other two edges. Since $n > 3$, then the endpoints of the three consecutive edges are four distinct vertices of K, and these vertices lie in the plane P. This contradicts that K is in general position.

Proposition 4 has the following immediate corollary.

COROLLARY 1. There are no generic quadrisecants of K intersecting four distinct consecutive edges of K.

The following proposition complements Proposition 3.

PROPOSITION 5. Let $\mathcal{E}_4$ be a collection of four distinct edges of K that contains no pair of consecutive edges. Then there are at most two generic quadrisecants of K that intersect all edges in $\mathcal{E}_4$.

PROOF. If all edges in $\mathcal{E}_4$ are pairwise skew, then Proposition 1 implies that there are at most two generic quadrisecants of K intersecting all edges in $\mathcal{E}_4$. If the collection $\mathcal{E}_4$ contains a pair of coplanar edges, then Proposition 2 implies that there is at most one generic quadrisecant of K intersecting all edges in $\mathcal{E}_4$.

4 Combinatorial Results

For a collection $\mathcal{E}_4$ of four distinct edges of the knot K, the following theorem gives an upper bound for the number of generic quadrisecants of K that intersect all edges in $\mathcal{E}_4$.

THEOREM 2. Let K be a polygonal knot in general position. Given a collection $\mathcal{E}_4$ of four distinct edges of K, consider the union $X_{\mathcal{E}_4}$ of the edges in $\mathcal{E}_4$ (with the subspace topology induced from K). Let c be the number of connected components of the space $X_{\mathcal{E}_4}$.

(i) If $c = 1$, then there are no generic quadrisecants intersecting all edges in $\mathcal{E}_4$.

(ii) If $c = 2$, and one of the connected components of $X_{\mathcal{E}_4}$ consists of a single edge of K, then there are no generic quadrisecants intersecting all edges in $\mathcal{E}_4$.

(iii) If $c = 2$, and each of the connected components of $X_{\mathcal{E}_4}$ is the union of exactly two consecutive edges of K, then there is at most one generic quadrisecant intersecting all edges in $\mathcal{E}_4$.

(iv) If $c = 3$, then there is at most one generic quadrisecant intersecting all edges in $\mathcal{E}_4$.

(v) If $c = 4$, then there are at most two generic quadrisecants intersecting all edges in $\mathcal{E}_4$.

PROOF. We divide the proof into four cases.

Case 1: $c = 1$. In this case Corollary 1 implies the result.

Case 2: $c = 2$. If one of the connected components of $X_{\mathcal{E}_4}$ consists of a single edge, then the result follows from Proposition 4. Otherwise, the result follows from Proposition 3.

Case 3: $c = 3$. Since the collection $\mathcal{E}_4$ contains a pair of consecutive edges, then Proposition 3 implies the result.

Case 4: $c = 4$. Since $\mathcal{E}_4$ contains no pair of consecutive edges, then the result follows from Proposition 5.

To obtain an upper bound for the number of generic quadrisecants of a knot, we need to consider the number of collections of four distinct edges of the knot for each of the cases stated in Theorem 2. These numbers are defined as follows.

DEFINITION 4. Let K be a polygonal knot in general position with exactly n edges. For a collection $\mathcal{E}_4$ of four distinct edges of K, consider the union $X_{\mathcal{E}_4}$ of the edges in $\mathcal{E}_4$ (with the subspace topology induced from K).

(i) For $c = 1, 2, 3, 4$, let $S_c^{(n)}(K)$ be the number of collections $\mathcal{E}_4$ of four distinct edges of K such that $X_{\mathcal{E}_4}$ has exactly c connected components.

(ii) For $c = 2$ we also define the following.

  (a) Let $S_{2,1}^{(n)}(K)$ be the number of collections $\mathcal{E}_4$ of four distinct edges of K such that $X_{\mathcal{E}_4}$ has exactly two connected components, and one of these components consists of a single edge.

  (b) Let $S_{2,2}^{(n)}(K)$ be the number of collections $\mathcal{E}_4$ of four distinct edges of K such that $X_{\mathcal{E}_4}$ has exactly two connected components, and each of these components is the union of exactly two consecutive edges.

By definition,

$$S_2^{(n)}(K) = S_{2,1}^{(n)}(K) + S_{2,2}^{(n)}(K); \tag{1}$$

$$S_1^{(n)}(K) + S_{2,1}^{(n)}(K) + S_{2,2}^{(n)}(K) + S_3^{(n)}(K) + S_4^{(n)}(K) = \binom{n}{4}. \tag{2}$$

Combining Theorem 2 with Definition 4, we obtain an upper bound for the number of generic quadrisecants of a polygonal knot in general position.

COROLLARY 2. Let K be a polygonal knot in general position with exactly n edges. Then the number $U_n = S_{2,2}^{(n)}(K) + S_3^{(n)}(K) + 2S_4^{(n)}(K)$ is an upper bound for the number of generic quadrisecants of K.

In our next result we find explicit formulas for the numbers $S_c^{(n)}(K)$'s.

THEOREM 3. Let K be a polygonal knot in general position with exactly n edges. Then

$$S_1^{(n)}(K) = \begin{cases} 0 & \text{if } n = 3 \\ 1 & \text{if } n = 4 \\ n & \text{if } n \geq 5; \end{cases} \tag{3}$$

$$S_{2,1}^{(n)}(K) = \begin{cases} 0 & \text{if } n \leq 5 \\ n(n-5) & \text{if } n \geq 6; \end{cases} \tag{4}$$

$$S_{2,2}^{(n)}(K) = \begin{cases} 0 & \text{if } n \leq 5 \\ \frac{n(n-5)}{2} & \text{if } n \geq 6; \end{cases} \tag{5}$$

$$S_3^{(n)}(K) = \begin{cases} 0 & \text{if } n \leq 6 \\ \frac{n(n-5)(n-6)}{2} & \text{if } n \geq 7; \end{cases} \tag{6}$$

$$S_4^{(n)}(K) = \begin{cases} 0 & \text{if } n \leq 7 \\ \binom{n}{4} - \frac{n(n-5)(n-6)}{2} - \frac{n(n-5)}{2} - n(n-5) - n & \text{if } n \geq 8. \end{cases} \tag{7}$$

PROOF. Fix an orientation of K and an edge e_1 of K. Suppose that $e_1, e_2, \cdots, e_n$ (in that order) are all the distinct edges of K that we encounter when we follow the orientation of K, starting and ending at the initial point of e_1. For the rest of the proof, the subindices of the edges e_j's are understood modulo n.

Proof of equation 3. Clearly, $S_1^{(n)}(K) = 0$ for $n = 3$ and $S_1^{(n)}(K) = 1$ for $n = 4$. Suppose that $n \geq 5$. Let $\mathcal{E}_4$ be a collection of four distinct edges of K such that $X_{\mathcal{E}_4}$ is connected. The collection $\mathcal{E}_4$ is completely determined by the only integer $i \in \{1, 2, \cdots, n\}$ such that $\mathcal{E}_4 = \{e_i, e_{i+1}, e_{i+2}, e_{i+3}\}$. Since this number i can be chosen in n different ways, then $S_1^{(n)}(K) = n$.

Proof of equation 4. If $n \leq 5$, then clearly $S_{2,1}^{(n)}(K) = 0$ and $S_{2,2}^{(n)}(K) = 0$. For the proof of equations 4 and 5, we will assume that $n \geq 6$.

Let $\mathcal{E}_4$ be a collection of four distinct edges of K such that $X_{\mathcal{E}_4}$ has exactly two connected components, X_1 and X_2, with X_1 consisting of a single edge of K. Let $\mathcal{E}_3$ be the collection of the three consecutive edges in X_2. There are n different ways to choose the collection $\mathcal{E}_3$. Once we have chosen the three edges e_i, e_{i+1}, e_{i+2} in X_2, the edge in X_1 has to be different from the edges $e_{i-1}, e_i, e_{i+1}, e_{i+2}, e_{i+3}$. Thus, given the edges in X_2, the edge in X_1 can be chosen in $n-5$ different ways. Hence, the number $S_{2,1}^{(n)}(K)$ is equal to $n(n-5)$.

Proof of equation 5. We may assume that $n \geq 6$. Let $\mathcal{E}_4$ be a collection of four distinct edges of K such that $X_{\mathcal{E}_4}$ has exactly two connected components, X_1 and X_2, with each X_i being the union of exactly two consecutive edges of K. There are n different ways to choose the collection of edges in X_1. Once we have chosen the two edges in X_1, the edges in X_2 can be chosen in $n-5$ different ways. However we are double-counting, as interchanging the collections X_1 and X_2 produces the same collection $X_{\mathcal{E}_4}$. Therefore, $S_{2,2}^{(n)}(K) = \frac{n(n-5)}{2}$.

Proof of equation 6. We may assume that $n \geq 7$. Let $\mathcal{E}_4$ be a collection of four distinct edges of K such that $X_{\mathcal{E}_4}$ has exactly three connected components, X_1, X_2 and X_3, with X_1 being the union of exactly two edges of K. There are n different ways to choose the collection of edges in X_1. Once we have chosen the two edges in X_1, the two edges in $X_2 \sqcup X_3$ can be chosen in $\binom{n-4}{2} - k$ different ways, where k is the number of different ways to choose a collection of two consecutive edges out of $n-4$ edges. Since $k = n-5$, then the collection of edges in $X_2 \sqcup X_3$ can be chosen in $\binom{n-4}{2} - (n-5) = \frac{(n-5)(n-6)}{2}$ different ways. Hence, the number $S_3^{(n)}(K)$ is equal to $\frac{n(n-5)(n-6)}{2}$.

Proof of equation 7. We may assume that $n \geq 8$. By equation 2,

$$S_4^{(n)}(K) = \binom{n}{4} - S_1^{(n)}(K) - S_{2,1}^{(n)}(K) - S_{2,2}^{(n)}(K) - S_3^{(n)}(K).$$

Thus, equation 7 follows from equations 3 to 6.

5 The Main Result

Combining Corollary 2 with Theorem 3, we obtain an *explicit* upper bound for the number of generic quadrisecants of a polygonal knot in general position.

COROLLARY 3. Let K be a polygonal knot in general position with exactly n edges.

1. If $n \leq 5$, then K has no generic quadrisecant.
2. If $n = 6$, then K has at most three generic quadrisecants.

3. If $n = 7$, then K has at most 14 generic quadrisecants.

4. If $n \geq 8$, then K has at most $\dfrac{n}{12}(n-3)(n-4)(n-5)$ generic quadrisecants.

PROOF. By Corollary 2, the knot K has at most $U_n = S_{2,2}^{(n)}(K) + S_3^{(n)}(K) + 2S_4^{(n)}(K)$ generic quadrisecants.

1. Suppose that $n \leq 5$. Then $S_{2,2}^{(n)}(K) = 0 = S_3^{(n)}(K) = S_4^{(n)}(K)$, and so $U_n = 0$.

2. Suppose that $n = 6$. Then $S_{2,2}^{(6)}(K) = 3$, $S_3^{(6)}(K) = 0$ and $S_4^{(6)}(K) = 0$, so $U_n = 3$.

3. Suppose that $n = 7$. Then $S_{2,2}^{(7)}(K) = 7$, $S_3^{(7)}(K) = 7$ and $S_4^{(7)}(K) = 0$, so $U_n = 14$.

4. Suppose that $n \geq 8$. By equation 2,

$$S_4^{(n)}(K) = \binom{n}{4} - S_1^{(n)}(K) - S_{2,1}^{(n)}(K) - S_{2,2}^{(n)}(K) - S_3^{(n)}(K).$$

   Thus,

$$U_n = 2\binom{n}{4} - 2S_1^{(n)}(K) - 2S_{2,1}^{(n)}(K) - S_{2,2}^{(n)}(K) - S_3^{(n)}(K). \tag{8}$$

   By Theorem 3, equation 8 becomes:

$$U_n = \frac{1}{12}n(n-1)(n-2)(n-3) - 2n - 2n(n-5) - \frac{n(n-5)}{2} - \frac{n(n-5)(n-6)}{2}. \tag{9}$$

   Equation 9 can be written as $\frac{n}{12}(n-3)(n-4)(n-5)$.

Notice that the expression $\dfrac{n}{12}(n-3)(n-4)(n-5)$ from Corollary 3 is equal to zero for $n = 3, 4, 5$; it is equal to three for $n = 6$, and it is equal to 14 for $n = 7$. This means that Corollary 3 can be reformulates as follows.

THEOREM 4. Let K be a polygonal knot in general position with exactly n edges. Then K has at most $U_n = \dfrac{n}{12}(n-3)(n-4)(n-5)$ generic quadrisecants.

References

[1] E. J. Denne, Alternating quadrisecants of knots, ProQuest LLC, Ann Arbor, MI, 2004, Thesis (Ph.D.)–University of Illinois at Urbana-Champaign.

[2] D. Hilbert and S. Cohn-Vossen, Geometry and the Imagination, Chelsea Publishing Company, New York, N. Y., 1952, Translated by P. Neményi.

[3] G. T. Jin and S. Park, Quadrisecant approximation of hexagonal trefoil knot, J. Knot Theory Ramifications, 20(2011), 1685–1693.

[4] G. Kuperberg, Quadrisecants of knots and links, J. Knot Theory Ramifications, 3(1994), 41–50.

[5] H. R. Morton and D. M. Q. Mond, Closed curves with no quadrisecants, Topology, 21(1982), 235–243.

[6] J.-P. Otal, Une propriété de géométrie élémentaire des nœuds, Le journal de maths des élèves, 1(1994), 34–39.

[7] E. Pannwitz, Eine elementargeometrische Eigenschaft von Verschlingungen und Knoten, Math. Ann., 108(1933), 629–672.

[8] H. Pottmann and J. Wallner, Computational Line Geometry, Paperback edition [of MR1849803]. Mathematics and Visualization, Springer-Verlag, Berlin, 2010. x+563 pp.