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Application Of Dominating Sets In Vague Graphs*

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Abstract

The concept of vague graph was introduced by Ramakrishna in [12]. A vague graph is a generalized structure of a fuzzy graph that gives more precision, flexibility, and compatibility to a system when compared with systems that are designed using fuzzy graphs. The main purpose of this paper is to introduce the concept of dominating set, perfect dominating set, minimal perfect dominating set and independent dominating set in vague graphs and obtain some interesting results for these new parameters. Finally, we have given some applications of dominating sets in vague graphs and other sciences.

1 Introduction

In 1736, Euler first introduced the notion of graph theory. In the history of mathematics, the solution given by Euler of the well known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of a combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science. In 1965, Zadeh [24] first proposed the theory of fuzzy sets. In 1975, Rosenfeld [17] introduced the concept of fuzzy graph theory as a generalization of Euler's graph. Gau and Buehrer [10] proposed the concept of vague set in 1993, by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, a true-membership function $t_v(x)$ and a false membership function $f_v(x)$ are used to describe the boundaries of the membership degree. Ramakrishna [12] introduced the concept of vague graphs and studied some of their properties. Akram et al. [1] introduced certain types of vague graphs. A. Somasundaram and S. Somasundaram [18] defined domination in fuzzy graphs. Domination in intuitionistic fuzzy graphs are introduced by Parvathi and Thamizhendhi [11]. Rashmanlou et al. [13, 14, 15] studied bipolar fuzzy graphs. Rashmanlou and Jun [16] investigated complete interval-valued fuzzy graphs. Borzooei and Rashmanlou [2, 3, 4, 5, 6, 7, 8, 9] discussed ring sum in product intuitionistic fuzzy graphs, degree of vertices, and new concepts of domination sets in vague graphs. Samanta and Pal [19, 20, 21, 22, 23] defined fuzzy threshold graphs, fuzzy tolerance graphs, and new concepts on bipolar fuzzy graphs. In this paper, we study different types of dominating

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set in vague graphs. A characterization of minimal perfect dominating sets in a vague graph is given. Finally, we have introduced some new applications of domination in vague graph and other sciences.

2 Preliminaries

In this section, we review briefly some definitions in vague graphs and introduce some new notations.

A graph is an ordered pair $G = (V, E)$, where V is the set of vertices of G and E is the set of edges of G. A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$, for all $u, v \in V$, where V is a finite non-empty set and $\wedge$ denote minimum. A vague set A in an ordinary finite non-empty set X, is a pair (t_A, f_A), where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$ are true and false membership functions, respectively such that $0 \leq t_A(x) + f_A(x) \leq 1$, for all $x \in X$. Note that $t_A(x)$ is considered as the lower bound for degree of membership of x in A and $f_A(x)$ is the lower bound for negative of membership of x in A. So, the degree of membership of x in the vague set A, is characterized by the interval $[t_A(x), 1 - f_A(x)]$. Hence, a vague set is a special case of interval-valued sets studied by many mathematicians and applied in many branches of mathematics.

We mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval-valued membership value is assigned to each element of the universe considering the evidence for x only, without considering evidence against x. In vague sets both are independently proposed by the decision maker. This makes a major difference in the judgment about the grade of membership. A vague relation is a generalization of a fuzzy relation. Let X and Y be ordinary finite non-empty sets. We call a vague relation to be a vague subset of $X \times Y$, that is an expression R defined by

$$R = \{\langle (x, y),\ t_R(x, y), f_R(x, y)\rangle : x \in X,\ y \in Y\}$$

where $t_R : X \times Y \to [0, 1]$, $f_R : X \times Y \to [0, 1]$, which satisfies the condition $0 \leq t_R(x, y) + f_R(x, y) \leq 1$, for all $(x, y) \in X \times Y$.

DEFINITION 1 ([12]). A vague graph is defined to be a pair $G = (A, B)$, where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that

$$t_B(xy) \leq \min(t_A(x), t_A(y)) \quad \text{and} \quad f_B(xy) \geq \max(f_A(x), f_A(y))$$

for each edge $xy \in E$.

The underlying crisp graph of a vague graph $G = (A, B)$, is the graph $G = (V, E)$ where $V = \{v : t_A(v) > 0 \text{ and } f_A(v) > 0\}$ and $E = \{\{u, v\} : t_B(\{u, v\}) > 0,\ f_B(\{u, v\}) > 0\}$. V is called the vertex set and E is called the edge set. A vague graph may be also denoted as $G = (V, E)$.

A vague graph G is said to be strong if $t_B(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\}$ and $f_B(v_iv_j) = \max\{f_A(v_i), f_A(v_j)\}$ for every edge $v_iv_j \in E$. A vague graph G is said to be complete if $t_B(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\}$ and $f_B(v_iv_j) = \max\{f_A(v_i), f_A(v_j)\}$ for all $v_i, v_j \in V$.

Let $G = (V, E)$ be a vague graph.

(i) The cardinality of G is defined to be

$$|G| = \left| \sum_{v_i \in V} \frac{1 + t_A(v_i) - f_A(v_i)}{2} + \sum_{v_i v_j \in E} \frac{1 + t_B(v_i v_j) - f_B(v_i v_j)}{2} \right|.$$

(ii) The vertex cardinality of G is defined by

$$|V| = \sum_{v_i \in V} \frac{1 + t_A(v_i) - f_A(v_i)}{2}, \quad \text{for all } v_i \in V.$$

(iii) The edge cardinality of G is defined by

$$|E| = \sum_{v_i v_j \in E} \left(\frac{1 + t_B(v_i v_j) - f_B(v_i v_j)}{2} \right), \quad \text{for all } v_i v_j \in E.$$

(iv) The number of vertices (The cardinality of V) is called the order of a vague graph and is denoted by

$$O(G) = \sum_{v_i \in V} \left(\frac{1 + t_A(v_i) - f_A(v_i)}{2} \right), \quad \text{for all } v_i \in V.$$

(v) The number of edges (the cardinality of E) is called the size of a vague graph and is denoted by

$$S(G) = \sum_{v_i v_j \in E} \left(\frac{1 + t_B(v_i v_j) - f_B(v_i v_j)}{2} \right), \quad \text{for all } v_i v_j \in E.$$

The complement of a vague graph $G = (A, B)$ is a vague graph $\overline{G} = (\overline{A}, \overline{B})$, where $\overline{A} = A$,

$$\overline{t_B}(v_i v_j) = \min(t_A(v_i), t_A(v_j)) - t_B(v_i v_j)$$

and

$$\overline{f_B}(v_i v_j) = f_B(v_i v_j) - \max(f_A(v_i), f_A(v_j)).$$

A vague graph $G = (V, E)$ is said to be n-partite (multipartite) if the vertex et V can be partitioned into n subsets $V_1, V_2, \cdots, V_n$ such that:

(i) For every $v_i, v_j \in V_i$ and $1 \leq i, j \leq n$, we have $t_B(v_i v_j) = 0$ and $f_B(v_i v_j) = 0$.

(ii) For every $v_i \in V$, $v_j \in V$ and $1 \leq i, j \leq n$, we have $t_B(v_i v_j) = 0$, $f_B(v_i v_j) > 0$ or $t_B(v_i v_j) > 0$, $f_B(v_i v_j) = 0$.

An edge $e = (u, v)$ of a vague graph is called an effective edge if

$$t_B(uv) = \min\{t_A(u), t_A(v)\} \text{ and } f_B(uv) = \max\{f_A(u), f_A(v)\}.$$

The effective degree of a vertex v in vague graph $G = (V, E)$ is defined to be sum of the weights of the effective edges incident at v and it is denoted by $d_E(v)$. The minimum effective degree of G is $\delta_E(G) = \min\{d_E(v) : v \in V\}$. The maximum effective degree of G is $\triangle_E(G) = \max\{d_E(v) : v \in V\}$. The effective edge degree of an edge $e = (u, v)$ in a vague graph $G = (V, E)$ is defined to be $d_E(e) = d_E(u) + d_E(v) - 1$, if e is an effective edge and $d_E(e) = d_E(u) + d_E(v)$ otherwise.

Two vertices v_i and v_j are said to be neighbors in vague graph $G = (V, E)$ if either one of the following conditions hold:

(i) $t_B(v_iv_j) > 0,\ f_B(v_iv_j) > 0,$

(ii) $t_B(v_iv_j) = 0,\ f_B(v_iv_j) > 0,$

(iii) $t_B(v_iv_j) > 0,\ f_B(v_iv_j) = 0,\ v_i, v_j \in V.$

Two vertices v_i and v_j are said to be strong neighbors in vague graph $G = (V, E)$ if

$$t_B(v_iv_j) = \min\{t_A(v_i), t_A(v_j)\} \text{ and } f_B(v_iv_j) = \max\{f_A(v_i), f_A(v_j)\},$$

for all $(v_i, v_j) \in E$. A vertex subset $N(v) = \{u \in V : v \text{ adjacent to } u\}$ is called the open neighborhood set of a vertex v and $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood set of V. The neighborhood degree of a vertex v in a vague graph G is defined to be sum of the weights of the vertices adjacent to v, and it is denoted by $d_N(v)$, that is mean that $d_N(v) = |N(v)|$. The minimum neighborhood degree of G is $\delta_N(G) = \min\{d_N(v) : v \in V\}$. The maximum neighborhood degree of G is $\triangle_N(G) = \max\{d_N(v) : v \in V\}$.

DEFINITION 2 ([4]). An edge (u, v) is said to be strong edge in vague graph G if $t_B(uv) \geq (t_B)^\infty(uv)$ and $f_B(uv) \leq (f_B)^\infty(uv)$, where

$$(t_B)^\infty(uv) = \max\{(t_B)^k(uv) : k = 1, 2, \cdots, n\}$$

and

$$(f_B)^\infty(uv) = \min\{(f_B)^k(uv) : k = 1, 2, \cdots, n\}.$$

Given $u, v \in V$. We say that u dominates v in G if there exists a strong edge between them.

For any $u, v \in V$, we see that if u dominates v, then v dominates u. Hence domination is a symmetric relation on V. A subset S of V is said to be independent set if $t_B(uv) < (t_B)^\infty(uv)$ and $f_B(uv) > (f_B)^\infty(uv)$ for all $u, v \in S$.

DEFINITION 3 ([4]). A subset S of V in a vague graph G is said to be vertex covering set of G if for every edge e of G there is a vertex v in S incident with e. A vertex covering set S of G is called a minimal vertex covering set of G if for every $u \in S$, $S - [u]$ is not a vertex covering set of G. The minimum cardinality among all minimal vertex covering set in G is called the vertex covering number of G and is denoted by $\alpha(G)$.

3 Perfect Domination in Vague Graphs

Domination in graphs has applications in several fields. Domination arises in facility location problems, where the number of facilities (e.g., hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a facility is fixed and one attempts to minimize the number of facilities necessary so that every one is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveyor must stand in order to take height measurements for an entire region. In this section, we define the concepts of dominating set, perfect dominating set, independent dominating set and irredundant set of vague graphs.

DEFINITION 4. Let u be a vertex in a vague graph $G = (A, B)$. Then

$$N(u) = \{v : v \in V \text{ and } (u, v) \text{ is strong edge in } G\}$$

is called neighborhood of u in G. A vertex $u \in V$ of a vague graph G is said to be an isolated vertex if $t_B(uv) = 0$ and $f_B(uv) = 0$, for all $v \in V, u \neq v$, that is $N(u) = \emptyset$. Hence, an isolated vertex does not dominate any other vertex of G.

DEFINITION 5. Let $G = (A, B)$ be a vague graph and $u, v \in V$. Then we say that u dominates v in G if there exists a strong edge between them.

REMARK 6.

(i) For any $u, v \in V$, if u dominates v then v dominates u. Hence domination is a symmetric relation on V.

(ii) For any $v \in V$, $N(v)$ is precisely the set of all vertices in V which are dominated by v.

(iii) If $t_B(uv) < (t_B)^\infty(uv)$ and $f_B(uv) > (f_B)^\infty(uv)$, for all $u, v \in V$, then the dominating set of G is V.

DEFINITION 7.

(i) A subset D of V is called a dominating set in G if for every $v \in V - D$, there exists $u \in D$ such that u dominates v.

(ii) A dominating set D of a vague graph is said to be minimal dominating set if no proper subset of D is a dominating set.

(iii) Minimum cardinality among all minimal dominating set is called lower domination number of G, and is denoted by $d_v(G)$.

(iv) Maximum cardinality among all minimal dominating set is called upper domination number of G, and is denoted by $D_v(G)$.

EXAMPLE 8. Consider a vague graph $G = (V, E)$ such that $V = \{u_1, u_2, u_3, u_4, u_5\}$ and

$$E = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_1), (u_1, u_4), (u_2, u_4)\}.$$

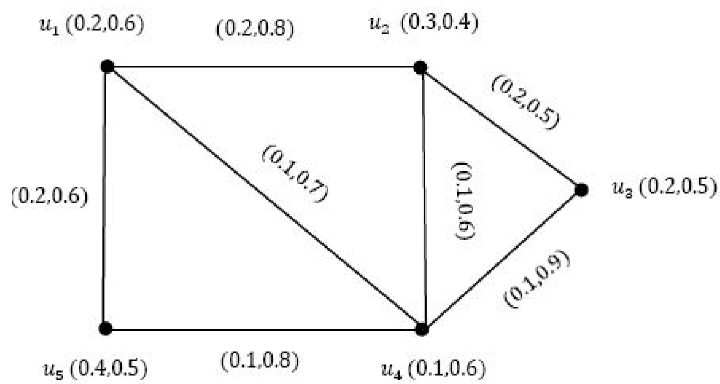

Figure 1: Vague graph G.

In Figure 1, $\{u_2, u_5\}$, $\{u_1, u_3, u_5\}$ and $\{u_1, u_2, u_3, u_5\}$ are three dominating sets, $\{(u_1, u_4)\}$ is a minimal dominating set of minimal cardinality and $d_v(G) = 0.55$, $\{(u_2, u_3)\}$ is a minimal dominating set of maximal cardinality and $D_v(G) = 0.8$.

DEFINITION 9.

(i) A dominating set D in a vague graph $G = (V, E)$ is called perfect dominating set in G if for each vertex $v \in V - D$, there exists exactly one vertex $u \in D$ such that u dominates v.

(ii) A perfect dominating set D in a vague graph $G = (V, E)$ is said to be minimal perfect dominating set if for each $v \in D$, $D - \{v\}$ is not a perfect dominating set in G.

(iii) The minimum cardinality among all minimal perfect dominating sets in G is called the perfect domination number of G and is denoted by $\gamma_p(G)$ or simply γ_p.

(iv) A perfect dominating set D with smallest cardinality equal to $\gamma_p(G)$ is called the minimum perfect dominating set and is denoted by γ_p-set.

EXAMPLE 10. In Figure 1, $\{u_2, u_5\}$, $\{u_1, u_2, u_4\}$, $\{u_2, u_3, u_5\}$ and $\{u_1, u_3, u_5\}$ are four perfect dominating sets, $\{u_1, u_2, u_4\}$ and $\{u_1, u_3, u_5\}$ are minimal perfect dominating sets in G. Hence, $\gamma_p(G) = \min\{1, 1.1\} = 1$.

THEOREM 11. Every dominating set in a complete vague graph $G = (V, E)$ is a perfect dominating set.

PROOF. Let D be a minimum dominating set of a vague graph G. Since G is complete, then every edge in G is one effective edge and every vertex $v \in V - D$ is adjacent to exactly on vertex $u \in D$. Therefore, every dominating set in G is a perfect dominating set.

REMARK 12. Let $G = (V, E)$ be a vague graph. Then,

(i) if $G = K_p$ is a complete vague graph, then $\gamma_p(G) = \min\{|v_i| : \forall v_i \in V(G)\}$,

(ii) if $G = K_{n,m}$ is a bipartite vague graph, then $\gamma_p(G) = \min\{|n|, |m|\}$, where $n = |V_1|$ and $m = |V_2|$.

DEFINITION 13. Let $G = (V, E)$ be a vague graph and D be a set of vertices.

(i) A vertex v is said to be a vague private neighbor of $u \in D$ with respect to D, if $N[v] \cap D = \{u\}$. Furthermore, we define vague private neighborhood of $u \in D$ with respect to D, by $PN[u, D] = \{v : N[v] \cap D = \{u\}\}$. In the other words, $PN[u, D] = N[u] - N[D - \{u\}]$.

(ii) A vertex u in $D \subseteq V$ is said to be vague redundant vertex, if $PN[u, D] = \emptyset$. Equivalently u is redundant in D if $N[u] \subseteq N[D - \{u\}]$. Otherwise u is said to be vague irredundant vertex.

(iii) A set $D \subseteq V$ is said to be vague irredundant set, if $PN[u, D] \neq \emptyset$, for every vertex in D. Equivalently, D is irredundant if every vertex of D is D-perfect or adjacent with an D-perfect vertex.

(iv) A vague irredundant set D is maximal vague irredundant set if for every vertex $v \in V - D$, the set $D \cup \{v\}$ is not vague irredundant set, which means that there is at least one vertex $w \in D \cup \{u\}$ which does not have any private neighbor.

(v) Maximum cardinality among all maximal vague irredundant set is called upper vague irredundance number and is denoted by $IR_V(G)$.

(vi) Minimum cardinality among all maximal vague irredundant set is called lower vague irredundance number and is denoted by $ir_V(G)$.

EXAMPLE 14. Consider a vague graph $G = (V, E)$ such that $V = \{u_1, u_2, u_3, u_4, u_5\}$ and

$$\begin{aligned} E &= \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_1), (u_1, u_6), (u_4, u_6), \\ &\quad (u_5, u_6), (u_2, u_5), (u_3, u_5)\} \end{aligned}$$

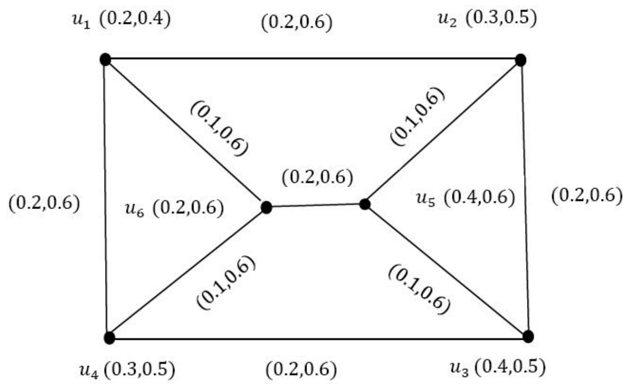

Figure 2: Vague graph G.

In Figure 2, $D_1 = \{u_1, u_4, u_6\}$, $D_2 = \{u_2, u_3, u_5\}$ and $D_3 = \{u_1, u_2\}$ are three vague irredundant sets, $PN[u_1, D_1] = \{u_2\}$, $PN[u_4, D_1] = \{u_3\}$ and $PN[u_6, D_1] = \{u_5\}$. $\{u_5, u_6\}$ has minimum cardinality among all maximal vague irredundant sets and $ir_V(G) = 0.7$. $\{u_2, u_3, u_5\}$ has maximum cardinality among all maximal vague irredundant sets and $IR_V(G) = 1.25$.

THEOREM 15. A perfect dominating set D in a vague graph $G = (V, E)$ is minimal perfect dominating set if and only if for each vertex $v \in D$, $Pnf[v, D] \neq \emptyset$.

PROOF. Suppose D is minimal and $v \in D$. Hence there is a vertex $u \notin D - \{v\}$ such that u is adjacent to no vertex of $D - \{v\}$ or u is adjacent to atleast two vertices of $D - \{v\}$. If $u = v$ then this implies that $v \in Pnf[v, D]$. If $u \neq v$ then it is impossible that u is adjacent to atleast two vertices of $D - \{v\}$, because D is a perfect dominating set. Therefore u is not adjacent to any vertex of $D - \{v\}$. Since D is a perfect dominating set, hence u is adjacent to only $v \in D$ i.e. $N(u) \cap D = v$. Thus, $u \in Pnf[v, D]$ and so $Pnf[v, D] \neq \emptyset$. Conversely, suppose $v \in D$ and $Pnf[v, D]$ contains vertex u of G. If $u = v$ then u is either adjacent to atleast two vertices of $D - \{v\}$ or u is adjacent to no vertex of $D - \{v\}$. Thus, $D - \{v\}$ is not a perfect dominating set. If $u \neq v$ then $N(w) \cap D = v$ implies that u is not adjacent to any vertex of $D - \{v\}$. Hence, in all cases $D - \{v\}$ is not a perfect dominating set if $v \in D$. Therefore„ D is minimal.
The following theorem gives a characterization of minimal perfect dominating sets in a vague graph.

THEOREM 16. A perfect dominating set D in a vague graph $G = (A, B)$ is a minimal perfect dominating set if and only if for each vertex $v \in D$, one of the following conditions holds

(i) $N(v) \cap D = \emptyset$,

(ii) there is a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$.

PROOF. Let D be a minimal perfect dominating set and $v \in D$. Then, $D = D - \{v\}$ is not a dominating set and hence there exists a vertex $u \in V - D$ such that u is not dominated by an element of D. If $u = v$ we get (i) and if $u \neq v$ we get (ii). The converse is true according to definition.

THEOREM 17. Let $G = (V, E)$ be a connected vague graph and D be a minimal perfect dominating set of G. Then, $V - D$ is a dominating set of G.

PROOF. Let D be a minimal perfect dominating set of G and $V - D$ is not a dominating set. Then there exists a vertex $v \in D$ such that v is not a dominated by any vertex in $V - D$. Since G is connected, v is a strong neighbor of atleast one vertex in $D - \{v\}$. Then $D - \{v\}$ is a dominating set, which contradicts the minimality of D. Thus for every vertex v in D there is atleast one vertex u in $V - D$ such that $t_B(uv) = t_A(u) \wedge t_A(v)$ and $f_B(uv) = f_A(u) \vee f_A(v)$. Hence $V - D$ is a dominating set.

DEFINITION 18.

(i) A dominating set D in a vague graph $G = (V, E)$ is said to be independent dominating set in G if D is independent.

(ii) An independent dominating set D in a vague graph G is said to be minimal independent dominating set, if for every $v \in D$, $D - \{v\}$ is not dominating set in G. The minimum cardinality among all minimal independent dominating set in G is called the independence dominating number of G and is denoted by $\gamma_i(G)$ or simply γ_i.

(iii) An independent dominating set D in a vague graph G is said to be maximal independent dominating set, if for every vertex $v \in V - D$, the set $D \cup \{v\}$ is not independent.

(iv) The minimum cardinality among all maximal independent dominating set is called lower independent number of G, and it is denoted by $i(G)$.

(v) The maximum cardinality among all maximal independent dominating set is called upper independent number of G, and it is denoted by $\beta(G)$.

EXAMPLE 19. Consider a vague graph $G = (V, E)$ such that $V = \{u_1, u_2, u_3, u_4\}$ and

$$E = \{(u_1, u_2), (u_2, u_3), (u_1, u_3), (u_3, u_4), (u_2, u_4)\}.$$

In Figure 3, $\{\{u_1, u_2\}, \{u_2, u_3\}\}$ is independent dominating set, $\{u_2, u_3\}$ is a maximal independent dominating set of minimal cardinality and $i(G) = 0.9$, $\{u_1, u_2\}$ is a maximal independent dominating set of maximal cardinality and $\beta(G) = 0.95$.

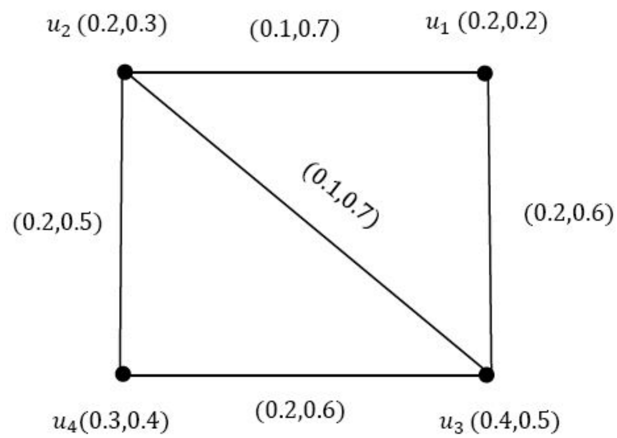

Figure 3: Vague graph G.

REMARK 20. Let G be any vague graph. Then $\gamma_p(G) \leq \gamma_i(G)$.

THEOREM 21. Let G be a vague graph and D be a γ_p-set. If $t_B(uv) \leq t_A(u) \wedge t_A(v)$ and $f_B(uv) > f_A(u) \vee f_A(v)$, for all $u, v \in D$, then $\gamma_p(G) = \gamma_i(G)$.

PROOF. It is clear that from the definition, the minimum perfect dominating set D has the smallest cardinality among all minimal perfect dominating sets. Since $t_B(uv) < t_A(u) \wedge t_A(v)$ and $f_B(uv) > f_A(u) \vee f_A(v)$, for all $u, v \in D$, we implies that D is an independent. Hence $\gamma_p(G) = \gamma_i(G)$. In comparing to the crisp case, $\gamma_p(G) = \gamma_i(G)$ if the crisp graph G is claw free but that is not required for vague graph.

COROLLARY 22. If G is a complete vague graph then $\gamma_i(G) = \gamma_p(G)$.

PROOF. Since G is a complete vague graph, every edge in G is an effective edge. Hence $\gamma_i(G) = \min\{|v_i| : \forall v_i \in V(G)\}$ and $\gamma_p(G) = \min\{|v_i| : \forall v_i \in V(G)\}$. It is clear that $\gamma_i(G) = \gamma_p(G)$.

THEOREM 23. For any vague graph G, $\gamma_p(G) \leq p - \triangle_E(G)$.

PROOF. Let D be a γ_p-set in a vague graph G and $u \in V$ such that $d_E(u) = \triangle_E(G)$. Then $V - N(u)$ is a perfect dominating set in G and so $\gamma_p(G) \leq |V - N(u)|$. Hence $\gamma_p(G) \leq p - \triangle_E(G)$.

REMARK 24. For any vague graph $G = (V, E)$, $\gamma_p(G) \leq p - \delta_E(G)$.

THEOREM 25. Let G be a connected vague graph. If G^* is not cycle or does not have an induced cycle subgraph. Then, $\gamma_p(G) \leq p - \triangle_N(G)$.

PROOF. Let G be a connected vague graph such that G^* is not a cycle or does not have an induced cycle subgraph. Let D be a minimum perfect dominating set in G and $u \in V$ such that $d_N(u) = \triangle_N(G)$. Then, $V - N(u)$ is a perfect dominating set of G and $\gamma_p(G) \leq |V - N(u)|$. Hence, $\gamma_p(G) \leq p - \triangle_N(G)$.

THEOREM 26. Let G be a connected vague graph without isolated vertices. If G^* is not cycle or does not have an induced cycle subgraph, then $\gamma_p(G) \leq p - \alpha(G)$.

PROOF. Let G be a vague graph without isolated vertices, such that G^* is not a cycle and D be a minimal vertex covering set in G. Then $V - D$ is an independent set in G and therefor, $V - D$ is a perfect dominating set in G. Hence, $\gamma_p(G) \leq |V - D| = p - \alpha(G)$.

THEOREM 27. Let G be a connected vague graph without isolated vertices. If G^* is not cycle or does not have an induced cycle subgraph, then $\gamma_p(G) \leq p - \beta(G)$.

PROOF. Let G be a vague graph without isolated vertices, such that G^* is not a cycle and D be a maximal independent set in G. Then $V - D$ is a vertex covering set and so $V - D$ is a perfect dominating set in G. Hence, $\gamma_p(G) \leq |V - D| = p - \beta(G)$.

COROLLARY 28. Let G be a vague graph without isolated vertices. If G^* is not cycle or does not have an induced cycle subgraph, then $\gamma_p(G) \leq \alpha(G)$.

4 Application of Dominating Sets

A vague set is an extension of Zadeh's fuzzy set theory whose range of membership degree is $[0, 1]$. The vague graph is a generalized structure of a fuzzy graph which gives more precision, flexibility, and compatibility with a system when compared with the fuzzy graphs. Domination is a rapidly developing area of research in graph theory, and its various applications to ad hoc networks, distributed computing, social networks and web graphs partly explain the increased interest. Now we give an application of domination in vague graph.

These days, graph models are finding many applications in different fields of science and technology such as computer science, topology, operation research, biological and social sciences. It is observed that in a social group some people can influence others and it can happen only when there is a strong relationship between them. Now, let us consider a vague graph of a social group as given in Figure 4.

The nodes are depicting the degree of power of a person belonging to a set of social group. The degree of power of a person is defined in terms of its true membership and false membership. Degree of true membership can be interpreted as how much power a person possesses and false membership can be interpreted as how much power a person losses. A has 30% power within the social group but he losses 50% power in the same group. In this example, $\{C, D\}$, $\{D, E\}$, $\{A, E\}$ and $\{B, D\}$ are strong edges. So, we can say there is a strong relation between C and D. It is true for D, E and A, E too. Hence, we see that D dominates C, E dominates D and E dominates A. It is clear

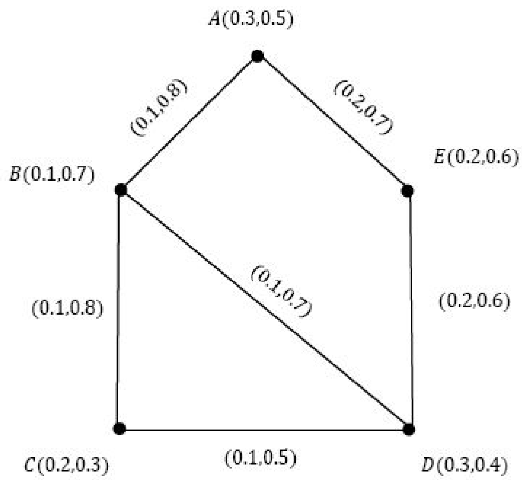

Figure 4: Vague graph G of social relations.

that $\{A, B, D\}$, $\{B, E, D\}$, $\{A, C, D\}$ are dominating sets. The edges of a vague graph show the influence of one person to another person. The degree of true membership and false membership of edges can be interpreted as the percentage of true membership and false membership influence. For example, D follows 10% B's suggestions but he does not follow 70% his suggestions. Therefore, domination can help us to find people who have strong relation in a social group. Domination can be applied in other areas which we have listed below.

4.1 Queens Problem

This problem was mentioned by Ore. According to the rules of chess a queen can, in one move, advance any number of squares horizontally, diagonally, or vertically (assuming that no other chess figure is on its way). How to place a minimum number of queens on a chessboard so that each square is controlled by at least one queen? See one of the solutions in Figure 5.

4.2 Radio Stations

Suppose that we have a collection of small villages in a remote part of the world. We would like to locate radio stations in some of these villages so that massages can be broadcast to all of the village in the region. Since each radio station has a limited broadcasting range, we must use several stations to reach all villages. But since radio stations are costly, we want to locate as few as possible which can reach all other villages. Let each village be represented by a vertex. An edge between two villages is labeled with the distance, say in kilometers, between the two villages. Let us assume

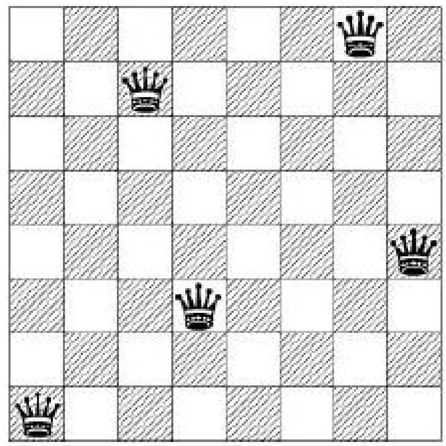

Figure 5: Queens dominating the chessboard .

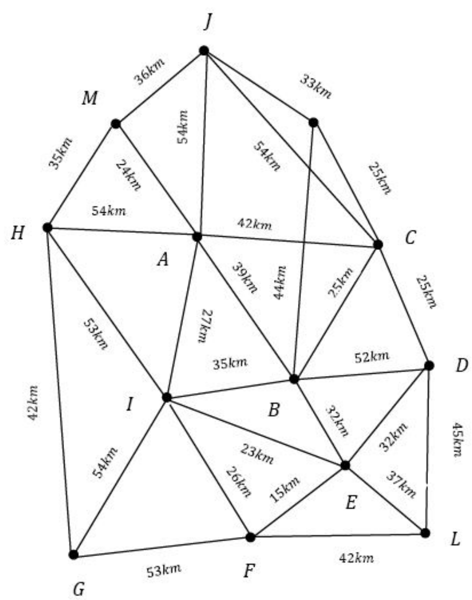

Figure 6: Distance between villages.

that a radio station has a broadcast range of fifty kilometers. What is the least number of stations in a set which dominate (within distance 50) all other vertices in this graph? A set $\{B, F, H, J\}$ of cardinality four is indicated in the following figure. Here we have

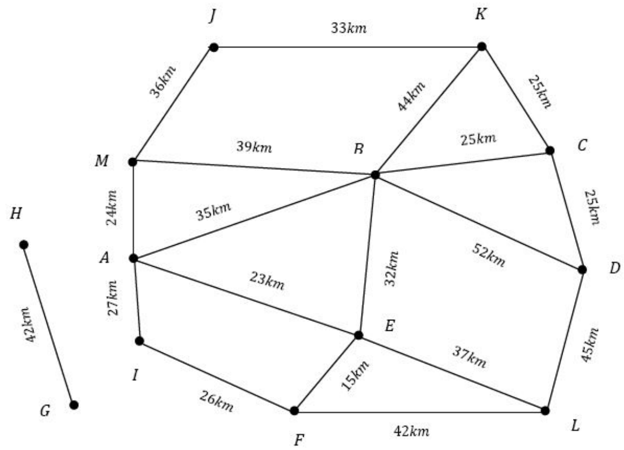

Figure 7: A set of cardinality four.

assumed that a radio station has a broadcast range of only fifty kilometers, we can essentially remove all edges in the graph, which represent a distance of more than fifty kilometers. We need only to find a dominating set in this graph. Notice that if we could afford radio stations which have a broadcast range of seventy kilometers, three radio stations would be sufficient.

4.3 Facility Location Problems

The dominating set in vague graphs are natural models for facility location problems in operational research. Facility location problems are concerned with the location of one or more facilities in a way that optimizes a certain objective such as minimizing transportation cost, providing equitable service to customers and capturing the largest market share.

4.4 Multiple Domination Problems

An important role is played by multiple domination. Multiple domination can be used to construct hierarchical overlay networks in peer-to-peer applications for more efficient index searching. The hierarchical overlay networks usually serve as distributed databases for index searching, e.g. in modern file sharing and instant messaging computer network applications. Dominating sets of several kinds are used for balancing efficiency and fault tolerance as well as in the distributed construction of minimum spanning trees. Another good example of direct, important and quickly developing applications of multiple domination in modern computer networks is a wireless sensor network.

4.5 Dominating Set in Mobile Ad-hoc Network

A Mobile Ad-Hoc Network (MANET) is a self-configurable infrastructureless network connecting the mobile devices in wireless mode. The dominating set has been commonly used for routing and broadcasting the information to the mobile devices in mobile ad networks. The connected dominating set (CDS) is widely used as a virtual backbone for MANETs to provide different communication primitive such as routing, broadcasting etc. The construction of virtual backbone is very important and it is approximated by a minimum CDS with a unit disk graph as shown in the Figure 8. The connected

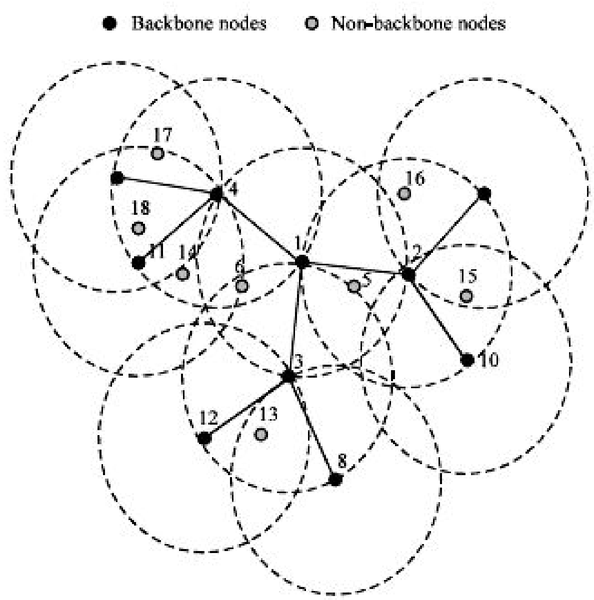

Figure 8: CDS Construction in MANETs.

dominating set is defined as a subset of nodes in a wireless network such that each node in the network is either present in the set or as a neighbor of some node in the set and the induced graph with the nodes in the set is connected. The CDS is useful in reducing the communication and the storage overhead by keeping its size to be minimal. However it is well known that the problem of finding the minimal CDS is NP-hard and it can be solved practically through preprocessing.

5 Conclusion

Graph theory has several interesting applications in system analysis, operation research, computer applications, and economics. Since most of the time the aspects of graph problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy systems. It is known that fuzzy graph theory has numerous applications in modern science and engineering, especially in the fields of information theory, neural networks, expert systems, town planning, and control theory. We have studied various forms of dominating set in vague graphs. Finally, some applications of domination in vague graph are given. In our future work, we will focus on double domination of vague graphs and some interesting result on $\gamma_{dd}(G)$ using some known parameter of vague graph G will be given.

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