A Krasnoselskii Existence Result For Nonlinear Delay Caputo q–Fractional Difference Equations With Applications to Lotka–Volterra Competition Model*

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Abstract

In this paper, we investigate the existence of solutions for nonlinear delay Caputo q–fractional difference equations. The main result is proved by means of Krasnoselskii's fixed point theorem. As an application, we link the conclusion of the main theorem to an existence result for Lotka–Volterra model.

1   Introduction

The theory of q-calculus, on one hand, which is dated back to the late of nineties deals with continuous functions which do not need to be smooth. Despite the early exploration, its investigation was lagged to the beginning of the twentieth century when discussed in Jackson's paper [1]. Following the discovery of its demonstrated applications in the fields of combinatorics and fluid mechanics, Al Salam re-introduced this theory in his paper [2] and then it has continued to develop until these days [3-12]. The theory of fractional calculus, on the other hand, generalizes integer-order analysis by considering derivatives of non-integer order [13, 14]. Notable contributions have been made to both theory and application of fractional calculus during the last years when some rather special properties of derivatives of arbitrary order were examined for arbitrary functions. Applications including problems in rheology, electrochemistry, physics and engineering are amongst those which can be described using equations of fractional order; the reader is invited to see the paper [15] for more topics considered as an applications of fractional calculus.

The natural extension, which we investigate here, is to consider a q-fractional calculus which unifies these two theories by considering q-derivatives of non-integer order. In

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the recent years, there have appeared many results dealing with the qualitative properties of solutions of q-fractional equations; for instance one can consult [16-23]. In the analysis of differential equations, particularly, the determination of whether there is a solution or not is one of the main concepts that must be taken into consideration before proceeding to the investigations of other properties of solutions. The existence of solutions for q-fractional equations involving boundary conditions has been the object of many researchers who carried out the investigations under different conditions and by using various methods [24-29]. Less contributions, never the less, have been conducted for q-fractional equations with initial conditions [30]. The Lotka-Volterra model has been extensively investigated through different approaches [33, 34, 35]. However, all the above mentioned papers studied the integer order Lotka-Volterra model. Despite its significance, there are few papers studied the fractional order Lotka-Volterra model [36, 37]. As far as we know, however, there is no literature achieved in the direction of q-fractional Lotka-Volterra model.

For $0 < q < 1$, we define the time scale $\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of integers. For $a = q^{n_0}$ and $n_0 \in \mathbb{Z}$, we denote $\mathbb{T}_a = [a, \infty)_q = \{q^{-i}a : i = 0, 1, 2, \ldots\}$. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space and define $\mathbb{I}_\tau = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \ldots, a\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{T}_{\tau a} = [\tau a, \infty)_q = \{\tau a, q^{-1}\tau a, q^{-2}\tau a, \ldots\}$ where $\tau = q^d \in \mathbb{T}_q$, $d \in \mathbb{N}_0$ and $\mathbb{I}_\tau = \{a\}$ with $d = 0$ is the non-delay case .

The objective of this paper is to study the existence of solutions for equations of the form

$$\begin{cases} {}_qC_a^\alpha x(t) = f(t, x(t), x(\tau t)) & t \in \mathbb{T}_a, \\ x(t) = \phi(t) & t \in \mathbb{I}_\tau, \end{cases} \tag{1}$$

where $f : \mathbb{T}_a \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\phi : \mathbb{I}_\tau \to \mathbb{R}$ and ${}_qC_a^\alpha$ denotes the Caputo's q-fractional difference operator of order $\alpha \in (0, 1)$. To prove our main results, we employ Krasnoselskii fixed point theorem and Arzela-Ascoli's theorem. As an application, we link the conclusion of the main result to an existence result for q-fractional Lotka-Volterra model. To the best of authors' realization, no published paper exists regarding the existence of solutions of initial value q-fractional problem and its applications to q-fractional Lotka-Volterra model.

2   Preliminaries

In this section, we set forth some basic nabla notations, definitions and lemmas that will be used in the sequel. However, before proceeding we state the following two theorems that play an important role in the proof of the main theorem.

THEOREM 1 ([31] Arzela-Ascoli's Theorem). A bounded, uniformly Cauchy subset D of $l_\infty(\mathbb{T}_a)$ (all bounded real-valued sequences with domain $\mathbb{T}_a$) is relatively compact.

THEOREM 2 ([32] Krasnoselskii's Fixed Point Theorem). Let D be a nonempty, closed, convex and bounded subset of a Banach space $(X, \|x\|)$. Suppose that $A : X \to X$ and $B : D \to X$ are two operators such that

(i) A is a contraction.

(ii) B is continuous and $B(D)$ resides in a compact subset of X,

(iii) for any $x, y \in D$, $Ax + By \in D$.

Then the operator equation $Ax + Bx = x$ has a solution $x \in D$.

For a function $f : \mathbb{T}_q \to \mathbb{R}$, its nabla q-derivative of f is written as

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \in \mathbb{T}_q - \{0\}. \tag{2}$$

The nabla q-integral of f has the following form

$$\int_0^t f(s)\nabla_q s = (1 - q)t \sum_{i=0}^{\infty} q^i f(tq^i). \tag{3}$$

For $a \in \mathbb{T}_q$, however, (3) becomes

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s.$$

The definition of the q-factorial function for $n \in \mathbb{N}$ is given below

$$(t - s)_q^n = \prod_{i=0}^{n-1} (t - q^i s). \tag{4}$$

In case α is a non positive integer, the q-factorial function is defined by

$$(t - s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t}q^i}{1 - \frac{s}{t}q^{i+\alpha}}. \tag{5}$$

In Lemma 1, we present some properties of q-factorial functions.

LEMMA 1 ([16]). For $\alpha, \gamma, \beta \in \mathbb{R}$, we have

(i) $(t - s)_q^{\beta+\gamma} = (t - s)_q^\beta (t - q^\beta s)_q^\gamma$.

(ii) $(at - as)_q^\beta = a^\beta (t - s)_q^\beta$.

(iii) The nabla q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t - s)_q^\alpha = \frac{1 - q^\alpha}{1 - q}(t - s)_q^{\alpha-1}.$$

(iv) The nabla q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t - s)_q^\alpha = -\frac{1 - q^\alpha}{1 - q}(t - qs)_q^{\alpha-1}.$$

DEFINITION 1. For a function $f : \mathbb{T}_q \to \mathbb{R}$, the left q-fractional integral ${}_q\nabla_a^{-\alpha}$ of order $\alpha \neq 0, -1, -2, \ldots$ and starting at $a = q^{n_0} \in \mathbb{T}_q$, $n_0 \in \mathbb{Z}$, is defined by

$$
\begin{aligned}
{}_q\nabla_a^{-\alpha} f(t) &= \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f(s) \nabla_q s \\
&= \frac{1-q}{\Gamma_q(\alpha)} \sum_{i=n}^{n_0-1} q^i (q^n - q^{i+1})_q^{\alpha-1} f(q^i),
\end{aligned}
\tag{6}
$$

where

$$
\Gamma_q(\alpha + 1) = \frac{1 - q^\alpha}{1 - q} \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \ \alpha > 0.
\tag{7}
$$

REMARK 1. The left q-fractional integral ${}_q\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{T}_q$ to functions defined on $\mathbb{T}_q$.

DEFINITION 2 ([30]). Let $0 < \alpha \notin \mathbb{N}$. Then the Caputo left q-fractional derivative of order α of a function f defined on $\mathbb{T}_q$ is defined by

$$
{}_qC_a^\alpha f(t) \triangleq {}_q\nabla_a^{-(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s,
\tag{8}
$$

where $n = [\alpha] + 1$. In case $\alpha \in \mathbb{N}$, then we may write ${}_qC_a^\alpha f(t) \triangleq \nabla_q^n f(t)$. The (left) Riemann q-fractional derivative is defined by $({}_q\nabla_a^\alpha f)(t) = (\nabla_q \ {}_q\nabla_a^{-(n-\alpha)} f)(t)$. In virtue of [30], the Riemann and Caputo q-fractional derivatives are related by

$$
({}_qC_a^\alpha f)(t) = ({}_q\nabla_a^\alpha f)(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} f(a).
\tag{9}
$$

LEMMA 2 ([30]). Let $\alpha > 0$ and f be defined in a suitable domain. Thus

$$
{}_q\nabla_a^{-\alpha} \ {}_qC_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a)
\tag{10}
$$

and if $0 < \alpha \leq 1$ we have

$$
{}_q\nabla_a^{-\alpha} \ {}_qC_a^\alpha f(t) = f(t) - f(a).
\tag{11}
$$

The following identity is crucial in solving the linear q-fractional equations

$$
{}_q\nabla_a^{-\alpha}(x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha}, \quad (0 < a < x < b),
\tag{12}
$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$. The q-analogue of Mittag-Leffler function with double index (α, β) is introduced in [30]. It was defined as follows.

DEFINITION 3 ([30]). For $z, z_0 \in \mathbf{C}$ and $\mathfrak{R}(\alpha) > 0$, the q-Mittag-Leffler function is defined by

$$_qE_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z - z_0)_q^{\alpha k}}{\Gamma_q(\alpha k + \beta)}. \tag{13}$$

In case $\beta = 1$, we use $_qE_\alpha(\lambda, z - z_0) := {_qE_{\alpha,1}(\lambda, z - z_0)}$.

EXAMPLE 1 ([30]). Let $0 < \alpha \le 1$ and consider the left Caputo q-fractional difference equation

$$_qC_a^\alpha y(t) = \lambda y(t) + f(t), \quad y(a) = a_0, \ t \in T_q. \tag{14}$$

The solution of (14) is given by

$$y(t) = a_0 \, {_qE_\alpha(\lambda, t - a)} + \int_a^t (t - qs)_q^{\alpha-1} \, {_qE_{\alpha,\alpha}(\lambda, t - q^\alpha s)} f(s) \nabla_q s. \tag{15}$$

If instead we use the modified q-Mittag-Leffler function

$$_qe_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z - z_0)_q^{\alpha k + (\beta-1)}}{\Gamma_q(\alpha k + \beta)},$$

then the solution representation (15) becomes

$$y(t) = a_0 \, {_qe_\alpha(\lambda, t - a)} + \int_a^t {_qe_{\alpha,\alpha}(\lambda, t - qs)} f(s) \nabla_q s.$$

REMARK 2 ([30]). If we set $\alpha = 1$, $\lambda = 1$, $a = 0$ and $f(t) = 0$ in (14), we reach at a q-exponential formula $e_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma_q(k+1)}$ on the time scale $\mathbb{T}_q$, where $\Gamma_q(k+1) = [k]_q! = [1]_q[2]_q \ldots [k]_q$ with $[r]_q = \frac{1-q^r}{1-q}$. We recall that $e_q(t) = E_q((1-q)t)$, where $E_q(t)$ denotes a special case of the basic hypergeormetric series, namely,

$$E_q(t) = {_1\phi_0(0; q, t)} = \prod_{n=0}^{\infty} (1 - q^n t)^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n},$$

where $(q)_n = (1 - q)(1 - q^2) \ldots (1 - q^n)$ is the q-Pochhammer symbol.

3   The Main Result

We prove our main result under the following assumptions:

(I) $f(t, x(t), y(t)) = f_1(t, x(t)) + f_2(t, x(t), y(t))$, where f_i are Lipschitz functions with Lipschitz constants L_{f_i}, $i = 1, 2$.

(II) $\big|f_1(t, x(t))\big| \le M_1 |x(t)|$ and $|f_2(t, x(t), y(t))| \le M_2 |x(t)| \times |y(t)|$ for some positive numbers M_1 and M_2.

Let $B(\mathbb{T}_{\tau a}, \mathbb{R}) = l_\infty(\mathbb{T}_{\tau a})$ and $\mathbb{T}_{\tau a} = [\tau a, \infty)_q$ denote the set of all bounded functions (sequences) on $\mathbb{T}_{\tau a}$. Define the set

$$D = \big\{ x : \ x \in B(\mathbb{T}_{\tau a}, \mathbb{R}),\ |x(t)| \leq r,\ \text{for all } t \in \mathbb{T}_{\tau a} \big\},$$

where r satisfies

$$|\phi(a)| + \frac{M_1 r + M_2 r^2}{\Gamma_q(\alpha)} \leq r.$$

Define the operators F_1 and F_2 by

$$F_1 x(t) = \phi(a) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f_1(s, x(s)) \nabla_q s,$$

and

$$F_2 x(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)^{\alpha-1} f_2(s, x(s), x(s - \tau)) \nabla_q s.$$

It is clear that $x(t)$ is a solution of (1) if it is a fixed point of the operator $Fx = F_1 x + F_2 x$.

THEOREM 3. Let conditions (I)–(II) hold. Then, equation (1) has a solution in the set D provided that

$$\frac{L_{f_1} C(\alpha)}{\Gamma_q(\alpha)} < 1 \ \text{ and } \ |\phi(a)| + \frac{(M_1 r + M_2 r^2) C(\alpha)}{\Gamma_q(\alpha)} \leq r.$$

PROOF. First it is clear that the set D is a nonempty, closed, convex and bounded set. In light of Theorem 2, we present the proof in three steps.

Step 1: We prove that F_1 is contractive. We can easily see that

$$\begin{aligned}
\big|F_1 x(t) - F_1 y(t)\big| &= \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \big|f_1(s, x(s)) - f_1(s, y(s))\big| \nabla_q s \\
&\leq \frac{L_{f_1}}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} |x(s) - y(s)| \nabla_q s \\
&\leq \frac{L_{f_1}}{\Gamma_q(\alpha)} \|x - y\| \int_a^t (t - qs)_q^{\alpha-1} \nabla_q s. \qquad (16)
\end{aligned}$$

By virtue of (12) and since $(t - a)_q^{\overline{0}} = 1$, one can see that

$$\frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} (t - a)_q^0 \nabla_q s = {}_q \nabla_a^\alpha (t - a)_q^0 = \frac{\Gamma_q(1)(t - a)_q^\alpha}{\Gamma(\alpha + 1)}.$$

Therefore, (16) becomes

$$\big|F_1 x(t) - F_1 y(t)\big| \leq \frac{L_{f_1} C(\alpha)}{\Gamma_q(\alpha)} \|x - y\|, \ \ t < T_1,$$

where $C(\alpha) = \frac{(1-q)(T_1-a)_q^\alpha}{1-q^\alpha}$ is a positive constant depending on the order α. By the assumption that $\frac{L_{f_1}C(\alpha)}{\Gamma_q(\alpha)} < 1$, we conclude that F_1 is contractive. Furthermore, we obtain for $x \in D$:

$$\left|F_1x(t) + F_2x(t)\right|$$

$$\leq \quad |\phi(a)| + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \left|f_1(s,x(s)) + f_2(s,x(s),x(\tau s))\right| \nabla_q s$$

$$\leq \quad |\phi(a)| + \frac{M_1\|x\| + M_2\|x\|^2}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \nabla_q s$$

$$\leq \quad |\phi(a)| + \frac{(M_1 r + M_2 r^2)C(\alpha)}{\Gamma_q(\alpha)} \leq r,$$

which implies that $F_1 x + F_2 x \in D$. For $x \in D$, we also get

$$|F_2x(t)| \leq \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \left|f_2(s,x(s),x(\tau s))\right| \nabla_q s \leq \frac{(M_2 r^2)C(\alpha)}{\Gamma(\alpha)} \leq r,$$

which implies that $F_2(D) \subset D$.

Step 2: We prove that F_2 is continuous. Let a sequence x_n converge to x. Taking the norm of $F_2x_n(t) - F_2x(t)$, we have

$$\left|F_2x_n(t) - F_2x(t)\right|$$

$$\leq \quad \frac{1}{\Gamma(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \left|f_2(s,x_n(s),x_n(\tau s)) - f_2(s,x(s),x(\tau s))\right| \nabla_q s$$

$$\leq \quad \frac{L_{f_2}}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} \left(\left|x_n(s) - x(s)\right| + \left|x_n(\tau s)) - x(\tau s)\right|\right) \nabla_q s$$

$$\leq \quad \frac{2L_{f_2}}{\Gamma_q(\alpha)} \|x_n - x\| \int_a^t (t-qs)_q^{\alpha-1} \nabla_q s = \frac{(2L_{f_2})C(\alpha)}{\Gamma_q(\alpha)} \|x_n - x\|.$$

From the above inequalities, we conclude that whenever $x_n \to x$, then $F_2x_n \to F_2x$. This proves the continuity of F_2. To prove that $F_2(D)$ resides in a relatively compact subset of $l_\infty(\mathbb{T}_{\tau a})$, we let $t_1 \leq t_2 \leq H$ to get

$$\left|F_2x(t_2) - F_2x(t_1)\right| \quad \leq \quad \frac{1}{\Gamma_q(\alpha)} \left| \int_a^{t_2} (t_2-qs)_q^{\alpha-1} f_2(s,x(s),x(\tau s)) \right.$$

$$- \left. \int_a^{t_1} (t_1-qs)_q^{\alpha-1} f_2(s,x(s),x(\tau s)) \right| \nabla_q s$$

$$\leq \quad \frac{1}{\Gamma_q(\alpha)} \sum_a^{t_1} \left|(t_2-qs)_q^{\alpha-1} - (t_1-qs)^{\alpha-1}\right| |f_2(s,x(s),x(\tau s))| \nabla_q s$$

$$+ \quad \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} \left|(t_2-qs)_q^{\alpha-1}\right| |f_2(s,x(s),x(\tau s))| \nabla_q s.$$

Upon employing condition (II), we obtain

$$\left| F_2x(t_2) - F_2x(t_1) \right|$$
$$\leq \quad M_2r^2 \Big[\frac{1}{\Gamma_q(\alpha)} \int_a^{t_1} (t_2 - qs)_q^{\alpha-1} \nabla_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^{t_1} (t_1 - qs)_q^{\alpha-1} \nabla_q s$$
$$+ \quad \frac{1}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} \nabla_q s \Big].$$

Hence, we reach to

$$\left| F_2x(t_2) - F_2x(t_1) \right| \leq M_2r^2 \Big[{}_q\nabla_a^{-\alpha}(t_2 - a)_q^0 + {}_q\nabla_a^{-\alpha}(t_1 - a)_q^0 + {}_q\nabla_{t_1}^{-\alpha}(t_2 - t_1)_q^0 \Big].$$

From (12), it follows that

$$\left| F_2x(t_2) - F_2x(t_1) \right| \leq \frac{M_2r^2}{\Gamma_q(\alpha + 1)} \Big[(t_2 - a)_q^\alpha + (t_1 - a)_q^\alpha + (t_2 - t_1)_q^\alpha \Big].$$

This implies that F_2 is bounded and uniformly Cauchy subset of $l_\infty(\mathbb{T}_{\tau a})$. Thus, by virtue of the Arzela Ascoli's Theorem, we conclude that F_2 is relatively compact.

Step 3: It remains to show that for any $x, y \in D$, we have $F_1x(t) + F_2y(t) \in D$. If $z(\cdot) = F_1x(\cdot) + F_2y(\cdot)$, then we have

$$
\begin{aligned}
|z(t)| &\leq |\phi(a)| + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} \left| f_1(s, x(s)) + f_2(s, y(s), y(\tau s)) \right| \nabla_q s \\
&\leq |\phi(a)| + \frac{M_1\|x\| + M_2\|y\|^2}{\Gamma_q(\alpha)} \int_a^t (t - s)_q^{\alpha-1} \nabla_q s \\
&\leq |\phi(a)| + \frac{(M_1r + M_2r^2)C(\alpha)}{\Gamma_q(\alpha)},
\end{aligned}
$$

which implies that $z(t) \in D$.

By employing the Krasnoselskii Fixed Point Theorem, we conclude that there exists $x \in D$ such that $x = Fx = F_1x + F_2x$ which is a fixed point of F. Hence, equation (1) has at least one solution in D.

4 Applications

In this section, we employ Theorem 3 to prove an existence result for the solutions of Lotka-Volterra model

$$\begin{cases} {}_qC_a^\alpha x(t) = x(t)(\gamma(t) - \beta(t)x(\tau t)) & t \in \mathbb{T}_a, \\ x(t) = \phi(t) & t \in \mathbb{I}_\tau,\ 0 < \alpha < 1, \end{cases} \tag{17}$$

where $f(t, x(t), x(\tau t)) = x(t)\big(\gamma(t) - \beta(t)x(\tau t)\big)$ in equation (1) and the coefficients γ and β satisfy the boundedness relations

$$\inf_{t \in \mathbb{T}_a} \gamma(t) = \gamma^- \leq \gamma(t) \leq \gamma^+ = \sup_{t \in \mathbb{T}_a} \gamma(t) \ \text{ and } \ \inf_{t \in \mathbb{T}_a} \beta(t) = \beta^- \leq \beta(t) \leq \beta^+ = \sup_{t \in \mathbb{T}_a} \beta(t),$$

which are medically and biologically feasible. Model (17) represents the interspecific competition in single species with τ denotes the maturity time period.

Denote

$$\overline{f}_1(t, x(t)) = x(t)\gamma(t) \text{ and } \overline{f}_2(t, x(t), x(\tau t)) = -\beta(t)x(t)x(\tau t).$$

It follows that the functions $\overline{f}_1$ and $\overline{f}_2$ satisfy the conditions

(III) $|\overline{f}_1(t, x(t))| \leq \gamma^+ |x(t)|, \quad |\overline{f}_2(t, x(t), x(\tau t))| \leq \beta^+ |x(t)| \times |x(\tau t)|.$

(IV) $\overline{f}_i$ are Lipschitz functions with Lipschitz constants $\overline{L}_{f_i}, \; i = 1, 2.$

The solution of model (17) has the form

$$x(t) = \phi(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} x(s)\Big(\gamma(s) - \beta(s)x(\tau s)\Big)\nabla_q s, \;\; t \in \mathbb{T}_a, \qquad (18)$$

and $x(t) = \phi(t), \; t \in \mathbb{I}_\tau.$ Define a function G by

$$Gx(t) = G_1 x(t) + G_2 x(t),$$

where

$$G_1 x(t) = \phi(a) + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} x(s)\gamma(s)\nabla_q s,$$

and

$$G_2 x(t) = -\frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} x(s)\beta(s)x(s - \tau)\nabla_q s.$$

One can easily employ the same arguments used in the proof of Theorem 3 to complete the proof of the following theorem for equation (17).

THEOREM 4. Let conditions (III)–(IV) hold. Then, the model (17) has a solution in the set D provided that $\frac{L_{f_1} C(\alpha)}{\Gamma_q(\alpha)} < 1$ and $|\phi(a)| + \frac{(\gamma^+ r + \beta^+ r^2)C(\alpha)}{\Gamma_q(\alpha)} \leq r.$

REMARK 3. The above result can be extended to n species competitive Caputo q-fractional Lotka-Volterra system of the form

$$\begin{cases} {}_qC_a^\alpha x_i(t) = x_i(t)(\gamma_i(t) - \sum_{j=1}^n \beta_{ij}(t)x_j(\tau_{ij}t)) & t \in \mathbb{T}_a, \; i = 1, 2, ..., n, \\ x_i(t) = \phi_i(t) & t \in \mathbb{I}_\tau, \; 0 < \alpha < 1, \; \tau_i = \max\limits_{1 \leq j \leq n} \tau_{ij}. \end{cases}$$

REMARK 4. The results can be also carried out for the following system which allows a classical constant delay

$$\begin{cases} {}_qC_a^\alpha x(t) = f(t, x(t), x(t - \tau)) & t \in [a, b], \; b \leq \infty, \\ x_i(t) = \phi_i(t) & t \in [a - \tau, a]. \end{cases}$$

REMARK 5. The analysis carried out in this paper is based on the use of nabla rather than delta operators. Indeed, unlike the delta operator the range of nabla fractional sum and difference operators depends only of the starting point and independent of the order α. This provides exceptional ability to treat skillfully different circumstances throughout the proofs. The delta approach can be obtained from nabla operator through the implementation of the dual identities discussed in [38].

5   Conclusion

The q-fractional difference equations are regarded as fractional analogue of q-difference equations. Motivated by their widespread applications in many disciplines, the topic of q-fractional equations has attracted the attention of many researchers during the last three decades. In parallel with the recent interests in this topic, an existence result for a type of nonlinear delayed q-fractional difference equations is investigated in this paper. The main equation is constructed in the sense of Caputo such that it fits many real life applications. The proof of the main result is based on the employment of Krasnoselskii fixed point and Arzela-Ascoli's theorems. Prior to the main results, preliminary assertions are addressed on the properties of q-fractional difference equations. The applicability of the proposed results is discussed by linking the main theorem to an existence result for Lotka-Volterra model.

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