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Inequalities For The Polar Derivative Of A Polynomial*

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Abstract

In this paper certain inequalities for the polar derivative of a polynomial with restricted zeros are given, which generalize and refine some well-known polynomial inequalities due to Govil, Malik, Aziz and others.

1 Introduction

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z)$ of degree n. It was shown by Turan [14] that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq 2 \max_{|z|=1} |P'(z)| . \tag{1}$$

Equality in (1) holds for $P(z) = az^n + b$ where $|a| = |b|$. For the class of polynomials $P \in \mathcal{P}_n$ having all their zeros in $|z| \leq k$ where $k \leq 1$, Mailk [9] proved that

$$n \max_{|z|=1} |P(z)| \leq (1+k) \max_{|z|=1} |P'(z)| . \tag{2}$$

and where as Govil [4] showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k, k \geq 1$, then

$$n \max_{|z|=1} |P(z)| \leq (1+k^n) \max_{|z|=1} |P'(z)| . \tag{3}$$

Both the results are sharp and equalities in (2) and (3) hold for $P(z) = (z+k)^n$ and $P(z) = (z^n + k^n)$ respectively. Malik [10] obtained an extension of (1) in the sense that the left hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$ by proving that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)| . \tag{4}$$

Equality in (4) holds for $P(z) = az^n + b, |a| = |b|$.

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As generalizations of the inequalities (2)–(4), A. Aziz [1] considered the class of polynomials $P \in \mathcal{P}_n$ having all their zeros in $|z| \leq k$ and proved for each $q > 0$,

$$n\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1+ke^{i\theta})\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|P'(z)\right|, k \leq 1 \tag{5}$$

and

$$n\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1+k^n e^{i\theta})\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|P'(z)\right|, k \geq 1. \tag{6}$$

Equality in (6) holds for $P(z) = z^n + k^n$.

In the limiting case when $q \to \infty$, the inequalities (5) and (6) reduce to (2) and (3) respectively. Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P \in \mathcal{P}_n$ with respect to point $\alpha \in \mathbb{C}$, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

(see [8]). The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary $P'(z)$ of $P(z)$ in the sense that

$$Lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect z for $|z| \leq R, R > 0$. Aziz and Rather [2] extended inequalities (2) and (3) to the polar derivatives of polynomials and proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$n\left(|\alpha| - k\right) \max_{|z|=1} |P(z)| \leq (1+k) \max_{|z|=1} |D_\alpha P(z)|,$$

and if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$n\left(|\alpha| - k\right) \max_{|z|=1} |P(z)| \leq (1+k^n) \max_{|z|=1} |D_\alpha P(z)|. \tag{7}$$

Inequality (7) is sharp and equality holds for $P(z) = (z-k)^n$ where α is any real number with $\alpha \geq k$.

Recently Rather et al. [13] extended inequality (3) to the polar derivative of polynomials and proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for $|\gamma| \geq k$ and $q > 0$,

$$n\left(|\gamma| - k\right)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1+ke^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|D_\gamma P(z)\right| \tag{8}$$

and under the same hypothesis, Rather et al. [13] also showed that

$$n\left(|\gamma| - k\right)\left\{\int_0^{2\pi} \left|P(e^{i\theta}) + \beta m/k^{n-1}\right|^q d\theta\right\}^{1/q}$$

$$\leq \left\{ \int_0^{2\pi} \left|1 + ke^{i\theta}\right|^q d\theta \right\}^{1/q} \left(\max_{|z|=1} |D_\gamma P(z)| - m/k^{n-1} \right) \tag{9}$$

where $|\beta| \leq 1$ and $m = \min_{|z|=k} |P(z)|$.

The main aim of this paper is to extends the inequality (6) to the polar derivative of a polynomial and obtain a generalization of (7) in the sense that the left hand side of (7) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. More precisely we prove:

THEOREM 1. If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in C$ with $|\alpha| \geq k$ and for each $q > 0$,

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\alpha P(z)|. \tag{10}$$

REMARK 1. If we divide the two sides of (10) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequality (6). Further if make $q \to \infty$ in (10), we get inequality (7).

Next we prove:

THEOREM 2. If $P \in P_n$, $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $q > 0$,

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} \left|P(e^{i\theta}) + \beta m\right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1} \right\}.$$

For $\beta = 0$, Theorem 2 yields the following refinement of Theorem 1.

COROLLARY 1. If $P \in P_n$, $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq k$, $|\beta| \leq 1$ and for each $q > 0$,

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1} \right\}. \tag{11}$$

Letting $q \to \infty$ in (11) and choosing the argument of β with $|\beta| = 1$ suitably, we obtain the following refinement of inequality (7).

COROLLARY 2. If $P \in P_n$, $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha \in C$ with $|\alpha| \geq k$,

$$n\left(|\alpha| - k\right) \max_{|z|=1} |P(z)| + n\left(|\alpha| + 1/k^{n-1}\right) m \leq \left(1 + k^n\right) \max_{|z|=1} |D_\alpha P(z)| .$$

Finally we use Holder's inequality to establish a generalization of (10) in the sense that maximum on in the right hand side of (10) is replaced by factor involving the integral mean of $|D_\alpha P(z)|$ on $|z| = 1$.

THEOREM 3. If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in C$ with $|\alpha| \geq k$ and for $q > 0, r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$,

$$\begin{aligned} n\left(|\alpha| - k\right) &\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \\ &\leq B_q \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^{qr} d\theta\right\}^{1/qr} \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta\right\}^{1/qs} \end{aligned} \tag{12}$$

where

$$B_q = \frac{\left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q}}. \tag{13}$$

REMARK 2. By letting $s \to \infty$ (so that $r \to 1$) in (12), we get inequality (10).

The following result is an immediate consequence of Theorem 3.

COROLLARY 3. If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in C$ with $|\alpha| \geq k$ and for each $q > 0$,

$$n\left(|\alpha| - k\right) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq 2B_q \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^q d\theta\right\}^{1/q}, \tag{14}$$

where B_q is given by (13).

REMARK 3. Making $q \to \infty$ in (14), we get inequality (6).

2 Lemmas

For the proofs of these theorems we need the following lemmas. The first Lemma is a simple deduction from Maximum Modulus Principle (see [5] or [11]).

LEMMA 1. If $P \in P_n$, then for $R \geq 1$,

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| .$$

The next lemma is a simple deduction from a well-known result of G. H. Hardy [6].

LEMMA 2. If $P \in P_n$, then for $q > 0, R \geq 1$,

$$\left\{\int_0^{2\pi} \left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leq R^n \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}.$$

We also require the following result is due to Rahman and Schmeisser [12].

LEMMA 3. If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$ and $q > 0$,

$$\left\{\int_0^{2\pi} \left|P(Re^{i\theta})\right|^q d\theta\right\}^{1/q} \leq C_q \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}$$

where

$$C_q = \frac{\left\{\int_0^{2\pi} \left|1 + R^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q d\theta\right\}^{1/q}}.$$

3 Proofs of the Theorems

PROOF OF THEOREM 1. Since all the zeros of $P(z)$ lie in $|z| \leq k$, therefore, all the zeros of $F(z) = P(kz)$ lie in $|z| \leq 1$. Applying inequality (8) with $k = 1$ to the polynomial $F(z)$, it follows for each $q > 0$ and $|\gamma| \geq 1$,

$$n\left(|\gamma| - 1\right)\left\{\int_0^{2\pi} \left|F(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} |D_\gamma F(z)|.$$

Setting $\gamma = \frac{\alpha}{k}$ in above inequality and noting that $|\gamma| = \left|\frac{\alpha}{k}\right| \geq 1$, we get

$$n\left(\frac{|\alpha|}{k} - 1\right)\left\{\int_0^{2\pi} \left|F(e^{i\theta})\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q} \max_{|z|=1} \left|D_{\frac{\alpha}{k}} F(z)\right|. \quad (15)$$

Let $G(z) = z^n \overline{F(1/\bar{z})}$. Then

$$|G(z)| = |F(z)| \quad \text{for} \quad |z| = 1$$

and $G(z)$ does not vanish in $|z| < 1$. Therefore, by Lemma 3 applied to the polynomial $G(z)$ with $R = k \geq 1$, it follows that for each $q > 0$,

$$\int_0^{2\pi} \left|G(ke^{i\theta})\right|^q \leq B_q^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta = B_q^q \int_0^{2\pi} \left|F(e^{i\theta})\right|^q d\theta. \quad (16)$$

where B_q is given by (13).

Combining (15)and (16), we get for each $q > 0$,

$$n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|G(ke^{i\theta})\right|^q d\theta\right\}^{1/q}$$
$$\le kB_q\left\{\int_0^{2\pi}\left|1+e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|$$
$$= k\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|. \tag{17}$$

Also since

$$G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{P(k/\bar{z})},$$

we see that for $0 \le \theta < 2\pi$,

$$\left|G(ke^{i\theta}\right| = \left|k^n e^{in\theta}\overline{P(e^{i\theta})}\right| = k^n\left|P(e^{i\theta})\right|.$$

Using this in (17), we get

$$nk^n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q}$$
$$\le \quad k\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|. \tag{18}$$

Again, since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and

$$\begin{aligned}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right| &= \max_{|z|=1}\left|nF(z)+(\frac{\alpha}{k}-z)F'(z)\right|\\ &= \max_{|z|=1}\left|nP(kz)+(\frac{\alpha}{k}-z)kP'(kz)\right|\\ &= \max_{|z|=k}\left|nP(z)+(\alpha-z)P'(z)\right|\\ &= \max_{|z|=k}\left|D_\alpha P(z)\right|,\end{aligned}$$

by Lemma 1 for $R = k \ge 1$, we have

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right| = \max_{|z|=k}\left|D_\alpha P(z)\right| \le k^{n-1}\max_{|z|=1}\left|D_\alpha P(z)\right|. \tag{19}$$

This in **conjunction** with (18) gives

$$nk^n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le k^n\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_\alpha P(z)\right|$$

so that

$$n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} \le \left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\max_{|z|=1}\left|D_\alpha P(z)\right|.$$

This proves Theorem 1.

The proof of Theorem 2 follows on the lines of proof of Theorem 1. However, for the sake of completeness we present a proof.

PROOF OF THEOREM 2. The polynomial $F(z) = P(kz)$ has all its zeros in $|z| \leq 1$. By inequality (9) applied to the polynomial $F(z)$ (with $k = 1$), we get for each $q > 0, |\beta| \leq 1$ and $|\alpha| \geq k$,

$$n\left(\frac{|\alpha|}{k} - 1\right)\left\{\int_0^{2\pi}\left|F(e^{i\theta}) + \beta\min_{|z|=1}|F(z)|\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right| - n\min_{|z|=1}|F(z)|\right\}. \tag{20}$$

Since

$$m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|P(kz)| = \min_{|z|=1}|F(z)|,$$

from (20), we obtain for each $q > 0, |\beta| \leq 1$ and $|\alpha| \geq k$,

$$n(|\alpha| - k)\left\{\int_0^{2\pi}\left|F(e^{i\theta}) + \beta m\right|^q d\theta\right\}^{1/q} \leq k\left\{\int_0^{2\pi}\left|1 + e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right| - nm\right\}. \tag{21}$$

Further, since all the zeros of $F(z)$ lie in $|z| \leq 1$ and

$$m \leq |F(z)| \quad \text{for} \quad |z| = 1,$$

by the maximum modulus theorem for $m \neq 0$,

$$m < |F(z)| \quad \text{for} \quad |z| > 1. \tag{22}$$

We show all the zeros of polynomial $G(z) = F(z) + \beta m$ lie in $|z| \leq 1$ for every β with $|\beta| \leq 1$. This is obvious if $m = 0$. For $m \neq 0$, if there is a point $z = z_0$ with $|z_0| > 1$ such that $G(z_0) = F(z_0) + \beta m = 0$, then we have

$$|F(z_0)| = |\beta| m \leq m, \quad |z_0| > 1,$$

a contradiction to (22). Therefore, the polynomial $G(z)$ has all its zeros in $|z| \leq 1$ and hence the polynomial $H(z) = z^n\overline{G(1/\bar{z})}$ does not vanish in $|z| < 1$. Applying Lemma 3 to the polynomial $H(z)$ with $R = k \geq 1$, it follows that for each $q > 0$,

$$\int_0^{2\pi}\left|H(ke^{i\theta})\right|^q d\theta \leq B_q^q\int_0^{2\pi}\left|H(e^{i\theta})\right|^q d\theta = B_q^q\int_0^{2\pi}\left|G(e^{i\theta})\right|^q d\theta = B_q^q\int_0^{2\pi}\left|F(e^{i\theta}) + \beta m\right|^q d\theta,$$

where B_q is the same as given by (13). By this in (21), we obtain for each $q > 0$,

$$n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|H(ke^{i\theta})\right|^q d\theta\right\}^{1/q} \leq k\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|-nm\right\}. \quad (23)$$

But,

$$H(z)=z^n\overline{G(1/\bar{z})}=z^n\overline{F(1/\bar{z})}+\bar{\beta}z^n m,$$

therefore, for $|z|=1$, we get

$$|H(kz)|=\left|k^n z^n\overline{F(1/k\bar{z})}+\bar{\beta}z^n mk^n\right|=k^n\left|F(z/k)+\beta m\right|=k^n\left|P(z)+\beta m\right|. \quad (24)$$

Further by inequality (19), we have

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}F(z)\right|\leq k^{n-1}\max_{|z|=k}|D_\alpha P(z)| \quad \text{for } |z|=1. \quad (25)$$

From (23)–(25), we deduce for each $q>0, |\beta|\leq 1$ and $|\alpha|\geq k$,

$$n\left(|\alpha|-k\right)\left\{\int_0^{2\pi}\left|P(e^{i\theta})+\beta m\right|^q d\theta\right\}^{1/q} \leq \left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}\left\{\max_{|z|=1}|D_\alpha P(z)|-nm/k^{n-1}\right\}.$$

This completes the proof of Theorem 2.

PROOF OF THEOREM 3. Let $F(z)=P(kz)$. Since all the zeros of $P(z)$ lie in $|z|\leq k$, therefore, all the zeros of $F(z)$ lie in $|z|\leq 1$. Hence the polynomial $G(z)=z^n\overline{F(1/\bar{z})}$ has all its zeros in $|z|\geq 1$ and

$$|G(z)|=|F(z)| \quad \text{for } |z|=1.$$

By a result of De Bruijn (see [3, Theorem 1, p. 1265]), if follows that

$$|G'(z)|\leq|F'(z)| \quad \text{for } |z|=1. \quad (26)$$

Since $G(z)=z^n\overline{F(1/\bar{z})}$, we see that $F(z)=z^n\overline{G(1/\bar{z})}$ and it can be easily seen that

$$|G'(z)|=|nF(z)-zF'(z)| \quad \text{and} \quad |F'(z)|=|nG(z)-zG'(z)| \quad \text{for } |z|=1. \quad (27)$$

Combining (26) and (27), we get

$$|nF(z)-zF'(z)|\leq|F'(z)| \quad \text{for } |z|=1. \quad (28)$$

Also since $F(z)$ has all its zeros in $|z|\leq 1$, by Gauss-Lucas theorem all the zeros of $F'(z)$ also lie in $|z|\leq 1$. This implies that the polynomial

$$z^{n-1}\overline{F'(1/\bar{z})}\equiv nG(z)-zG'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (28) that the function

$$w(z) = \frac{zG'(z)}{nG(z) - zG'(z)}$$

is analytic for $|z| \le 1$ and $|w(z)| \le 1$ for $|z| \le 1$. Furthermore, $w(0) = 0$. Thus the function $1 + w(z)$ is subordinate to the function $1 + z$. Hence by a well-known property of subordination[7, p. 422], we have for each $q > 0$,

$$\int_0^{2\pi} \left|1 + w(e^{i\theta})\right|^q d\theta \le \int_0^{2\pi} \left|1 + e^{i\theta}\right|^q d\theta.$$

Now

$$1 + w(z) = \frac{nG(z)}{nG(z) - zG'(z)},$$

which gives with the help of (27),

$$n\,|G(z)| = |1 + w(z)|\,|nG(z) - zG'(z)| = |1 + w(z)|\,|F'(z)| \text{ for } |z| = 1.$$

This implies for each $q > 0$,

$$n^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta = \int_0^{2\pi} \left|1 + w(e^{i\theta})\right|^q \left|F'(e^{i\theta})\right|^q d\theta. \tag{29}$$

Also, by using (26) and (27), we have for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$ and for $|z| = 1$,

$$\begin{aligned} \left|D_{\alpha/k}F(z)\right| &= \left|nF(z) + \left(\frac{\alpha}{k} - z\right) F'(z)\right| \\ &\ge \frac{|\alpha|}{k} |F'(z)| - |nF(z) - zF'(z)| \\ &= \frac{|\alpha|}{k} |F'(z)| - |G'(z)| \\ &\ge \frac{|\alpha|}{k} |F'(z)| - |F'(z)| = \left(\frac{|\alpha|}{k} - 1\right) |F'(z)|\,. \end{aligned} \tag{30}$$

Combining (29) and (30), we have for each $q > 0$,

$$n^q \left(|\alpha| - k\right)^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta \le \int_0^{2\pi} \left|1 + w(e^{i\theta})\right|^q k^q \left|D_{\frac{\alpha}{k}}F(e^{i\theta})\right|^q d\theta.$$

This gives with the help of Holder's inequality for $r > 1$, $s > 1$ with $r^{-1} + s^{-1} = 1$,

$$\begin{aligned} &n^q \left(|\alpha| - k\right)^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta \\ &\qquad \le k^q \left\{\int_0^{2\pi} \left|1 + w(e^{i\theta})\right|^{qr} d\theta\right\}^{1/r} \left\{\int_0^{2\pi} \left|D_{\frac{\alpha}{k}}F(e^{i\theta})\right|^{qs} d\theta\right\}^{1/s}. \end{aligned} \tag{31}$$

Further, since $G(z) \neq 0$ in $|z| < 1$ and $k \geq 1$, by taking $R = k \geq 1$ in Lemma 3, we have for each $q > 0$,

$$\int_0^{2\pi} \left|G(ke^{i\theta})\right|^q \leq B_q^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta \tag{32}$$

where B_q is given by (13). Using (32) in (31) and noting that $\left|G(ke^{i\theta})\right| = k^n|P(e^{i\theta})$, we get

$$\begin{aligned} n^q k^{nq} \left(|\alpha| - k\right)^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta & \\ \leq B_q^q n^q \left(|\alpha| - k\right)^q \int_0^{2\pi} \left|G(e^{i\theta})\right|^q d\theta & \\ \leq B_q^q \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^{qr} d\theta\right\}^{1/r} \left\{\int_0^{2\pi} \left|D_{\frac{\alpha}{k}} F(e^{i\theta})\right|^{qs} d\theta\right\}^{1/s}. & \quad (33) \end{aligned}$$

Further since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, it follows from Lemma 2 for $q > 0$ and $s > 0$ that

$$\begin{aligned} \int_0^{2\pi} \left|D_{\frac{\alpha}{k}} F(e^{i\theta})\right|^{qs} d\theta &= \int_0^{2\pi} \left|nP(ke^{i\theta}) + (\alpha - ke^{i\theta})P'(ke^{i\theta})\right|^{qs} d\theta \\ &\leq k^{(n-1)qs} \int_0^{2\pi} \left|nP(e^{i\theta}) + (\alpha - ke^{i\theta}P'(ke^{i\theta})\right|^{qs} d\theta. \\ &= k^{(n-1)qs} \int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta. \qquad (34) \end{aligned}$$

From (33) and (34), we deduce

$$\begin{aligned} n\left(|\alpha| - k\right) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta\right\}^{1/q} & \\ \leq B_q \left\{\int_0^{2\pi} \left|1 + e^{i\theta}\right|^{qr} d\theta\right\}^{1/qr} \left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^{qs} d\theta\right\}^{1/qs}. & \end{aligned}$$

This proves Theorem 3.

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