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Some Remarks On Traveling Wave Solutions In A Time-Delayed Population System With Stage Structure*

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Abstract

We show the existence of two traveling wave solutions in a time-delayed population system with stage structure by using the cross-iteration method.

1 Introduction

This work is a sequel to [1], we continue study the existence of traveling wave solutions for the two-species Lotka-Volterra competition model with age structure in the form

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + \alpha_1 \int_{\mathbb{R}} G_1(y) u(t-\tau_1, x-y) dy - \eta_1 u^2 - p_1 uv, \\ \frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + \alpha_2 \int_{\mathbb{R}} G_2(y) v(t-\tau_2, x-y) dy - \eta_2 v^2 - p_2 uv. \end{cases} \tag{1}$$

Here $u(t,x)$ and $v(t,x)$ represent densities of adult members of two species u and v at time t and point x, respectively. $d_1 > 0$ ($d_2 > 0$) is the diffusion coefficient of the adult population u (v). α_1 (α_2) is made up of two factors, the per capita birth rate and the survival rate of immature for the population u (v) during the immature stage. The two probability kernels G_1 and G_2 are given by

$$G_1(y) = \frac{e^{-y^2/4d_{i(u)}\tau_1}}{\sqrt{4\pi d_{i(u)}\tau_1}}, \ G_2(y) = \frac{e^{-y^2/4d_{i(v)}\tau_2}}{\sqrt{4\pi d_{i(v)}\tau_2}}.$$

For more details of model (1) see [1] and the references cited therein.

Model (1) has the trivial equilibrium $E_0 = (0,0)$, the mono-culture equilibria $E_u = (u^*, 0)$ and $E_v = (0, v^*)$ with

$$u^* = \frac{\alpha_1}{\eta_1}, \quad v^* = \frac{\alpha_2}{\eta_2},$$

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Table 1: Summary of local stability of system (1)

Steady state	Criteria for existence	Criteria for asymptotic stability
E_0	always exists	unstable
E_u	always exists	$\alpha_1 p_2 > \alpha_2\eta_1$
E_v	always exists	$\alpha_2 p_1 > \alpha_1\eta_2$
E_+	$\alpha_2 p_1 < \alpha_1\eta_2\alpha_1 p_2 < \alpha_2\eta_1$ or $\alpha_2 p_1 > \alpha_1\eta_2\alpha_1 p_2 > \alpha_2\eta_1$	$\alpha_1 p_2 < \alpha_2\eta_1$ and $\alpha_2 p_1 < \alpha_1\eta_2$

and the coexistence equilibrium $E_+ = (e_1, e_2)$ with

$$e_1 = \frac{\alpha_2 p_1 - \alpha_1\eta_2}{p_1 p_2 - \eta_1\eta_2},\ e_2 = \frac{\alpha_1 p_2 - \alpha_2\eta_1}{p_1 p_2 - \eta_1\eta_2}.$$

E_+ exists if and only if $\alpha_2 p_1 < \alpha_1\eta_2$ and $\alpha_1 p_2 < \alpha_2\eta_1$ or $\alpha_2 p_1 > \alpha_1\eta_2$ and $\alpha_1 p_2 > \alpha_2\eta_1$. We showed that if $\alpha_1 p_2 < \alpha_2\eta_1$ and $\alpha_2 p_1 < \alpha_1\eta_2$, then the unique coexistence E_+ is globally asymptotically stable. We summarized the stability of the equilibria in Table 1.1. A traveling wave solution of (1) connecting E_0 to E_+ takes the form of $u(t,x) = \phi(x+ct)$, $v(x,t) = \psi(x+ct)$, where $(\phi,\psi) \in C^2(\mathbb{R},\mathbb{R}^2)$ with $\phi(\xi)$ and $\psi(\xi)$ satisfying

$$d_1\phi''(\xi) - c\phi'(\xi) + \alpha_1\int_{\mathbb{R}} G_1(y)\phi(\xi - y - c\tau_1)dy - \eta_1\phi^2(\xi) - p_1\phi(\xi)\psi(\xi) = 0, \qquad (2)$$

$$d_2\psi''(\xi) - c\psi'(\xi) + \alpha_2\int_{\mathbb{R}} G_2(y)\psi(\xi - y - c\tau_1)dy - \eta_2\psi^2(\xi) - p_2\phi(\xi)\psi(\xi) = 0, \qquad (3)$$

$$\lim_{\xi\to-\infty}(\phi(\xi),\psi(\xi)) = E_0 \text{ and } \lim_{\xi\to\infty}(\phi(\xi),\psi(\xi)) = E_+.$$

We substitute $\phi(\xi) = e^{\lambda\xi}$ and $\psi(\xi) = e^{\lambda\xi}$ into the linearization equation of (2)–(3) to obtain the characteristic equations as follows

$$\Delta_i(\lambda, c) := d_i\lambda^2 - c\lambda + \alpha_i e^{-c\lambda\tau_i}\int_{\mathbb{R}} G_i(y)e^{-\lambda y}dy,\ i = 1, 2.$$

Then it is easy to verify the following properties:

i. $\Delta_i(0,c) = \alpha_i\int_{\mathbb{R}} G_i(y)e^{-\lambda y}dy > 0$;

ii. $\lim_{\lambda\to\infty}\Delta_i(\lambda,c) = \infty$ for all $c \geq 0$;

iii. $\frac{\partial^2\Delta_i(\lambda,c)}{\partial\lambda^2} = 2d_i > 0$ and

$$\frac{\partial\Delta_i(\lambda,c)}{\partial c} = -\lambda - \lambda\alpha_i\tau_i\int_{\mathbb{R}} G_i(y)e^{-\lambda y}dy < 0$$

for all $\lambda > 0$;

iv. $\lim_{c\to\infty} \Delta_i(\lambda, c) = -\infty$ for all $\lambda > 0$ and $\Delta_i(\lambda, 0) > 0$.

By the properties of $\Delta_i(\lambda, c)$ we know that there exist $c_i^* > 0$, $i = 1, 2$ such that the following statements are valid.

i. If $c \geq c_i^*$, then there exist four positive numbers Λ_{i1}, Λ_{i2}, $i = 1, 2$ (which are independent on c) with $\Lambda_{i1} \leq \Lambda_{i2}$ such that $\Delta_i(\Lambda_{i1}, c) = \Delta_i(\Lambda_{i2}, c) = 0$.

ii. If $c < c_i^*$, then $\Delta_i(\lambda, c) > 0$ for all $\lambda > 0$.

iii. If $c = c_i^*$, then $\Lambda_{i1} = \Lambda_{i2}$; and if $c > c_i^*$, then $\Lambda_{i1} < \Lambda_{i2}$, $\Delta_i(\lambda, c) < 0$ for all $\lambda \in (\Lambda_{i1}, \Lambda_{i2})$, $\Delta_i(\lambda, c) > 0$ for all $\lambda \in [0, \infty)\backslash[\Lambda_{i1}, \Lambda_{i2}]$.

Define

$$c^* = \max\{c_1^*, c_2^*\}.$$

By Liang and Zhao [2], c_i^* may be viewed as the spreading speeds of species u if $i = 1$ and species v if $i = 2$ in the absence of its rival. The existence of one traveling wave solution has been studied in [1]. In this remark, we show the existence of two traveling wave solutions by employing the cross-iteration method, which has been successfully used in many literatures, see e.g., [1, 3, 4, 5] and the references cited therein.

Our main theorem is now in the following:

THEOREM 1. Suppose that $\alpha_1 p_2 < \alpha_2 \eta_1$, and $\alpha_2 p_1 < \alpha_1 \eta_2$ in (1). Then for $c > c^*$, there exist two traveling wave solution $(u(t, x), v(t, x)) = (\phi(x + ct), \psi(x + ct))$ with

$$\lim_{\xi\to-\infty} (\phi(\xi), \psi(\xi)) = (0, 0), \quad \lim_{\xi\to\infty} (\phi(\xi), \psi(\xi)) = (e_1, e_2),$$

and

$$\lim_{\xi\to-\infty} \phi(\xi) e^{\Lambda_{11}(\xi)} = \lim_{\xi\to-\infty} \psi(\xi) e^{\Lambda_{21}(\xi)} = 1,$$

where $\Lambda_{11}(\xi)$ and $\Lambda_{21}(\xi)$ are small eigenvalues of $\Delta_1(\lambda, c)$ and $\Delta_2(\lambda, c)$, respectively.

2 Proofs

In [1], in order to prove the existence of traveling wave solutions for model (1), we constructed two pairs of functions $(\bar{\phi}(\xi), \bar{\psi}(\xi))$ and $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ as follows:

$$\bar{\phi}(\xi) = \begin{cases} e^{\Lambda_{11}\xi} & \text{for } \xi \leq \xi_3, \\ e_1 + \epsilon_3 e^{-\lambda\xi} & \text{for } \xi \geq \xi_3, \end{cases} \quad \bar{\psi}(\xi) = \begin{cases} e^{\Lambda_{21}\xi} & \text{for } \xi \leq \xi_4, \\ e_2 + \epsilon_4 e^{-\lambda\xi} & \text{for } \xi \geq \xi_4, \end{cases}$$

$$\underline{\phi}(\xi) = \begin{cases} e^{\Lambda_{11}\xi} - q_1 e^{\eta\Lambda_{11}\xi} & \text{for } \xi \leq \xi_1, \\ e_1 - \epsilon_1 e^{-\lambda\xi}, & \text{for } \xi \geq \xi_1, \end{cases} \quad \underline{\psi}(\xi) = \begin{cases} e^{\Lambda_{21}\xi} - q_2 e^{\eta\Lambda_{21}\xi} & \text{for } \xi \leq \xi_2, \\ e_2 - \epsilon_2 e^{-\lambda\xi} & \text{for } \xi \geq \xi_2, \end{cases}$$

where each $q_i > 1$ is sufficiently large and $\lambda > 0$ is sufficiently small.

We use the usual Banach space $\mathscr{B} := C(\mathbb{R}, \mathbb{R}^2)$ of bounded continuous functions endowed with the maximum norm $\|(\phi, \psi)\| = \sup_{\xi\in\mathbb{R}}(|\phi(\xi)| + |\psi(\xi)|)$. For any $c > c^*$, let

$$\mathscr{S}_c = \left\{(\phi, \psi) : (\phi, \psi) \in \mathscr{B},\ \underline{\phi}(\xi) \leq \phi(\xi) \leq \bar{\phi}(\xi),\ \underline{\psi}(\xi) \leq \psi(\xi) \leq \bar{\psi}(\xi)\right\}.$$

Clearly, $\mathscr{S}_c$ is a bounded nonempty closed convex subset of $\mathscr{B}$.

Define the operator $F = (F_1, F_2) : \mathscr{S}_c \to \mathscr{B}$ by

$$F_1(\phi,\psi)(\xi) := \alpha_1 \int_{\mathbb{R}} G_1(y)\phi(\xi - y - c\tau_1)dy - \eta_1\phi^2(\xi) - p_1\phi(\xi)\psi(\xi) + \beta_1\phi(\xi),$$

$$F_2(\phi,\psi)(\xi) := \alpha_2 \int_{\mathbb{R}} G_2(y)\psi(\xi - y - c\tau_1)dy - \eta_2\psi^2(\xi) - p_2\phi(\xi)\psi(\xi) + \beta_2\psi(\xi),$$

where each β_i is a large positive number. Then system (2)–(3) now can be rewritten as

$$\begin{cases} d_1\phi''(\xi) - c\phi'(\xi) - \beta_1\phi(\xi) + F_1(\phi,\psi)(\xi) = 0, \\ d_2\psi''(\xi) - c\psi'(\xi) - \beta_2\psi(\xi) + F_2(\phi,\psi)(\xi) = 0. \end{cases} \tag{4}$$

Let

$$\lambda_{11} = \frac{c - \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, \quad \lambda_{12} = \frac{c + \sqrt{c^2 + 4\beta_1 d_1}}{2d_1},$$
$$\lambda_{21} = \frac{c - \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}, \quad \lambda_{22} = \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}.$$

Clearly,

$$\lambda_{11} < 0 < \lambda_{12}, \ \lambda_{21} < 0 < \lambda_{22},$$
$$d_1\lambda_{1j}^2 - c\lambda_{1j} - \beta_1 = 0 \text{ and } d_2\lambda_{2j}^2 - c\lambda_{2j} - \beta_2 = 0 \text{ for } j = 1, 2.$$

Define the operator $Q = (Q_1, Q_2) : \mathscr{S}_c \to \mathscr{B}$ by

$$\begin{aligned} Q_1(\phi,\psi)(\xi) \quad &= \quad \frac{1}{d_1(\lambda_{12} - \lambda_{11})}\left(\int_{-\infty}^{\xi} e^{\lambda_{11}(\xi - s)}F_1(\phi,\psi)(s)ds \right. \\ &\quad \left. + \int_{\xi}^{\infty} e^{\lambda_{12}(\xi - s)}F_1(\phi,\psi)(s)ds\right) \end{aligned} \tag{5}$$

$$\begin{aligned} Q_2(\phi,\psi)(\xi) \quad &= \quad \frac{1}{d_2(\lambda_{22} - \lambda_{21})}\left(\int_{-\infty}^{\xi} e^{\lambda_{21}(\xi - s)}F_2(\phi,\psi)(s)ds \right. \\ &\quad \left. + \int_{\xi}^{\infty} e^{\lambda_{22}(\xi - s)}F_2(\phi,\psi)(s)ds\right) \end{aligned} \tag{6}$$

It is easily verified that the operator Q is well defined for $(\phi,\psi) \in \mathscr{S}_c$ and

$$\begin{cases} d_1Q_1(\phi,\psi)''(\xi) - cQ_1(\phi,\psi)'(\xi) - \beta_1Q_1(\phi,\psi)(\xi) + F_1(\phi,\psi)(\xi) = 0, \\ d_2Q_2(\phi,\psi)''(\xi) - cQ_2(\phi,\psi)'(\xi) - \beta_2Q_2(\phi,\psi)(\xi) + F_2(\phi,\psi)(\xi) = 0. \end{cases}$$

Thus the fixed of Q is the solution of (4), which is the travelling solution of (1).

We showed that $(\bar{\phi}(z), \bar{\psi}(z))$ is an upper solution and $(\underline{\phi}(z), \underline{\psi}(z))$ is a lower solution of the operator Q defined by (5) and (6) in the sense that

$$Q_1(\bar{\phi}, \underline{\psi})(\xi) \le \bar{\phi}(\xi), \ Q_2(\underline{\phi}, \bar{\psi})(z) \le \bar{\psi}(\xi), \tag{7}$$

$$Q_1(\underline{\phi}, \bar{\psi})(\xi) \geq \underline{\phi}(\xi),\ \ Q_2(\bar{\phi}, \underline{\psi})(z) \geq \underline{\psi}(\xi). \tag{8}$$

We also showed that for any $(\phi, \psi) \in \mathscr{S}_c$, $Q_1(\phi, \psi)$ is nondecreasing in ϕ and nonincreasing in ψ, and $Q_2(\phi, \psi)$ is nondecreasing in ψ and nonincreasing in ϕ. Define

$$(\underline{\phi}, \underline{\psi})(\xi) = (\underline{\phi}_0, \underline{\psi}_0)(\xi),\ \ (\bar{\phi}, \bar{\psi})(\xi) = (\bar{\phi}_0, \bar{\psi}_0)(\xi),$$

$$\underline{\phi}_1(\xi) = Q_1[\underline{\phi}_0, \bar{\psi}_0](\xi),\ \ \bar{\phi}_1(\xi) = Q_1[\bar{\phi}_0, \underline{\psi}_0](\xi),$$

$$\underline{\psi}_1(\xi) = Q_2[\bar{\phi}_0, \underline{\psi}_0](\xi),\ \ \bar{\psi}_1(\xi) = Q_2[\underline{\phi}_0, \bar{\psi}_0](\xi).$$

By (5)–(8), it follows that

$$(\underline{\phi}_0, \underline{\psi}_0)(\xi) \leq (\underline{\phi}_1, \underline{\psi}_1)(\xi) \leq (\bar{\phi}_1, \bar{\psi}_1)(\xi) \leq (\bar{\phi}_0, \bar{\psi}_0)(\xi).$$

For general cases we define

$$\begin{cases} \underline{\phi}_{k+1}(\xi) = Q_1[\underline{\phi}_k, \bar{\psi}_k](\xi),\ \ \bar{\phi}_{k+1}(\xi) = Q_1[\bar{\phi}_k, \underline{\psi}_k](\xi), \\ \underline{\psi}_{k+1}(\xi) = Q_2[\bar{\phi}_k, \underline{\psi}_k](\xi),\ \ \bar{\psi}_{k+1}(\xi) = Q_2[\underline{\phi}_k, \bar{\psi}_k](\xi), \end{cases} \tag{9}$$

for $k = 0, 1, 2, ...$ The inductive method show that

$$(\underline{\phi}_k, \underline{\psi}_k)(\xi) \leq (\underline{\phi}_{k+1}, \underline{\psi}_{k+1})(\xi) \leq (\bar{\phi}_{k+1}, \bar{\psi}_{k+1})(\xi) \leq (\bar{\phi}_k, \bar{\psi}_k)(\xi), \tag{10}$$

for $k = 0, 1, 2, ...$ and $\xi \in \mathbb{R}$.

One can easily check that $\underline{\phi}_k(\xi)$, $\underline{\psi}_k(\xi)$, $\bar{\phi}_k(\xi)$, and $\bar{\psi}_k(\xi)$ equicontinuous for $k = 0, 1, 2, ...$ and $\xi \in \mathbb{R}$. Furthermore, for $\xi \in \mathbb{R}$, the monotonicity of function sequences $\{\underline{\phi}_k(\xi)\}_{k=0}^{\infty}$, $\{\underline{\psi}_k(\xi)\}_{k=0}^{\infty}$, $\{\bar{\phi}_k(\xi)\}_{k=0}^{\infty}$, and $\{\bar{\psi}_k(\xi)\}_{k=0}^{\infty}$ implies that there exist two pairs of continuous functions $(\phi_\star, \psi^\star)(\xi)$ and $(\phi^\star, \psi_\star)(\xi)$ such that

$$\lim_{k\to\infty} \underline{\phi}_k(\xi) = \phi_\star(\xi),\ \ \lim_{k\to\infty} \underline{\psi}_k(\xi) = \psi_\star(\xi),$$

$$\lim_{k\to\infty} \bar{\phi}_k(\xi) = \phi^\star(\xi),\ \ \lim_{k\to\infty} \bar{\psi}_k(\xi) = \psi^\star(\xi),$$

convergence uniformly for all $\xi \in \mathbb{R}$ with respect to the super norm. In fact, for given $\varepsilon > 0$, by the construction of $(\bar{\phi}(\xi), \bar{\psi}(\xi))$ and $(\underline{\phi}(\xi), \underline{\psi}(\xi))$ there exists $M(\varepsilon) > 0$ such that

$$\sup_{|\xi|>M(\varepsilon)} \left|\bar{\phi}(\xi) - \underline{\phi}(\xi) + \bar{\psi}(\xi) - \underline{\psi}(\xi)\right| < \varepsilon.$$

Since $\bar{\phi}_k(\xi)$, $\bar{\psi}_k(\xi)$, $\underline{\phi}_k(\xi)$, and $\underline{\phi}_k(\xi)$ are equicontinuous, there exists $N(\varepsilon) > 0$ such that for any $m, n > N(\varepsilon)$,

$$\max_{|\xi|\leq T(\varepsilon)} \Big\{\left|\bar{\phi}_m(\xi) - \bar{\phi}_n(\xi))\right| + \left|\underline{\phi}_m(\xi) - \underline{\phi}_n(\xi))\right| + \left|\bar{\psi}_m(\xi) - \bar{\psi}_n(\xi))\right| \\ + \left|\underline{\psi}_m(\xi) - \underline{\psi}_n(\xi))\right|\Big\} < \varepsilon$$

Hence,

$$\sup_{\xi\in\mathbb{R}} \Big\{\left|\bar{\phi}_m(\xi) - \bar{\phi}_n(\xi))\right| + \left|\underline{\phi}_m(\xi) - \underline{\phi}_n(\xi))\right| + \left|\bar{\psi}_m(\xi) - \bar{\psi}_n(\xi))\right|$$

$$+\left|\underline{\psi}_m(\xi) - \underline{\psi}_n(\xi))\right|\Big\} < \varepsilon$$

It follows from the dominated convergence theorem and (9) that

$$\begin{cases} \phi_\star(\xi) = Q_1[\phi_\star, \psi^\star](\xi), \ \phi^\star(\xi) = Q_1[\phi^\star, \psi_\star](\xi), \\ \psi_\star(\xi) = Q_2[\phi^\star, \psi_\star](\xi), \ \psi^\star(\xi) = Q_2[\phi_\star, \psi^\star](\xi) \end{cases} \tag{11}$$

for all $\xi \in \mathbb{R}$. By (10) we obtain that

$$(\underline{\phi}, \underline{\psi})(\xi) \leq (\phi_\star, \psi_\star)(\xi) \leq (\phi^\star, \psi^\star)(\xi) \leq (\bar{\phi}, \bar{\psi})(\xi).$$

The operator Q defined by (5) and (6), and (11) show that the wave system (2)–(3) has two traveling wave solutions $(\phi_\star, \psi^\star)(\xi)$ and $(\phi^\star, \psi_\star)(\xi)$ between the super solution $(\bar{\phi}, \bar{\psi})(\xi)$ and the lower solution $(\underline{\phi}, \underline{\psi})(\xi)$. Moreover, by the definitions of $(\bar{\phi}, \bar{\psi})(\xi)$ and $(\underline{\phi}, \underline{\psi})(\xi)$ one can see that the traveling waves have the following decay rate

$$\lim_{z\to-\infty} \phi_\star(z)R_1(\xi) = \lim_{z\to-\infty} \psi_\star(z)R_2(\xi) = \lim_{z\to-\infty} \phi^\star(z)R_1(\xi) = \lim_{z\to-\infty} \psi^\star(z)R_2(\xi) = 1$$

with $R_1(\xi) = e^{-\Lambda_{11}(c)(\xi)}$ and $R_2(\xi) = e^{-\Lambda_{21}(c)(\xi)}$. The proof for Theorem 1 is complete.

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