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Inequalities Of (k, s), (k, h)-Type For Riemann-Liouville Fractional Integrals *

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Abstract

The main objective of this paper is to establish some new integral inequalities by using the $(k,h),(k,s)$-Riemann-Liouville fractional integrals in the case of synchronous functions.

1 Introduction

During the last two decades, many authors have studied some well-known inequalities and their applications using Riemann-Liouville fractional derivative and integral. For more about these, see [1–10] and the related references therein.

DEFINITION 1 ([13]). Two integrable functions f and g are said to be synchronous on $[a,b]$ if

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \text{ for all } x \in [a,b].$$

Recently, in [13] Dahmani gave the following fractional integral inequalities, using standard Riemann-Liouville fractional integral:

THEOREM 1 ([13, Theorem 2]). Let f,g be two synchronous functions on $[0,\infty)$ and let $p,q,r : [0,\infty) \to [0,\infty)$, then for all $t > a \geq 0$, $\alpha > 0$ the following (k,h)-fractional integral inequality

$$\begin{aligned} & 2J^\alpha r(t)\left[J^\alpha p(t)\, J^\alpha(qfg)(t) + J^\alpha q(t)\, J^\alpha(pfg)(t)\right] + 2J^\alpha p(t) J^\alpha q(t) J^\alpha(rfg)(t) \\ \geq\ & J^\alpha r(t)\left[J^\alpha(pf)(t)\, J^\alpha(qg)(t) + J^\alpha(qf)(t)\, J^\alpha(pg)(t)\right] \\ & +J^\alpha p(t)\left[J^\alpha(rf)(t)\, J^\alpha(qg)(t) + J^\alpha(qf)(t)\, J^\alpha(rg)(t)\right] \\ & +J^\alpha q(t)\left[J^\alpha(rf)(t)\, J^\alpha(pg)(t) + J^\alpha(pf)(t)\, J^\alpha(rg)(t)\right] \end{aligned}$$

holds.

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THEOREM 2 ([13, Theorem 4]). Let f,g be two synchronous functions on $[0,\infty)$ and let $p,q,r:[0,\infty)\rightarrow[0,\infty)$, then for all $t>a\geq 0$, $\alpha>0$, $\beta>0$ the following (k,h)-fractional integral inequality

$$\begin{aligned} & J^{\alpha}r(t)\Big[J^{\alpha}q(t)\,J^{\beta}(pfg)(t)+2J^{\alpha}p(t)\,J^{\beta}(qfg)(t)+J^{\beta}q(t)\,J^{\alpha}(pfg)(t)\Big] \\ & +\Big[J^{\alpha}p(t)J^{\beta}q(t)+J^{\beta}p(t)J^{\alpha}q(t)\Big]J^{\alpha}(rfg)(t) \\ \geq\ & J^{\alpha}r(t)\left[J^{\alpha}(pf)(t)\,J^{\beta}(qg)(t)+J^{\beta}(qf)(t)\,J^{\alpha}(pg)(t)\right] \\ & +J^{\alpha}p(t)\left[J^{\alpha}(rf)(t)\,J^{\beta}(qg)(t)+J^{\beta}(qf)(t)\,J^{\alpha}(rg)(t)\right] \\ & +J^{\alpha}q(t)\left[J^{\alpha}(rf)(t)\,J^{\beta}(pg)(t)+J^{\beta}(pf)(t)\,J^{\alpha}(rg)(t)\right] \end{aligned}$$

holds.

In literature few results have been obtained on some fractional integral inequalities for k-fractional integrals in [14, 15, 16]. Motivated by [16, 17, 18], our purpose in this work is to establish some inequalities for generalized k-fractional integrals that are called in the literature by (k,s) and (k,h)-Riemann-Liouville fractional integrals which are stated in Theorems 3 and 4 of the last section.

2 Preliminaries

Here, we will give the necessary notation and basic definitions. Due to page restrictions, only the basic definitions of the (k,s), (k,h)-Riemann-Liouville fractional integrals are given, and the reader is referred to [14–18] for more details.

DEFINITION 2. Let $0<x\leq b$, $\alpha>0$ and $f\in L_1(a,b)$, then the k-fractional integral of the Riemann-Liouville type is defined as follows:

$${}_kJ^{\alpha}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_0^x(x-t)^{\frac{\alpha}{k}-1}f(t)dt,$$

where k-gamma function is defined by

$$\Gamma_k(x)=\int_0^{\infty}t^{\frac{x}{k}-1}e^{-\frac{t^k}{k}}dt.$$

Note that when $k\rightarrow 1$, then the k-fractional integral reduces to the classical Riemann-liouville fractional integral [11, 12].

DEFINITION 3. Let $a\leq x\leq b$ and $f\in L_1(a,b)$, then the (k,s)-Riemann-Liouville fractional integral of f of order $\alpha>0$ is defined by

$${}^s_kJ^{\alpha}_af(x)=\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\alpha}{k}-1}t^s\,f(t)dt,$$

where $k>0$ and $s\in\mathbb{R}\setminus\{-1\}$.

DEFINITION 4. Let $a \leq x \leq b$ and $f \in L_1(a,b)$, then the (k,h)-Riemann-Liouville fractional integral of f of order $\alpha > 0$ is defined by

$$\left({}_kJ^{\alpha}_{a^+,h}\right) f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (h(x)-h(t))^{\frac{\alpha}{k}-1} h'(t) f(t) dt,$$

where $k > 0$.

3 Main Results

To obtain the first main theorem, we prove the following Lemma 1.

LEMMA 1. Let f, g be two synchronous functions on $[0,\infty)$ and let $y, z \geq 0$, then for all $t > a \geq 0$ and $\alpha > 0$, the following inequality for (k,h)-fractional integrals

$$\begin{aligned} & \left({}_kJ^{\alpha}_{a^+,h}\right) y(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (zfg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) z(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (yfg)(t) \\ \geq & \left({}_kJ^{\alpha}_{a^+,h}\right) (yf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (zg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) (zf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (yg)(t) \end{aligned} \quad (1)$$

holds.

PROOF. Since f and g are two synchronous functions on $[0,\infty)$ then for all $\tau, \xi \geq 0$, we have

$$(f(\xi)-f(\rho))(g(\xi)-g(\rho)) \geq 0.$$

This leads to

$$f(\xi)g(\xi) + f(\rho)g(\rho) \geq f(\xi)g(\rho) + f(\rho)g(\xi). \quad (2)$$

Multiplying both sides of (2) by $\frac{(h(t)-h(\xi))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} h'(\xi)y(\xi)$ for $\xi \in (a,t)$, then integrating the resulting inequalities with respect to ξ over (a,t), respectively, we obtain

$$\begin{aligned} & \left({}_kJ^{\alpha}_{a^+,h}\right) (yfg)(t) + f(\rho)g(\rho) \left({}_kJ^{\alpha}_{a^+,h}\right) y(t) \\ \geq \; & g(\rho) \left({}_kJ^{\alpha}_{a^+,h}\right) (yf)(t) + f(\rho) \left({}_kJ^{\alpha}_{a^+,h}\right) (yg)(t). \end{aligned} \quad (3)$$

Multiplying both sides of (3) by $\frac{(h(t)-h(\rho))^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)} h'(\rho)z(\rho)$ for $\rho \in (a,t)$, then integrating the resulting inequalities with respect to ρ over (a,t), we have

$$\begin{aligned} & \left({}_kJ^{\alpha}_{a^+,h}\right) y(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (zfg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) z(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (yfg)(t) \\ & \geq \left({}_kJ^{\alpha}_{a^+,h}\right) (yf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (zg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) (zf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (yg)(t), \end{aligned}$$

and so the proof is completed.

THEOREM 3. Let f, g be two synchronous functions on $[0,\infty)$ and let $y, z \geq 0$, then for all $t > a \geq 0$, $\alpha > 0$ the following (k,h)-fractional integral inequality

$$\begin{aligned}
& 2\left({}_kJ^{\alpha}_{a^+,h}\right) r(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right) p(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(qfg)(t) \\
& + \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pfg)(t)\Big] \\
& +2\left({}_kJ^{\alpha}_{a^+,h}\right) p(t)\left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\left({}_kJ^{\alpha}_{a^+,h}\right)(rfg)(t) \\
\geq\ & \left({}_kJ^{\alpha}_{a^+,h}\right) r(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right)(pf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(qg)(t) \\
& \qquad + \left({}_kJ^{\alpha}_{a^+,h}\right)(qf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pg)(t)\Big] \\
& + \left({}_kJ^{\alpha}_{a^+,h}\right) p(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right)(rf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(qg)(t) \\
& \qquad + \left({}_kJ^{\alpha}_{a^+,h}\right)(qf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(rg)(t)\Big] \\
& + \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right)(rf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pg)(t) \\
& \qquad + \left({}_kJ^{\alpha}_{a^+,h}\right)(pf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(rg)(t)\Big] \qquad (4)
\end{aligned}$$

holds.

PROOF. Put $v = p$ and $w = q$ into inequality (1), and then multiplying the resulting inequality by $\left({}_kJ^{\alpha}_{a^+,h}\right) r(t)$, we find

$$\begin{aligned}
& \left({}_kJ^{\alpha}_{a^+,h}\right) r(t)\left[\left({}_kJ^{\alpha}_{a^+,h}\right) p(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(qfg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pfg)(t)\right] \\
\geq\ & \left({}_kJ^{\alpha}_{a^+,h}\right) r(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right)(pf)(t)\left({}_kJ^{\alpha}_{a^+,h}\right)(qg)(t) \\
& + \left({}_kJ^{\alpha}_{a^+,h}\right)(qf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pg)(t)\Big]. \qquad (5)
\end{aligned}$$

Again, put $v = r$ and $w = q$ into inequality (1) and multiplying the result by $\left({}_kJ^{\alpha}_{a^+,h}\right) p(t)$, we get

$$\begin{aligned}
& \left({}_kJ^{\alpha}_{a^+,h}\right) p(t)\left[\left({}_kJ^{\alpha}_{a^+,h}\right) r(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(qfg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(rfg)(t)\right] \\
\geq\ & \left({}_kJ^{\alpha}_{a^+,h}\right) p(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right)(rf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(qg)(t) \\
& + \left({}_kJ^{\alpha}_{a^+,h}\right)(qf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(rg)(t)\Big]. \qquad (6)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
& \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\left[\left({}_kJ^{\alpha}_{a^+,h}\right) r(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pfg)(t) + \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(rfg)(t)\right] \\
\geq\ & \left({}_kJ^{\alpha}_{a^+,h}\right) q(t)\Big[\left({}_kJ^{\alpha}_{a^+,h}\right)(rf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(pg)(t) \\
& + \left({}_kJ^{\alpha}_{a^+,h}\right)(pf)(t)\ \left({}_kJ^{\alpha}_{a^+,h}\right)(rg)(t)\Big]. \qquad (7)
\end{aligned}$$

Adding the inequalities (5)–(7), we get the required inequality (4).

REMARK 1. If we choose $h(x)=\frac{x^{s+1}}{s+1}$, $s\neq -1$, then the inequality (4) reduces to the following (k,s)-fractional integral inequality

$$\begin{aligned}
& 2\,{}^s_kJ^\alpha_a r(t)\,[\,{}^s_kJ^\alpha_a p(t)\,{}^s_kJ^\alpha_a(qfg)(t)+\,{}^s_kJ^\alpha_a q(t)\,{}^s_kJ^\alpha_a(pfg)(t)] \\
& +2\,{}^s_kJ^\alpha_a p(t)\,{}^s_kJ^\alpha_a q(t)\,{}^s_kJ^\alpha_a(rfg)(t) \\
\geq\;& {}^s_kJ^\alpha_a r(t)\,[\,{}^s_kJ^\alpha_a(pf)(t)\,{}^s_kJ^\alpha_a(qg)(t)+\,{}^s_kJ^\alpha_a(qf)(t)\,{}^s_kJ^\alpha_a(pg)(t)] \\
& +{}^s_kJ^\alpha_a p(t)\,[\,{}^s_kJ^\alpha_a(rf)(t)\,{}^s_kJ^\alpha_a(qg)(t)+\,{}^s_kJ^\alpha_a(qf)(t)\,{}^s_kJ^\alpha_a(rg)(t)] \\
& +{}^s_kJ^\alpha_a q(t)\,[\,{}^s_kJ^\alpha_a(rf)(t)\,{}^s_kJ^\alpha_a(pg)(t)+\,{}^s_kJ^\alpha_a(pf)(t)\,{}^s_kJ^\alpha_a(rg)(t)]\,. \qquad (8)
\end{aligned}$$

To obtain the second theorem, we need the following Lemma 2.

LEMMA 2. Let f,g be two synchronous functions on $[0,\infty)$ and let $y,z\geq 0$. Then for all $t>a\geq 0,\alpha>0,\beta>0$, we have

$$\begin{aligned}
& \left({}_kJ^\alpha_{a^+,h}\right)y(t)\;\left({}_kJ^\beta_{a^+,h}\right)(zfg)(t)+\left({}_kJ^\beta_{a^+,h}\right)z(t)\;\left({}_kJ^\alpha_{a^+,h}\right)(yfg)(t) \\
\geq\;& \left({}_kJ^\alpha_{a^+,h}\right)(yf)(t)\;\left({}_kJ^\beta_{a^+,h}\right)(zg)(t)+\left({}_kJ^\beta_{a^+,h}\right)(zf)(t)\;\left({}_kJ^\alpha_{a^+,h}\right)(yg)(t) \qquad (9)
\end{aligned}$$

PROOF. Multiplying both sides of (3) by $\frac{(h(t)-h(\rho))^{\frac{\beta}{k}-1}}{k\Gamma_k(\beta)}h'(\rho)z(\rho)$, $\rho\in(a,t)$, then integrating the resulting inequalities with respect to ρ over (a,t), we have

$$\begin{aligned}
& \left({}_kJ^\alpha_{a^+,h}\right)y(t)\left({}_kJ^\beta_{a^+,h}\right)(zfg)(t)+\left({}_kJ^\beta_{a^+,h}\right)z(t)\;\left({}_kJ^\alpha_{a^+,h}\right)(yfg)(t) \\
\geq\;& \left({}_kJ^\alpha_{a^+,h}\right)(yf)(t)\;\left({}_kJ^\beta_{a^+,h}\right)(zg)(t)+\left({}_kJ^\beta_{a^+,h}\right)(zf)(t)\;\left({}_kJ^\alpha_{a^+,h}\right)(yg)(t).
\end{aligned}$$

This completes the proof of inequality (9).

THEOREM 4. Let f,g be two synchronous functions on $[0,\infty)$ and let $y,z\geq 0$. Then for all $t>a\geq 0,\alpha>0,\beta>0$ the following (k,h)-fractional integral inequality

$$\begin{aligned}
& \left({}_kJ^\alpha_{a^+,h}\right)r(t)\Big[\left({}_kJ^\alpha_{a^+,h}\right)q(t)\left({}_kJ^\beta_{a^+,h}\right)(pfg)(t) \\
& +2\left({}_kJ^\alpha_{a^+,h}\right)p(t)\left({}_kJ^\beta_{a^+,h}\right)(qfg)(t)+\left({}_kJ^\beta_{a^+,h}\right)q(t)\;\left({}_kJ^\alpha_{a^+,h}\right)(pfg)(t)\Big] \\
& +\Big[\left({}_kJ^\alpha_{a^+,h}\right)p(t)\left({}_kJ^\beta_{a^+,h}\right)q(t)+\left({}_kJ^\beta_{a^+,h}\right)p(t)\left({}_kJ^\alpha_{a^+,h}\right)q(t)\Big]\left({}_kJ^\alpha_{a^+,h}\right)(rfg)(t) \\
\geq\;& \left({}_kJ^\alpha_{a^+,h}\right)r(t)\Big[\left({}_kJ^\alpha_{a^+,h}\right)(pf)(t)\left({}_kJ^\beta_{a^+,h}\right)(qg)(t) \\
& +\left({}_kJ^\beta_{a^+,h}\right)(qf)(t)\left({}_kJ^\alpha_{a^+,h}\right)(pg)(t)\Big] \\
& +\left({}_kJ^\alpha_{a^+,h}\right)p(t)\Big[\left({}_kJ^\alpha_{a^+,h}\right)(rf)(t)\left({}_kJ^\beta_{a^+,h}\right)(qg)(t) \\
& +\left({}_kJ^\beta_{a^+,h}\right)(qf)(t)\left({}_kJ^\alpha_{a^+,h}\right)(rg)(t)\Big] \\
& +\left({}_kJ^\alpha_{a^+,h}\right)q(t)\Big[\left({}_kJ^\alpha_{a^+,h}\right)(rf)(t)\left({}_kJ^\beta_{a^+,h}\right)(pg)(t) \\
& +\left({}_kJ^\beta_{a^+,h}\right)(pf)(t)\;\left({}_kJ^\alpha_{a^+,h}\right)(rg)(t)\Big] \qquad (10)
\end{aligned}$$

holds.

PROOF. Using inequality (9) with $y = p$ and $z = q$, and then multiplying the resulting inequality by $\left({}_kJ^{\alpha}_{a^+,h}\right) r(t)$, we find

$$\begin{aligned}
&\left({}_kJ^{\alpha}_{a^+,h}\right) r(t) \left[\left({}_kJ^{\alpha}_{a^+,h}\right) p(t) \left({}_kJ^{\beta}_{a^+,h}\right) (qfg)(t) + \left({}_kJ^{\beta}_{a^+,h}\right) q(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (pfg)(t)\right] \\
\geq\; &\left({}_kJ^{\alpha}_{a^+,h}\right) r(t) \Big[\left({}_kJ^{\alpha}_{a^+,h}\right) (pf)(t) \left({}_kJ^{\beta}_{a^+,h}\right) (qg)(t) \\
&+ \left({}_kJ^{\beta}_{a^+,h}\right) (qf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (pg)(t)\Big]. \qquad (11)
\end{aligned}$$

Again, using inequality (9) with $y = r$ and $z = q$, we obtain

$$\begin{aligned}
&\left({}_kJ^{\alpha}_{a^+,h}\right) r(t) \left({}_kJ^{\beta}_{a^+,h}\right) (qfg)(t) + \left({}_kJ^{\beta}_{a^+,h}\right) q(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (rfg)(t) \\
\geq\; &\left({}_kJ^{\alpha}_{a^+,h}\right) (rf)(t) \left({}_kJ^{\beta}_{a^+,h}\right) (qg)(t) + \left({}_kJ^{\beta}_{a^+,h}\right) (qf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (rg)(t) \quad (12)
\end{aligned}$$

Multiplying both sides of (12) by $\left({}_kJ^{\alpha}_{a^+,h}\right) p(t)$, we get

$$\begin{aligned}
&\left({}_kJ^{\alpha}_{a^+,h}\right) p(t) \left[\left({}_kJ^{\alpha}_{a^+,h}\right) r(t) \left({}_kJ^{\beta}_{a^+,h}\right) (qfg)(t) + \left({}_kJ^{\beta}_{a^+,h}\right) q(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (rfg)(t)\right] \\
\geq\; &\left({}_kJ^{\alpha}_{a^+,h}\right) p(t) \Big[\left({}_kJ^{\alpha}_{a^+,h}\right) (rf)(t) \left({}_kJ^{\beta}_{a^+,h}\right) (qg)(t) \\
&+ \left({}_kJ^{\beta}_{a^+,h}\right) (qf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (rg)(t)\Big]. \qquad (13)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&\left({}_kJ^{\alpha}_{a^+,h}\right) q(t) \left[\left({}_kJ^{\alpha}_{a^+,h}\right) r(t) \left({}_kJ^{\beta}_{a^+,h}\right) (pfg)(t) + \left({}_kJ^{\beta}_{a^+,h}\right) q(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (rfg)(t)\right] \\
\geq\; &\left({}_kJ^{\alpha}_{a^+,h}\right) q(t) \Big[\left({}_kJ^{\alpha}_{a^+,h}\right) (rf)(t) \left({}_kJ^{\beta}_{a^+,h}\right) (pg)(t) \\
&+ \left({}_kJ^{\beta}_{a^+,h}\right) (pf)(t) \left({}_kJ^{\alpha}_{a^+,h}\right) (rg)(t)\Big]. \qquad (14)
\end{aligned}$$

Adding the inequalities (11)–(14), we get the inequality (9).

REMARK 2. If we choose $h(x) = \frac{x^{s+1}}{s+1}$, $s \neq -1$, then the equality (10) reduces to the following (k, s)-fractional integral

$$\begin{aligned}
&{}^s_kJ^{\alpha}_a r(t)\Big[{}^s_kJ^{\alpha}_a q(t)\, {}^s_kJ^{\alpha}_a (pfg)(t) + 2\, {}^s_kJ^{\alpha}_a p(t)\, {}^s_kJ^{\alpha}_a (qfg)(t) + {}^s_kJ^{\alpha}_a q(t)\, {}^s_kJ^{\alpha}_a (pfg)(t)\Big] \\
&+\Big[{}^s_kJ^{\alpha}_a p(t)\, {}^s_kJ^{\alpha}_a q(t) + {}^s_kJ^{\alpha}_a p(t)\, {}^s_kJ^{\alpha}_a q(t)\Big]\, {}^s_kJ^{\alpha}_a (rfg)(t) \\
\geq\; &{}^s_kJ^{\alpha}_a r(t)\, [{}^s_kJ^{\alpha}_a (pf)(t)\, {}^s_kJ^{\alpha}_a (qg)(t) + {}^s_kJ^{\alpha}_a (qf)(t)\, {}^s_kJ^{\alpha}_a (pg)(t)] \\
&+ {}^s_kJ^{\alpha}_a p(t)\, [{}^s_kJ^{\alpha}_a (rf)(t)\, {}^s_kJ^{\alpha}_a (qg)(t) + {}^s_kJ^{\alpha}_a (qf)(t)\, {}^s_kJ^{\alpha}_a (rg)(t)] \\
&+ {}^s_kJ^{\alpha}_a q(t)\, [{}^s_kJ^{\alpha}_a (rf)(t)\, {}^s_kJ^{\alpha}_a (pg)(t) + {}^s_kJ^{\alpha}_a (pf)(t)\, {}^s_kJ^{\alpha}_a (rg)(t)]\, .
\end{aligned}$$

REMARK 3. If f, g, r, p and q satisfy the following conditions,

(i) The function f and g are asynchronous on $[0, \infty)$.

(ii) The function r, p, q are negative on $[0, \infty)$.

(iii) Two of the function r, p, q are positive and the third is negative on $[0, \infty)$.

then the inequality (4) and (10) are reversed.

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