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On q-Integral Representations For q-Hahn And Askey-Wilson Polynomials And Certain Generating Functions*

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Abstract

In this paper, q-integral representations for q-Hahn and Askey-Wilson polynomials are given by the method of q-difference equation. Moreover, Srivastava-Agarwal type generating function for Al-Salam-Chihara polynomials and mixed generating function for Andrews-Askey polynomials are deduced by means of q-integral. At last, duality property of q-Hahn polynomials is obtained by the technique of transformation.

1 Introduction

The q-polynomials and their generating functions are very important and interesting sets of special functions and more specially of orthogonal polynomials. They appear in several branches of mathematics [2, 16, 29], e.g., continued fractions, Eulerian series, elliptic functions, quantum groups and algebras, discrete mathematics (combinatorics, graph theory), coding theory, etc. For more information, we refer to [1, 2, 15, 16, 24, 29, 32, 33, 34, 35, 38].

In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. The q-series and its compact factorials are defined respectively by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} \left(1 - aq^k\right), \quad (a;q)_\infty = \prod_{k=0}^{\infty} \left(1 - aq^k\right)$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where m is a positive integer and n is a non-negative integer or ∞.

The basic hypergeometric series ${}_r\phi_s$ [16, Eq. (1.2.22)] is given by

$${}_r\phi_s \left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left[(-1)^n q^{n(n-1)/2}\right]^{s+1-r},$$

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which is convergent for either $|q| < 1$ and $|z| < \infty$ when $r \leq s$ or $|q| < 1$ and $|z| < 1$ when $r = s + 1$, provided that no zero appears in the denominator.

The continuous q-Hermite polynomials [23, Eq. (3.26.1)]

$$H_n(x|q) = e^{in\theta}{}_2\phi_0 \left[\begin{matrix} q^{-n}, 0 \\ - \end{matrix} ; q, q^n e^{-2i\theta} \right], \quad x = \cos\theta \tag{1}$$

and the big q-Hermite polynomials [23, Eq. (3.18.1)]

$$H_n(x; a|q) = \frac{1}{a^n}{}_3\phi_2 \left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix} ; q, q \right], \quad x = \cos\theta. \tag{2}$$

The Al-Salam-Chihara polynomials [23, Eq. (3.8.1)]

$$Q_n(x; a, b|q) = \frac{(ab; q)_n}{a^n}{}_3\phi_2 \left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{matrix} ; q, q \right], \quad x = \cos\theta.$$

The continuous dual q-Hahn polynomials [23, Eq. (3.3.1)]

$$p_n(x; a, b, c|q) = \frac{(ab, ac; q)_n}{a^n}{}_3\phi_2 \left[\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{matrix} ; q, q \right], \quad x = \cos\theta. \tag{3}$$

Those q-polynomials are based on the Askey-scheme [23]. For more information, we refer to [16, 23].

The Askey-Wilson polynomials [23, Eq. (3.1.1)]

$$p_n(a, b, c, d; \cos\theta) = \frac{(ab, ac, ad; q)_n}{a^n}{}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right]$$

are the ${}_4\phi_3$ polynomials. Note that the product of the denominator parameters in this ${}_4\phi_3$ series is q times the product of the numerator parameters, and the argument of the function is q. Basic hypergeometric series with this property are called balanced [16]. The Askey-Wilson polynomials are the most general family of orthogonal polynomials that share the properties of the classical polynomials of Jacobi, Hermite and Laguerre. The Askey-Wilson polynomials provide a basic generalization or a q-analog of Wigner's $6 - j$ symbols and the Racah coefficients. For more information, please refer to [4, 18, 19, 20, 21, 26, 27, 36].

The concept of the q-integral has proved to be very useful in analyzing q-special functions. Ismail and Stanton [20] deduced the following q-integral representations for q-Hermite and Al-Salam-Chihara polynomials [20, Theorem 2.1 and 4.1]

$$H(\cos\theta|q) = \frac{1}{(1-q)e^{i\theta}\left(q, qe^{2i\theta}, e^{-2i\theta}; q\right)_\infty} \int_{e^{-i\theta}}^{e^{i\theta}} y^n \left(qye^{i\theta}, qye^{-i\theta}; q\right)_\infty \mathrm{d}_q y,$$

$$Q_n(\cos\theta; t_1, t_2|q) = \frac{t_1^n\left(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta}; q\right)_\infty}{(1-q)e^{i\theta}\left(q, t_1t_2, qe^{2i\theta}, e^{-2i\theta}; q\right)_\infty} \int_{e^{-i\theta}}^{e^{i\theta}} y^n \frac{\left(qye^{i\theta}, qye^{-i\theta}; q\right)_\infty}{(t_1y, t_2y; q)_\infty} \mathrm{d}_q y.$$

The author [9] given the q-integral representations for Andrews-Askey polynomials [9, Eq. (40)]

$$\begin{aligned} A_n^{(a,b)}(x,y|q) &\triangleq \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay,by;q)_k}{(abxy;q)_k} x^k y^{n-k} \\ &= \frac{(ax,ay,bx,by;q)_\infty}{y(1-q)\big(q,qy/x,x/y,abxy;q\big)_\infty} \int_x^y \frac{t^n\big(qt/x,qt/y;q\big)_\infty}{(at,bt;q)_\infty} \mathrm{d}_q t. \end{aligned} \tag{4}$$

Motivated by [20], one may ask naturally a question: Whether there exist q-integral representations for q-Hahn and Askey-Wilson polynomials? For more information, please refer to [7, 9, 10, 20, 30, 31, 37].

The purpose of this paper is to answer the above question and represent Askey-Wilson polynomials in terms of q-integrals as follows.

THEOREM 1. For $n \in \mathbb{N}$, we have

$$\begin{aligned} &{}_4\phi_3\left[\begin{matrix} q^{-n}, ax, ay, acdxy\alpha/q \\ abxy, acxy, adxy \end{matrix}; q, q\right] \\ =\; & \frac{a^n\big(ax,ay,bx,by,acdxy\alpha/q,cy,dy;q\big)_\infty}{y(1-q)\big(q,qy/x,x/y,abxy,acxy,adxy,cdy\alpha;q\big)_\infty} \int_x^y \frac{t^n\big(qt/x,qt/y;q\big)_\infty}{(at,bt;q)_\infty} \\ &\times \left\{ \sum_{j=0}^{\infty} \frac{\big(x/t,q/(by);q\big)_j (cy)^j}{\big(q,q/(bt);q\big)_j} {}_3\phi_2\left[\begin{matrix} q^{-j}, q/(cy), q/(c\alpha) \\ q^2/(cdy\alpha), 0 \end{matrix}; q, q\right] \right\} \mathrm{d}_q t. \end{aligned} \tag{5}$$

COROLLARY 2. For $n \in \mathbb{N}$, we have

$$\begin{aligned} &p_n(a,b,c,d;\cos\theta) \\ =\; & \frac{\big(ae^{-i\theta},ae^{i\theta},be^{i\theta},be^{-i\theta},abcdq^{n-1},ce^{i\theta},de^{i\theta};q\big)_\infty}{e^{i\theta}(1-q)\big(q,qe^{-2i\theta},e^{2i\theta},abq^n,acq^n,adq^n,bcdq^ne^{i\theta};q\big)_\infty} \int_{e^{-i\theta}}^{e^{i\theta}} \frac{t^n\big(qte^{i\theta},qte^{-i\theta};q\big)_\infty}{(at,bt;q)_\infty} \\ &\times \left\{ \sum_{j=0}^{\infty} \frac{\big(e^{-i\theta}/t,qe^{-i\theta}/b;q\big)_j (ce^{i\theta})^j}{\big(q,q/(bt);q\big)_j} {}_3\phi_2\left[\begin{matrix} q^{-j}, qe^{-i\theta}/c, q^{1-n}/(bc) \\ q^{2-n}e^{-i\theta}/(bcd), 0 \end{matrix}; q, q\right] \right\} \mathrm{d}_q t. \end{aligned} \tag{6}$$

COROLLARY 3. For $n \in \mathbb{N}$, we have

$$\begin{aligned} &p_n(x;a,b,c|q) \\ =\; & \frac{\big(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta};q\big)_\infty}{e^{in\theta}(1-q)\big(q,qe^{2i\theta},e^{-2i\theta},abq^n,acq^n;q\big)_\infty} \\ &\times \int_{e^{-i\theta}}^{e^{i\theta}} \frac{t^n\big(qte^{i\theta},qte^{-i\theta};q\big)_\infty}{(at,bt;q)_\infty} {}_2\phi_1\left[\begin{matrix} e^{-i\theta}/t, qe^{-i\theta}/b \\ q/(bt) \end{matrix}; q, ce^{i\theta}\right] \mathrm{d}_q t. \end{aligned} \tag{7}$$

REMARK 4. Letting $(x, y, \alpha) = \left(e^{-i\theta}, e^{i\theta}, bq^n\right)$ and $(x, y, d, \alpha) = \left(e^{-i\theta}, e^{i\theta}, 0, bq^n\right)$ in Theorem 1, equation (5) reduces to (6) and (7) respectively.

The rest of the paper will be organized as follows. In section 2, we deduce the main theorem by the method of q-difference equation. In section 3, we gain a new Srivastava-Agarwal type generating function for Al-Salam-Chihara polynomials by using q-integral. In section 4, we obtain duality property of q-Hahn polynomials by technique of transformation. In section 5, we derive a mixed generating function for Andrews-Askey polynomials by means of q-integral.

2 Proof of the Main Theorem

In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. The q-series and its compact factorials are defined respectively by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right), \quad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1 - aq^k\right)$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n$, where m is a positive integer and n is a non-negative integer or ∞.

Jackson defined the q-integral by [22]

$$\int_0^\infty f(t)\mathrm{d}_q t = d(1-q)\sum_{n=0}^{\infty} f\left(dq^n\right)q^n, \quad \int_c^d f(t)\mathrm{d}_q t = \int_0^d f(t)\mathrm{d}_q t - \int_0^c f(t)\mathrm{d}_q t.$$

For more information about q-series and q-integrals, please refer to [9, 16]. The following two q-difference operators are defined by [14, 16]

$$D_a\{f(a)\} = \frac{f(a) - f(aq)}{a}, \quad \theta_a\{f(a)\} = \frac{f\left(aq^{-1}\right) - f(a)}{q^{-1}a}.$$

Chen and Liu [13, 14] employed the technique of Parameter Augmentation by constructed the following two q-exponential operators

$$\mathbb{T}(bD_a) = \sum_{n=0}^{\infty}\frac{\left(bD_a\right)^n}{(q;q)_n}, \quad \mathbb{E}(b\theta_a) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}\left(b\theta_a\right)^n}{(q;q)_n}.$$

Later, authors [12, 15] researched on the following general q-exponential operators

$$\mathbb{T}(a, bD_c) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(bD_c)^n, \quad \mathbb{E}(a, -b\theta_c) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(-b\theta_c)^n.$$

To prove Theorem 1, the following lemmas are necessary.

LEMMA 5 ([25, Theorems 1 and 2]). Let $f(a,b)$ be a two variables analytic function in a neighbourhood of $(a,b) = (0,0) \in \mathbb{C}^2$.

(A.1) If $f(a,b)$ satisfies the difference equation

$$bf(aq,b)-af(a,bq)=(b-a)f(a,b),$$

then we have

$$f(a,b)=\mathbb{T}(bD_a)\{f(a,0)\}.$$

(A.2) If $f(a,b)$ satisfies the difference equation

$$af(aq,b)-bf(a,bq)=(a-b)f(aq,bq), \tag{8}$$

then we have

$$f(a,b)=\mathbb{E}(b\theta_a)\{f(a,0)\}.$$

LEMMA 6 ([26, Eq. (1.5)]). For $\max\{|bt|,|cy|\}<1$, we have

$$\mathbb{E}(c\theta_b)\left\{\frac{(bx,by;q)_\infty}{(bt;q)_\infty}\right\}=\frac{(bx,by,cy;q)_\infty}{(bt;q)_\infty}{}_2\phi_1\left[\begin{matrix}x/t,q/(by)\\q/(bt)\end{matrix};q,cy\right]. \tag{9}$$

LEMMA 7 ([40, Eq. (11)]). For $n\in\mathbb{N}$ and $|adst/q|<1$, we have

$$\mathbb{E}(d\theta_a)\{a^n(as,at;q)_\infty\}=\frac{a^n(as,at,ds,dt;q)_\infty}{\big(adst/q;q\big)_\infty}{}_3\phi_2\left[\begin{matrix}q^{-n},q/(as),q/(at)\\q^2/(adst),0\end{matrix};q,q\right]. \tag{10}$$

PROOF OF THEOREM 1. Equation (4) may be written equivalently as

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\big(q^{-n},ax,ay;q\big)_kq^k}{(q;q)_k}\cdot\big(abxyq^k;q\big)_\infty\\ =\;&\frac{a^n(ax,ay;q)_\infty}{y(1-q)\big(q,qy/x,x/y;q\big)_\infty}\int_x^y\frac{t^n\big(qt/x,qt/y;q\big)_\infty}{(at;q)_\infty}\cdot\frac{(bx,by;q)_\infty}{(bt;q)_\infty}\mathrm{d}_qt.\end{aligned} \tag{11}$$

Let $f(b,c)$ denote

$$f(b,c)\triangleq\sum_{k=0}^{n}\frac{\big(q^{-n},ax,ay;q\big)_kq^k}{(q;q)_k}\cdot\big(abxyq^k,acxyq^k;q\big)_\infty. \tag{12}$$

It's easy to verify that $f(b,c)$ satisfies equation (8), so by (11) and (9), we have

$$\begin{aligned}&f(b,c)\\ =\;&\mathbb{E}(b\theta_a)\{f(a,0)\}\\ =\;&\mathbb{E}(b\theta_a)\left\{\sum_{k=0}^{n}\frac{\big(q^{-n},ax,ay;q\big)_kq^k}{(q;q)_k}\cdot\big(abxyq^k;q\big)_\infty\right\}\\ =\;&\frac{a^n(ax,ay;q)_\infty}{y(1-q)\big(q,qy/x,x/y;q\big)_\infty}\int_x^y\frac{t^n\big(qt/x,qt/y;q\big)_\infty}{(at;q)_\infty}\cdot\mathbb{E}(b\theta_a)\left\{\frac{(bx,by;q)_\infty}{(bt;q)_\infty}\right\}\mathrm{d}_qt\\ =\;&\frac{a^n(ax,ay;q)_\infty}{y(1-q)\big(q,qy/x,x/y;q\big)_\infty}\\ &\times\int_x^y\frac{t^n\big(qt/x,qt/y;q\big)_\infty}{(at;q)_\infty}\cdot\frac{(bx,by,cy;q)_\infty}{(bt;q)_\infty}{}_2\phi_1\left[\begin{matrix}x/t,q/(by)\\q/(bt)\end{matrix};q,cy\right]\mathrm{d}_qt.\end{aligned} \tag{13}$$

By simplification of equation (12) and (13), we have

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{\left(q^{-n},ax,ay;q\right)_k q^k}{(q,abxy;q)_k}\cdot\left(acxyq^k;q\right)_\infty \\
= \quad &\frac{a^n(ax,ay,bx,by;q)_\infty}{y(1-q)\left(q,qy/x,x/y,abxy;q\right)_\infty} \\
&\times\int_x^y \frac{t^n\left(qt/x,qt/y;q\right)_\infty}{(at,bt;q)_\infty}\sum_{j=0}^{\infty}\frac{\left(x/t,q/(by);q\right)_j y^j}{\left(q,q/(bt);q\right)_j}\cdot c^j(cy;q)_\infty \mathrm{d}_q t. \qquad (14)
\end{aligned}$$

Equation (5) can also be written as

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{\left(q^{-n},ax,ay;q\right)_k q^k}{(q,abxy;q)_k}\cdot\frac{\left(acxyq^k,adxyq^k,c\alpha,d\alpha;q\right)_\infty}{\left(acdxy\alpha q^{k-1};q\right)_\infty} \\
= \quad &\frac{a^n(ax,ay,bx,by;q)_\infty}{y(1-q)\left(q,qy/x,x/y,abxy;q\right)_\infty}\int_x^y \frac{t^n\left(qt/x,qt/y;q\right)_\infty}{(at,bt;q)_\infty}\sum_{j=0}^{\infty}\frac{\left(x/t,q/(by);q\right)_j y^j}{\left(q,q/(bt);q\right)_j} \\
&\times\frac{c^j(cy,c\alpha,dy,d\alpha;q)_\infty}{\left(cdy\alpha/q;q\right)_\infty}{}_3\phi_2\left[\begin{matrix} q^{-j},q/(cy),q/(c\alpha) \\ q^2/(cdy\alpha),0\end{matrix};q,q\right]\mathrm{d}_q t. \qquad (15)
\end{aligned}$$

We use $g(c,d)$ to denote the left-right hand side of (15), it's checked that $g(c,d)$ also satisfies equation (8). So by (14), we have

$$\begin{aligned}
g(c,d) &= \mathbb{E}(d\theta_c)\{g(c,0)\} \\
&= \mathbb{E}(d\theta_c)\left\{\sum_{k=0}^{n}\frac{\left(q^{-n},ax,ay;q\right)_k q^k}{(q,abxy;q)_k}\cdot\left(acxyq^k,c\alpha;q\right)_\infty\right\} \\
&= \frac{a^n(ax,ay,bx,by;q)_\infty}{y(1-q)\left(q,qy/x,x/y,abxy;q\right)_\infty} \\
&\quad\times\int_x^y \frac{t^n\left(qt/x,qt/y;q\right)_\infty}{(at,bt;q)_\infty}\sum_{j=0}^{\infty}\frac{\left(x/t,q/(by);q\right)_j y^j}{\left(q,q/(bt);q\right)_j}\mathbb{E}(d\theta_c)\left\{c^j(cy,c\alpha;q)_\infty\right\}\mathrm{d}_q t,
\end{aligned}$$

which is the right-hand side of (15) after using formula (10). The proof is complete.

3 A New Srivastava-Agarwal Type Generating Function for Al-Salam-Chihara Polynomials

Note that as $q \to 1^{-1}$, the Al-Salam-Chihara polynomials $Q_n(x;a,b|q)$ become the simple monomials $(2x-a-b)^n$. To see another interesting limits by setting one or both parameters a and b to be zero. In these cases the polynomials $Q_n(x;a,b|q)$ reduce to q-generalizations of the classical Hermite polynomials, the so-called continuous big q-Hermite polynomials (2) and the continuous q-Hermite polynomials (1). For more information, please refer to [5, 23].

The author [8] considered Srivastava-Agarwal type generating functions by the method of q-exponential decomposition. For more information, please refer to [8].

In this section, we study the following Srivastava-Agarwal type generating functions for Al-Salam-Chihara polynomials by the way of q-integral.

THEOREM 8. For $\max\{|t_1t|, |st_1t/t_2|\} < 1$, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(s;q)_n t^n}{(q;q)_n} Q_n(\cos\theta; t_1, t_2|q) \\ =\ & \frac{\big(t_1t_2t, st_1t/t_2; q\big)_\infty}{\big(t_1te^{-i\theta}, t_1te^{i\theta}; q\big)_\infty} {}_3\phi_2\left[\begin{matrix} t_2e^{i\theta}, t_2e^{-i\theta}, t_1t_2/s \\ t_1t_2, t_1t_2t \end{matrix}; q, \frac{st_1t}{t_2}\right]. \end{aligned} \tag{16}$$

COROLLARY 9 ([23, Eq. (3.8.13)]). We have

$$\sum_{n=0}^{\infty} \frac{(t_1t_2;q)_n t^n}{(q;q)_n t_1^n} Q_n(\cos\theta; t_1, t_2|q) = \frac{(tt_1, tt_2; q)_\infty}{\big(te^{i\theta}, te^{-i\theta}; q\big)_\infty}. \tag{17}$$

REMARK 10. Let $(s,t) = \big(t_1t_2, t/t_1\big)$ in Theorem 3, equation (16) reduces to (17) directly.

Before the proof, the following lemma is necessary.

LEMMA 11 ([25, Eq. (4.2)]). We have

$$\begin{aligned} &\int_c^d \frac{(qt/c, qt/d, ft; q)_\infty}{(at, bt, et; q)_\infty} \mathrm{d}_q t \\ =\ & \frac{d(1-q)\big(q, c/d, qd/c, abcd, bcde, f/b; q\big)_\infty}{(ac, ad, bc, bd, ce, de; q)_\infty} {}_3\phi_2\left[\begin{matrix} bc, bd, abcde/f \\ abcd, bcde \end{matrix}; q, \frac{f}{b}\right]. \end{aligned} \tag{18}$$

PROOF OF THEOREM 8. The left-hand side of (16) is equal to

$$\begin{aligned} & \frac{\big(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta}; q\big)_\infty}{(1-q)e^{i\theta}\big(q, t_1t_2, qe^{2i\theta}, e^{-2i\theta}; q\big)_\infty} \int_{e^{-i\theta}}^{e^{i\theta}} \left\{\sum_{n=0}^{\infty} \frac{(s;q)_n (yt_1t)^n}{(q;q)_n}\right\} \frac{\big(qye^{i\theta}, qye^{-i\theta}; q\big)_\infty}{(t_1y, t_2y; q)_\infty} \mathrm{d}_q y \\ =\ & \frac{\big(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta}; q\big)_\infty}{(1-q)e^{i\theta}\big(q, t_1t_2, qe^{2i\theta}, e^{-2i\theta}; q\big)_\infty} \int_{e^{-i\theta}}^{e^{i\theta}} \frac{\big(qye^{i\theta}, qye^{-i\theta}, st_1ty; q\big)_\infty}{(t_1y, t_2y, t_1ty; q)_\infty} \mathrm{d}_q y \quad \text{by (18)} \\ =\ & \frac{\big(t_1e^{i\theta}, t_1e^{-i\theta}, t_2e^{i\theta}, t_2e^{-i\theta}; q\big)_\infty}{(1-q)e^{i\theta}\big(q, t_1t_2, qe^{2i\theta}, e^{-2i\theta}; q\big)_\infty} \frac{e^{i\theta}(1-q)\big(q, e^{-2i\theta}, qe^{2i\theta}, t_1t_2, t_1t_2t, st_1t/t_2; q\big)_\infty}{\big(t_1e^{-i\theta}, t_1e^{i\theta}, t_2e^{-i\theta}, t_2e^{i\theta}, t_1te^{-i\theta}, t_1te^{i\theta}; q\big)_\infty} \\ & \times {}_3\phi_2\left[\begin{matrix} t_2e^{i\theta}, t_2e^{-i\theta}, t_1t_2/s \\ t_1t_2, t_1t_2t \end{matrix}; q, \frac{st_1t}{t_2}\right], \end{aligned}$$

which is the right-hand side of (16) after simplification. The proof is complete.

4 Duality Property of q-Hahn Polynomials

The continuous dual q-Hahn polynomials (3) is a particular case of the Askey-Wilson polynomials. The limit process of $q \to 1$ gives the case of associated continuous Hahn polynomials which have been studied by Ismail et al. [21]. In both of these associated cases we made extensive use of contiguous relations for hypergeometric and q-hypergeometric functions. Although the use of contiguous relations in connection with continued fractions goes back to Gauss [28], the importance of contiguous relations in relation to the theory of orthogonal polynomials was first stressed by Wilson [39]. For more information, please refer to [17, 21, 23, 28, 39].

From the expression of (3), we find that the continuous dual q-Hahn polynomials are dual with respect to θ. In this section, we study duality property of q-Hahn polynomials from the view of their q-integral representations.

LEMMA 12 ([16, Eq. (Ⅲ.1)]). We have

$$ {}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};q,z\right] = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/b,z \\ az \end{matrix};q,b\right] \tag{19} $$

$$ = \frac{\big(abz/c;q\big)_\infty}{(z;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} c/a,c/b \\ c \end{matrix};q,\frac{abz}{c}\right]. \tag{20} $$

PROOF OF (3). Denoting $f(\theta)$ as the q-Hahn polynomials (7), by (20), we have

$$ \begin{aligned} & f(\theta) \\ = \ & \frac{\big(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta};q\big)_\infty}{e^{in\theta}(1-q)\big(q,qe^{2i\theta},e^{-2i\theta},abq^n,acq^n;q\big)_\infty} \\ & \times \int_{e^{-i\theta}}^{e^{i\theta}} \frac{t^n\big(qte^{i\theta},qte^{-i\theta};q\big)_\infty}{(at,bt;q)_\infty}\frac{\big(ce^{-i\theta};q\big)_\infty}{\big(ce^{i\theta};q\big)_\infty}\,{}_2\phi_1\left[\begin{matrix} e^{i\theta}/t,qe^{i\theta}/b \\ q/(bt) \end{matrix};q,ce^{-i\theta}\right]\mathrm{d}_q t. \end{aligned} $$

So we have

$$ \begin{aligned} f(-\theta) = & \frac{\big(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta};q\big)_\infty}{e^{-in\theta}(1-q)\big(q,qe^{-2i\theta},e^{2i\theta},abq^n,acq^n,ce^{-i\theta};q\big)_\infty} \\ & \times \int_{e^{i\theta}}^{e^{-i\theta}} \frac{t^n\big(qte^{i\theta},qte^{-i\theta};q\big)_\infty}{(at,bt;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} e^{-i\theta}/t,qe^{-i\theta}/b \\ q/(bt) \end{matrix};q,ce^{i\theta}\right]\mathrm{d}_q t \\ = & \frac{\big(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta};q\big)_\infty}{e^{in\theta}(1-q)\big(q,qe^{2i\theta},e^{-2i\theta},abq^n,acq^n;q\big)_\infty} \\ & \times \int_{e^{-i\theta}}^{e^{i\theta}} \frac{t^n\big(qte^{i\theta},qte^{-i\theta};q\big)_\infty}{(at,bt;q)_\infty}\,{}_2\phi_1\left[\begin{matrix} e^{-i\theta}/t,qe^{-i\theta}/b \\ q/(bt) \end{matrix};q,ce^{i\theta}\right]\mathrm{d}_q t \\ & \times \left\{-e^{2in\theta}\frac{\big(qe^{2in\theta},e^{-2i\theta};q\big)_\infty}{\big(qe^{-2i\theta},e^{2i\theta};q\big)_\infty}\right\}, \end{aligned} $$

which is equivalent to $f(\theta)$ after simplification. The proof is complete.

REMARK 13. We may deduce the duality property of Askey-Wilson polynomials by this method.

5 A Mixed Generating Function for Andrews-Askey Polynomials

The Andrews-Askey polynomials are important in the theory of q-polynomials [3, 9].

In this section, we deduce the following mixed generating function for Andrews-Askey polynomials.

THEOREM 14. For $M \in \mathbb{N}$ and $ay = q^{-M}$, we have

$$\begin{aligned} &\sum_{m,n=0}^{\infty} A_n^{(a,b)}(x,y|q) h_{m+n}(z|q) \frac{u^m v^n}{(q;q)_m (q;q)_n} \\ = &\frac{(uvyz;q)_\infty}{(u,uz,yvz,vy;q)_\infty} \sum_{n=0}^{\infty} \frac{(ay,yvz,by;q)_k (xv)^k}{(q,uvyz,abxy;q)_k} {}_3\phi_2 \left[\begin{matrix} u, byq^k, ayq^k \\ uvyzq^k, abxyq^k \end{matrix}; q, xvz \right] \end{aligned} \tag{21}$$

where $\max\{|u|, |uz|, |yvz|, |vy|, |xvz|, |xv|\} < 1$.

COROLLARY 15 ([11, Eq. (4.1)]). For $\max\{|uz|, |vtz|, |u|, |vt|\} < 1$, we have

$$\sum_{m,n=0}^{\infty} h_{m+n}(z|q) \frac{u^m (vt)^n}{(q;q)_m (q;q)_n} = \frac{(uvtz;q)_\infty}{(uz,vtz,u,vt;q)_\infty}. \tag{22}$$

COROLLARY 16 ([6, Eq. (1.4)]). For $\max\{|u|, |uz|, |v|, |xv|, |vz|, |xzv|\} < 1$, we have

$$\begin{aligned} &\sum_{m,n=0}^{\infty} h_n(x|q) h_{m+n}(z|q) \frac{u^m v^n}{(q;q)_m (q;q)_n} \\ = &\frac{(uvz;q)_\infty}{(u,uz,v,xv,vz;q)_\infty} {}_2\phi_1 \left[\begin{matrix} u, v \\ uvz \end{matrix}; q, xzv \right]. \end{aligned} \tag{23}$$

REMARK 17. Set $(a,b,x,y) = (0,0,0,t)$ and $(a,b,y) = (0,0,1)$ in Theorem 5, equation (21) reduces to (22) and (23) respectively.

Before the proof of the Theorem, the following lemmas are necessary.

LEMMA 18 ([9, Theorem 1]). For $M \in \mathbb{N}$ and $r = q^{-M}$, we have

$$\begin{aligned}
&\int_c^d \frac{\big(qt/c, qt/d, rst; q\big)_\infty}{(at, bt, et, st; q)_\infty} \mathrm{d}_q\, t \\
= \quad &\frac{d(1-q)\big(q, c/d, qd/c, abcd, rsd; q\big)_\infty}{(ac, bc, ad, bd, sd, de; q)_\infty} \\
&\times \sum_{n=0}^{\infty} \frac{(ad, sd, bd; q)_k (ce)^k}{(q, rsd, abcd; q)_k} {}_3\phi_2 \left[\begin{matrix} r, bdq^k, adq^k \\ rsdq^k, abcdq^k \end{matrix}; q, sc\right],
\end{aligned} \tag{24}$$

where $\max\{|ac|, |bc|, |ad|, |bd|, |sd|, |de|, |ce|\} < 1$.

PROOF. Setting $(f, g) = (0, c)$ in Theorem 1 of [9], we have the equation (24).

PROOF OF THEOREM 14. By (22), the left-hand side of (21) is equal to

$$\begin{aligned}
&\frac{(ax, ay, bx, by; q)_\infty}{y(1-q)\big(q, qy/x, x/y, abxy; q\big)_\infty} \\
&\quad \times \int_x^y \left\{ \sum_{m,n=0}^{\infty} h_{m+n}(z|q) \frac{u^m (vt)^n}{(q;q)_m (q;q)_n} \right\} \frac{\big(qt/x, qt/y; q\big)_\infty}{(at, bt; q)_\infty} \mathrm{d}_q t \\
&= \frac{(ax, ay, bx, by; q)_\infty}{y(1-q)\big(q, qy/x, x/y, abxy, u, uz; q\big)_\infty} \int_x^y \frac{\big(qt/x, qt/y, uvtz; q\big)_\infty}{(at, bt, vtz, vt; q)_\infty} \mathrm{d}_q t \\
&= \frac{(ax, ay, bx, by; q)_\infty}{y(1-q)\big(q, qy/x, x/y, abxy, u, uz; q\big)_\infty} \frac{y(1-q)\big(q, x/y, qy/x, abxy, uvyz; q\big)_\infty}{(ax, bx, ay, by, yvz, vy; q)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(ay, yvz, by; q)_k (xv)^k}{(q, uvyz, abxy; q)_k} {}_3\phi_2 \left[\begin{matrix} u, byq^k, ayq^k \\ uvyzq^k, abxyq^k \end{matrix}; q, xvz\right] \\
&= \frac{(uvyz; q)_\infty}{(u, uz, yvz, vy; q)_\infty} \sum_{n=0}^{\infty} \frac{(ay, yvz, by; q)_k (xv)^k}{(q, uvyz, abxy; q)_k} {}_3\phi_2 \left[\begin{matrix} u, byq^k, ayq^k \\ uvyzq^k, abxyq^k \end{matrix}; q, xvz\right],
\end{aligned}$$

which is the right-hand side of (21) after simplification. The proof is complete.

PROOF OF COROLLARY 16. By (19), the left-hand side of (23) is equal to

$$\begin{aligned}
&\frac{(uvz; q)_\infty}{(u, uz, vz, v; q)_\infty} \sum_{n=0}^{\infty} \frac{(vz; q)_k (xv)^k}{(q, uvz; q)_k} {}_2\phi_1 \left[\begin{matrix} u, 0 \\ uvzq^k \end{matrix}; q, xvz\right] \\
&= \frac{(uvz; q)_\infty}{(u, uz, vz, v; q)_\infty} \sum_{n=0}^{\infty} \frac{(vz; q)_k (xv)^k}{(q, uvz; q)_k} \frac{(u; q)_\infty}{\big(uvzq^k, xvz; q\big)_\infty} {}_2\phi_1 \left[\begin{matrix} vzq^k, xvz \\ 0 \end{matrix}; q, u\right] \\
&= \frac{1}{(uz, vz, v, xvz; q)_\infty} \sum_{n=0}^{\infty} \frac{(vz, xvz; q)_n u^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{\big(vzq^n; q\big)_k}{(q;q)_k} (xv)^k \\
&= \frac{\big(xv^2 z; q\big)_\infty}{(uz, vz, v, xvz, xv; q)_\infty} {}_2\phi_1 \left[\begin{matrix} vz, xvz \\ xv^2 z \end{matrix}; q, u\right],
\end{aligned}$$

which is the right-hand side of (23) after using (19) and simplification. The proof is complete.

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