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Coefficient Bounds For Certain Subclasses Of Analytic Functions*

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Abstract

In this paper, we introduce and investigate a subclass of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. Furthermore, we find upper bounds for the second and third coefficients for functions in this subclass. The results presented in this paper generalize and improve some recent works.

1 Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

The Koebe one-quarter Theorem [5] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of

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these functions. Recently there are interests to study the bi-univalent functions class Σ and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [12] (also [1, 3, 4, 13]). Many interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties and characteristics associated with the bi-univalent function class Σ (including also several open problems and conjectures involving estimates on the Taylor Maclaurin coefficients of functions in Σ), can be found in recent literatures [2, 7, 9, 10, 11, 15]. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2,3\})$ for each $f \in \Sigma$ is still an open problem. More recently Frasin [6] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

DEFINITION 1 ([6]). Let $0 < \eta \leq 1$ and $\lambda \geq 0$. A function $f(z)$ given by (1) is said to be in the class $H_\Sigma(\eta, \lambda)$ if the following conditions are satisfied

$$f \in \Sigma, \quad |arg(f'(z) + \lambda z f''(z))| < \frac{\eta\pi}{2}, \quad \text{and} \quad |arg(g'(w) + \lambda w g''(w))| < \frac{\eta\pi}{2},$$

where the function g is given by (2).

THEOREM 1 ([6]). Let $f(z)$ given by (1) be in the class $H_\Sigma(\eta, \lambda)$. Then

$$|a_2| \leq \frac{2\eta}{\sqrt{2(\eta+2) + 4\lambda(\eta + \lambda + 2 - \lambda\eta)}} \quad \text{and} \quad |a_3| \leq \frac{\eta^2}{(1+\lambda)^2} + \frac{2\eta}{3(1+2\lambda)}.$$

DEFINITION 2 ([6]). Let $0 \leq \beta < 1$ and $\lambda \geq 0$. A function $f(z)$ given by (1) is said to be in the class $H_\Sigma(\beta, \lambda)$ if the following conditions are satisfied

$$f \in \Sigma, \quad \mathfrak{Re}\,(f'(z) + \lambda z f''(z)) > \beta \quad \text{and} \quad \mathfrak{Re}\,(g'(w) + \lambda w g''(w)) > \beta,$$

where the function g is given by (2).

THEOREM 2 ([6]). Let $f(z)$ given by (1) be in the class $H_\Sigma(\beta, \lambda)$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3(1+2\lambda)}} \quad \text{and} \quad |a_3| \leq \frac{(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{3(1+2\lambda)}.$$

The purpose of our study is to investigate the bi-univalent function class $\mathcal{W}_\Sigma^{h,p}(\gamma, \lambda, \alpha)$ introduced here in Definition 3 and derive coefficient estimates on the first two Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$ for a function $f \in \mathcal{W}_\Sigma^{h,p}(\gamma, \lambda, \alpha)$ given by (1). Our results generalize and improve those in related works of several earlier authors.

2 The Subclass $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$

In this section, we introduce and investigate the general subclass $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$.

DEFINITION 3. Let the analytic functions $h, p : \mathbb{U} \to \mathbb{C}$ satisfying that

$$\min\{\mathfrak{Re}(h(z)), \mathfrak{Re}(p(z))\} > 0 \quad (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$

Let $\alpha \geqslant 0$, $\lambda \geqslant 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{A}$ given by (1) is said to be in the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ if the following conditions are satisfied

$$1 + \frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{f(z)}{z} + (\alpha-2\lambda)f'(z) + \lambda z f''(z) - 1\right] \in h(\mathbb{U}) \qquad (z \in \mathbb{U}), \quad (3)$$

and

$$1 + \frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{g(w)}{w} + (\alpha-2\lambda)g'(w) + \lambda w g''(w) - 1\right] \in p(\mathbb{U}) \qquad (w \in \mathbb{U}), \quad (4)$$

where the function g is defined by (2).

REMARK 1. There are many choices of h and p which would provide interesting subclasses of class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$. For example, if we take

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\eta} \qquad (0 < \eta \leq 1, \alpha \geqslant 0, \lambda \geqslant 0, \gamma \in \mathbb{C}\backslash\{0\}, z \in U),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$, then $f \in \Sigma$,

$$\left| arg\left(1 + \frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{f(z)}{z} + (\alpha-2\lambda)f'(z) + \lambda z f''(z) - 1\right]\right)\right| < \frac{\eta\pi}{2} \quad (z \in \mathbb{U}),$$

and

$$\left| arg\left(1 + \frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{g(w)}{w} + (\alpha-2\lambda)g'(w) + \lambda w g''(w) - 1\right]\right)\right| < \frac{\eta\pi}{2} \quad (w \in \mathbb{U}).$$

Therefore in this case we have the following items:

1. For $\gamma = 1,\ \alpha = 1 + 2\lambda$, the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ reduce to class $H_{\Sigma}(\eta,\lambda)$ in Definition 1.

2. For $\gamma = 1,\ \lambda = 0$, the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ reduce to class $B_{\Sigma}(\eta,\alpha)$ studied by Frasin and Aouf [7, Definition 2.1].

3. For $\gamma = 1,\ \lambda = 0,\ \alpha = 1$, the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ reduce to class H_{Σ}^{η} which studied by Srivastava [12, Definition 1].

If we take

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \quad (0 \le \beta < 1, \alpha \geqslant 0, \lambda \geqslant 0, \gamma \in \mathbb{C}\backslash\{0\}, z \in U),$$

then the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$, then

$$f \in \Sigma, \quad \mathfrak{Re}\left(1+\frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{f(z)}{z}+(\alpha-2\lambda)f'(z)+\lambda z f''(z)-1\right]\right) > \beta \quad (z \in \mathbb{U}),$$

and

$$\mathfrak{Re}\left(1+\frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{g(w)}{w}+(\alpha-2\lambda)g'(w)+\lambda w g''(w)-1\right]\right) > \beta, \quad (w \in \mathbb{U}).$$

Therefore in this case we have the following items:

1. For $\gamma = 1$ and $\alpha = 1+2\lambda$, the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ reduce to class $H_{\Sigma}(\beta,\lambda)$ in Definition 2.
2. For $\gamma = 1$ and $\lambda = 0$, the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ reduce to class $B_{\Sigma}(\beta,\alpha)$ studied by Frasin and Aouf [7, Definition 3.1].
3. For $\gamma = 1,\ \lambda = 0$ and $\alpha = 1$, the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$ reduce to class $H_{\Sigma}(\beta)$ which studied by Srivastava [12, Definition 2].

3 Coefficient Estimates

Now, we obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ for subclass $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$.

THEOREM 3. Let $f(z)$ given by (1) be in the class $\mathcal{W}_{\Sigma}^{h,p}(\gamma,\lambda,\alpha)$. Then

$$|a_2| \le \min\left\{\sqrt{\frac{|\gamma|^2(|h'|^2+|p'|^2)}{2(1+\alpha)^2}}, \sqrt{\frac{|\gamma|(|h''(0)|+|p''(0)|)}{4(1+2\alpha+2\lambda)}}\right\} \tag{5}$$

and

$$|a_3| \le \min\left\{\frac{|\gamma|^2(|h'|^2+|p'|^2)}{2(1+\alpha)^2}+\frac{|\gamma|(|h''(0)|+|p''(0)|)}{4(1+2\alpha+2\lambda)}, \frac{|\gamma||h''(0)|}{2(1+2\alpha+2\lambda)}\right\}. \tag{6}$$

PROOF. First of all, we write the argument inequalities in (3) and (4) in their equivalent forms as follows

$$1+\frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{f(z)}{z}+(\alpha-2\lambda)f'(z)+\lambda z f''(z)-1\right] = h(z) \qquad (z \in \mathbb{U}), \tag{7}$$

and

$$1+\frac{1}{\gamma}\left[(1-\alpha+2\lambda)\frac{g(w)}{w}+(\alpha-2\lambda)g'(w)+\lambda wg''(w)-1\right]=p(w) \qquad (w\in\mathbb{U}), \quad (8)$$

respectively, where functions h and p satisfy the conditions of Definition 3. Also, the functions h and p have the following Taylor-Maclaurin series expansions

$$h(z)=1+h_1z+h_2z^2+h_3z^3+\cdots, \quad (9)$$

and

$$p(w)=1+p_1w+p_2w^2+p_3w^3+\cdots. \quad (10)$$

Now, upon substituting from (9) and (10) into (7) and (8), respectively, and equating the coefficients, we get

$$(1+\alpha)a_2=\gamma h_1, \quad (11)$$

$$(1+2\alpha+2\lambda)a_3=\gamma h_2, \quad (12)$$

$$-(1+\alpha)a_2=\gamma p_1, \quad (13)$$

and

$$2(1+2\alpha+2\lambda)a_2^2-(1+2\alpha+2\lambda)a_3=\gamma p_2. \quad (14)$$

From (11) and (13), we get

$$h_1=-p_1 \ \text{ and } 2(1+\alpha)^2a_2^2=\gamma^2(h_1^2+p_1^2). \quad (15)$$

Adding (12) and (14), we get

$$2(1+2\alpha+2\lambda)a_2^2=\gamma(p_2+h_2). \quad (16)$$

Therefore, from (15) and (16), we have

$$a_2^2=\frac{\gamma^2(h_1^2+p_1^2)}{2(1+\alpha)^2} \ \text{ and } \ a_2^2=\frac{\gamma(p_2+h_2)}{2(1+2\alpha+2\lambda)}, \quad (17)$$

respectively. Therefore, we find from the equations (17), that

$$|a_2|^2\le\frac{|\gamma|^2(|h'|^2+|p'|^2)}{2(1+\alpha)^2} \ \text{ and } \ |a_2|^2\le\frac{|\gamma|(|h''(0)|+|p''(0)|)}{4(1+2\alpha+2\lambda)},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (5).

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (14) from (12), we get:

$$2(1+2\alpha+2\lambda)a_3-2(1+2\alpha+2\lambda)a_2^2=\gamma(h_2-p_2). \quad (18)$$

Upon substituting the value of a_2^2 from (17) into (18), it follows that

$$a_3=\frac{\gamma^2(h_1^2+p_1^2)}{2(1+\alpha)^2}+\frac{\gamma(h_2-p_2)}{2(1+2\alpha+2\lambda)}.$$

Therefore, we get

$$|a_3| \leq \frac{|\gamma|^2(|h'|^2 + |p'|^2)}{2(1+\alpha)^2} + \frac{|\gamma|(|h''(0)| + |p''(0)|)}{4(1+2\alpha+2\lambda)}. \tag{19}$$

On the other hand, upon substituting the value of a_2^2 from (17) into (18), it follows that

$$a_3 = \frac{\gamma(p_2 + h_2)}{2(1+2\alpha+2\lambda)} + \frac{\gamma(h_2 - p_2)}{2(1+2\alpha+2\lambda)} = \frac{\gamma h_2}{(1+2\alpha+2\lambda)}.$$

Therefore, we get:

$$|a_3| \leq \frac{|\gamma||h''(0)|}{2(1+2\alpha+2\lambda)}. \tag{20}$$

So we obtain from (19) and (20) the desired estimate on the coefficient $|a_3|$ as asserted in (6). This completes the proof.

4 Conclusions

If we take

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\eta \qquad (0 < \eta \leq 1,\ z \in \mathbb{U}),$$

in Theorem 3, we conclude the following result.

COROLLARY 1. Let the function $f(z)$ given by (1) be in the class $\mathcal{W}_\Sigma^\eta(\gamma, \lambda, \alpha)$. Then

$$|a_2| \leq \min\left\{\frac{2|\gamma|\eta}{\alpha+1}, \sqrt{\frac{2|\gamma|}{1+2\alpha+2\lambda}\eta}\right\},$$

and

$$|a_3| \leq \frac{2|\gamma|\eta^2}{1+2\alpha+2\lambda}.$$

By setting $\gamma = 1$ and $\alpha = 1 + 2\lambda$ in Corollary 1, we get the following corollary.

COROLLARY 2. Let the function f given by (1) be in the class $H_\Sigma(\eta, \lambda)$. Then

$$|a_2| \leq \min\left\{\frac{\eta}{\lambda+1}, \sqrt{\frac{2}{3(2\lambda+1)}\eta}\right\} \quad \text{and} \quad |a_3| \leq \frac{2\eta^2}{3(2\lambda+1)}.$$

REMARK 2. Corollary 2 is a refinement of Theorem 1.

If we set $\lambda = 0$ and $\gamma = 1$ in Corollary 1, then we have the following corollary.

COROLLARY 3. Let the function f given by (1) be in the class $B_\Sigma(\eta,\alpha)$. Then

$$|a_2| \le \min\left\{\frac{2\eta}{\alpha+1}, \sqrt{\frac{2}{2\alpha+1}}\eta\right\} \quad \text{and} \quad |a_3| \le \frac{2\eta^2}{2\alpha+1}.$$

REMARK 3. Corollary 3 provides an improvement of a result which obtained by Frasin and Aouf [7, Theorem 2.2].

If we take $\alpha = 1$ in Corollary 3, we get

COROLLARY 4. Let the function f given by (1) be in the class H_Σ^η. Then

$$|a_2| \le \sqrt{\frac{2}{3}}\eta \quad \text{and} \quad |a_3| \le \frac{2}{3}\eta^2.$$

REMARK 4. Corollary 4 provides a refinement of a result which obtained by Srivastava [12, Theorem 1].

By setting

$$h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z} \qquad (0 \le \beta < 1,\ z \in \mathbb{U}),$$

in Theorem 3, we deduce the following result.

COROLLARY 5. Let the function f given by (1) be in the class $\mathcal{W}_\Sigma(\gamma,\lambda,\alpha,\beta)$. Then

$$|a_2| \le \min\left\{\frac{2|\gamma|(1-\beta)}{\alpha+1}, \sqrt{\frac{2|\gamma|(1-\beta)}{1+2\alpha+2\lambda}}\right\} \quad \text{and} \quad |a_3| \le \frac{2|\gamma|(1-\beta)}{1+2\alpha+2\lambda}.$$

If we take $\gamma = 1$ and $\alpha = 1+2\lambda$ in Corollary 5, we get

COROLLARY 6. Let the function f given by (1) be in the class $H_\Sigma(\beta,\lambda)$. Then

$$|a_2| \le \min\left\{\frac{(1-\beta)}{\lambda+1}, \sqrt{\frac{2(1-\beta)}{3(2\lambda+1)}}\right\} \quad \text{and} \quad |a_3| \le \frac{2(1-\beta)}{3(2\lambda+1)}.$$

REMARK 5. Corollary 6 is a refinement of Theorem 2.

If we set $\lambda = 0$ and $\gamma = 1$ in Corollary 5, then we have the following corollary.

COROLLARY 7. Let the function f given by (1) be in the class $B_\Sigma(\beta,\alpha)$. Then

$$|a_2| \leq \min\left\{\frac{2(1-\beta)}{\alpha+1}, \sqrt{\frac{2(1-\beta)}{2\alpha+1}}\right\} \quad \text{and} \quad |a_3| \leq \frac{2(1-\beta)}{2\alpha+1}.$$

REMARK 6. Corollary 7 provides a refinement of a result which obtained by Frasin and Aouf [7, Theorem 3.2].

If we take $\alpha = 1$ in Corollary 7, then we get

COROLLARY 8. Let the function f given by (1) be in the class $H_\Sigma(\beta)$. Then

$$|a_2| \leq \begin{cases} \sqrt{\dfrac{2(1-\beta)}{3}} & \text{for } 0 \leq \beta \leq \frac{1}{3}, \\ (1-\beta), & \text{for } \frac{1}{3} \leq \beta < 1, \end{cases} \quad \text{and} \quad |a_3| \leq \frac{2(1-\beta)}{3}.$$

REMARK 7. Corollary 8 provides an improvement of a result which obtained by Srivastava [12, Theorem 2].

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