A Note On Uniqueness Of Meromorphic Functions And Their Derivatives Sharing Two Sets *

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Abstract

In this paper, on the basis of a specific question raised in [6], we further continue our investigations on the uniqueness of a meromorphic function with its higher derivatives sharing two sets and answer the question affirmatively. Moreover, we exhibit some examples to show the sharpness of some conditions used in our main result.

1 Introduction

By $\mathbb{C}$ and $\mathbb{N}$, we mean the set of complex numbers and set of positive integers respectively. We also assume that readers are familiar with the classical Nevanlinna theory [8].

In 1976, Gross [7] first generalized the concept of value sharing problem by proposing his famous question on set sharing. To understand Gross' contribution elaborately, we require the following definition of set sharing:

DEFINITION 1. Let f be a non constant meromorphic function and $S \subset \mathbb{C} \cup \{\infty\}$. We define

$$E_f(S) = \bigcup_{a \in S} \{(z,p) \in \mathbb{C} \times \mathbb{N} \mid f(z) = a \text{ with multiplicity } p\},$$

$$\overline{E}_f(S) = \bigcup_{a \in S} \{z \in \mathbb{C} \mid f(z) = a, \text{counting without multiplicity}\}.$$

Two meromorphic functions f and g are said to share the set S counting multiplicities (CM), if $E_f(S) = E_g(S)$. They are said to share the set S ignoring multiplicities (IM), if $\overline{E}_f(S) = \overline{E}_g(S)$.

In 1976, Gross ([7], Question 6) proposed a problem concerning the uniqueness of entire functions that share sets of distinct elements instead of values as follows:

QUESTION A. Can one find a finite set S such that any two non constant entire functions f and g satisfying $E_f(S) = E_g(S)$ must be identical?

In this directions, there are many elegant results in the literature but in the present scenario our prime focus will be on Gross' following question which deal with the two sets sharing problems:

QUESTION B. Can one find two finite set S_1 and S_2 such that any two non constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$ must be identical?

If the answer to this question is affirmative, it would be interesting to know how large both sets would have to be.

An affirmative answer of the above questions were provided by Yi [11] et al.. Since then, shared sets problems have been studied by many authors and a number of profound results have been obtained. Taking the question of Gross [7] into the background, the following question is natural:

QUESTION C ([12, 13, 15]). Can one find two finite sets S_j for $j = 1, 2$ such that any two non constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

In connection to the above question, a brief survey can be found in [6]. In this context, the possible best result is due to Yi [12].

In 2002, Yi [12] proved that there exist two finite sets S_1 with one element and S_2 with eight elements such that any two non constant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical.

In the mean time, Lahiri [9] introduced the notion of weighted sharing which is the scaling between CM sharing and IM sharing. As far as relaxations of the nature of sharing of the sets are concerned, this notion has a remarkable influence.

DEFINITION 2 ([9]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.

Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$. We denote by $E_f(S, k)$ the set $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$. If $E_f(S, k) = E_g(S, k)$, then we say that f and g share the set S with weight k.

In 2008, the present first author [1] improved the result of Yi [12] by relaxing the nature of sharing the range sets by the notion of weighted sharing. He established that there exist two finite sets S_1 with one element and S_2 with eight elements such that any two non constant meromorphic functions f and g satisfying $E_f(S_1, 0) = E_g(S_1, 0)$ and $E_f(S_2, 2) = E_g(S_2, 2)$ must be identical.

But after that no remarkable improvements were done regarding Question C. So the natural query would be whether one can get better result even for particular class of meromorphic functions. This possibility encouraged researchers to find the similar types range sets corresponding to the derivatives of two meromorphic functions. To proceed further, we first recall the existing results in this direction:

THEOREM A ([15]). Let $S_1 = \{z : z^n + az^{n-1} + b = 0\}$ and $S_2 = \{\infty\}$, where a, b are nonzero constants such that $z^n + az^{n-1} + b = 0$ has no repeated root and $n \ (\geq 7)$, k be two positive integers. Let f and g be two non constant meromorphic functions such that $E_{f^{(k)}}(S_1, \infty) = E_{g^{(k)}}(S_1, \infty)$ and $E_f(S_2, \infty) = E_g(S_2, \infty)$. Then $f^{(k)} \equiv g^{(k)}$.

In 2010, Banerjee-Bhattacharjee [3] improved *Theorem A* in the following way:

THEOREM B ([3]). Let $S_i \ (i = 1, 2)$ and k be given as in *Theorem A*. Let f and g be two non constant meromorphic functions such that either $E_{f^{(k)}}(S_1, l) = E_{g^{(k)}}(S_1, l)$ and $E_f(S_2, m) = E_g(S_2, m)$, where $(l, m) = (2, 1)$ or $(3, 0)$. Then $f^{(k)} \equiv g^{(k)}$.

In 2011, the same authors ([4]) further improved the above results as:

THEOREM C ([4]). Let $S_i \ (i = 1, 2)$ and k be given as in *Theorem A*. Let f and g be two non constant meromorphic functions such that $E_{f^{(k)}}(S_1, 2) = E_{g^{(k)}}(S_1, 2)$ and $E_f(S_2, 0) = E_g(S_2, 0)$. Then $f^{(k)} \equiv g^{(k)}$.

Recently the present authors [6] improved the above results at the cost of considering a new range sets instead of the previous. To discuss the results in [6], we first require the following polynomial. Suppose for two positive integers m, n

$$P_*(z) = z^n - \frac{2n}{n - m} z^{n-m} + \frac{n}{n - 2m} z^{n-2m} + c, \tag{1}$$

where c is any complex number satisfying $|c| \neq \frac{2m^2}{(n-m)(n-2m)}$ and $c \neq 0, -\frac{1 - \frac{2n}{n-m} + \frac{n}{n-2m}}{2}$.

THEOREM D ([6]). Let $n \geq 6$, $m = 1$ and $k \geq 1$ be three positive integers. Let $S_* = \{z : P_*(z) = 0\}$ where the polynomial $P_*(z)$ is defined by (1). Let f and g be two non constant meromorphic functions satisfying $E_{f^{(k)}}(S_*, 3) = E_{g^{(k)}}(S_*, 3)$ and $E_{f^{(k)}}(0, 0) = E_{g^{(k)}}(0, 0)$. Then $f^{(k)} \equiv g^{(k)}$.

In the same paper [6], the following question was asked:

QUESTION 1. Whether there exists two suitable sets S_1 (with one element) and S_2 (with five elements) such that when derivatives of any two non constant meromorphic functions share them with finite weight yield $f^{(k)} \equiv g^{(k)}$?

The motivation of writing this paper is to answer the Question 1 affirmatively. To this end, we recall some definitions which we need in this sequel.

DEFINITION 3 ([5]). Let z_0 be a zero of $f - a$ and $g - a$ of multiplicity p and q respectively. Then $N_E^{1)}(r, a; f)$ and $\overline{N}_E^{(2}(r, a; f)$ denote the reduced counting functions of those a-points of f and g where $p = q = 1$ and $p = q \geq 2$ respectively. Also $\overline{N}_L(r, a; f)$ and $\overline{N}_L(r, a; g)$ denote the reduced counting functions of those a-points of f and g where $p > q \geq 1$ and $q > p \geq 1$ respectively.

DEFINITION 4 ([5]). Let f and g share a value a-IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g, i.e, $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2   Main Result

For a positive integer n, we shall denote by $P(z)$ the following polynomial ([11]):

$$P(z) = z^n + az^{n-1} + b \quad \text{where} \quad ab \neq 0 \text{ and } \frac{b}{a^n} \neq (-1)^n \frac{(n-1)^{(n-1)}}{n^n}. \tag{2}$$

THEOREM 1. Suppose that $n(\geq 5)$ and $k(\geq 1)$ are two positive integers. Further suppose that $S = \{z : P(z) = 0\}$, where the polynomial $P(z)$ is defined by (2). For two non constant meromorphic functions f and g, if $E_{f^{(k)}}(S, 2) = E_{g^{(k)}}(S, 2)$ and $E_{f^{(k)}}(0, 1) = E_{g^{(k)}}(0, 1)$, then $f^{(k)} \equiv g^{(k)}$.

COROLLARY 1. Suppose that $n(\geq 3)$ and $k(\geq 1)$ are two positive integers. Further suppose that $S = \{z : P(z) = 0\}$, where the polynomial $P(z)$ is defined by (2). For two non constant entire functions f and g, if $E_{f^{(k)}}(S, 2) = E_{g^{(k)}}(S, 2)$ and $E_{f^{(k)}}(0, 0) = E_{g^{(k)}}(0, 0)$, then $f^{(k)} \equiv g^{(k)}$.

REMARK 1. Theorem 1 shows that there exists two sets S_1 with one element and S_2 with five elements such that when derivatives of any two non constant meromorphic functions share them with finite weight yields $f^{(k)} \equiv g^{(k)}$.

Thus the above theorem improves *Theorem D* in the direction of *Question 1.*

As Theorem 1 deals with specific class of meromorphic functions, so the general curiosity will be:

QUESTION 2. Does the Theorem 1 hold good for general class of meromorphic functions?

The next example shows that the answer is negative, i.e., $k \geq 1$ is sharp.

EXAMPLE 1. Let $n \in \mathbb{N}$ and $S = \{z : P(z) = 0\}$ where $P(z)$ is defined by (2). We choose $f(z) = -a\frac{1-h^{n-1}}{1-h^n}$ and, $g(z) = h(z)f(z)$, where $h(z) = \frac{e^z - 1 + sgn(n-5)}{e^z + 1}$ and $sgn(x)$ is defined as:

$$sgn(x) = \begin{cases} +1 \text{ for } x \geq 0, \\ -1 \text{ for } x < 0. \end{cases}$$

For $n \in \mathbb{N}$,

$$\left(\frac{f}{g}\right)^{n-1}\left(\frac{f+a}{g+a}\right) = \frac{1}{h^{n-1}}\left\{\frac{a\frac{h^{n-1}-h^n}{1-h^n}}{a\frac{1-h}{1-h^n}}\right\} = 1.$$

Thus $E_f(S, \infty) = E_g(S, \infty)$.

Again from the construction of h, it is clear that 0 is a Picard exceptional value of h only when $n \geq 5$. So $g = hf$ implies $E_f(0, \infty) = E_g(0, \infty)$ only when $n \geq 5$. Thus f and g satisfies all the conditions stated in Theorem 1, but $f \not\equiv g$.

Obviously the next natural query would be:

QUESTION 3. Whether the set S can be replaced by any arbitrary set of five elements in the same environment of Theorem 1?

The next example shows that the answer is negative.

EXAMPLE 2. Let

$$f(z) = \frac{1}{(\sqrt{\alpha\beta\gamma})^{k-1}}e^{\sqrt{\alpha\beta\gamma}\,z} \quad \text{and} \quad g(z) = \frac{(-1)^k}{(\sqrt{\alpha\beta\gamma})^{k-1}}e^{-\sqrt{\alpha\beta\gamma}\,z},$$

where $k \geq 1$. Also let $S = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\gamma}, \gamma\sqrt{\beta}, \sqrt{\alpha\beta\gamma}\}$, where α, β and γ are three nonzero distinct complex numbers. Clearly $E_{f^{(k)}}(S, \infty) = E_{g^{(k)}}(S, \infty)$ and $E_{f^{(k)}}(0, \infty) = E_{g^{(k)}}(0, \infty)$, but $f^{(k)} \not\equiv g^{(k)}$.

The following two examples show that $ab \neq 0$ is necessary in Theorem 1.

EXAMPLE 3. If $b = 0$, then $S = \{0, -a\}$. Thus for $f(z) = ae^z$ and, $g(z) = (-1)^k ae^{-z}$ $(k \geq 1)$, $E_{f^{(k)}}(S, \infty) = E_{g^{(k)}}(S, \infty)$ and $E_{f^{(k)}}(0, \infty) = E_{g^{(k)}}(0, \infty)$, but $f^{(k)} \not\equiv g^{(k)}$.

EXAMPLE 4. If $a = 0$, then $S = \{z \mid z^n + b = 0\}$. Take $n \geq 5$, $f(z) = \lambda e^z$ and, $g(z) = (-1)^k \lambda e^{-z}$ $(k \geq 1)$, where λ is one of the value of $(-b)^{\frac{1}{n}}$. Then $E_{f^{(k)}}(S, \infty) = E_{g^{(k)}}(S, \infty)$ and $E_{f^{(k)}}(0, \infty) = E_{g^{(k)}}(0, \infty)$, but $f^{(k)} \not\equiv g^{(k)}$.

The next example shows that when the derivative of two meromorphic functions share two sets, if the cardinality of one set is one, then the cardinality of another set

is at least three.

EXAMPLE 5. For three distinct complex numbers a, b and c, let $S_1 = \{a\}$ and $S_2 = \{b, c\}$. Choose $f(z) = p(z) + (b - a)e^z$ and, $g(z) = q(z) + (-1)^k(c - a)e^{-z}$, where $p(z) = \sum_{i=0}^{k-1} c_i z^i + \frac{a}{k!} z^k$ and $q(z) = \sum_{i=0}^{k-1} d_i z^i + \frac{a}{k!} z^k$, $c_i, d_i \in \mathbb{C}$. Clearly $E_{f^{(k)}}(S_j, \infty) = E_{g^{(k)}}(S_j, \infty)$ for $j = 1, 2$, but $f^{(k)} \not\equiv g^{(k)}$.

The next examples show that if we choose different sets other than the specific form of choosing the first set S with three or four elements Theorem 1 ceases to hold.

EXAMPLE 6. Choose four nonzero distinct complex numbers α, β, γ and δ such that $\alpha\beta = \gamma\delta$. Let $f(z) = -\alpha e^z$ and, $g(z) = (-1)^{k+1}\beta e^{-z}$ $(k \geq 1)$ and $S = \{\alpha, \beta, \gamma, \delta\}$. Clearly $E_{f^{(k)}}(S, \infty) = E_{g^{(k)}}(S, \infty)$ and $E_{f^{(k)}}(0, \infty) = E_{g^{(k)}}(0, \infty)$, but $f^{(k)} \not\equiv g^{(k)}$.

EXAMPLE 7. Choose three nonzero distinct complex numbers α, β, γ such that $\alpha\beta = \gamma^2$. Let $f(z) = -\alpha e^z$ and, $g(z) = (-1)^{k+1}\beta e^{-z}$ $(k \geq 1)$ and $S = \{\alpha, \beta, \gamma\}$. Clearly $E_{f^{(k)}}(S, \infty) = E_{g^{(k)}}(S, \infty)$ and $E_{f^{(k)}}(0, \infty) = E_{g^{(k)}}(0, \infty)$, but $f^{(k)} \not\equiv g^{(k)}$.

However the following question is inevitable from the above discussion:

QUESTION 4. Whether there exists two suitable sets S_1 with one element and S_2 with three or four elements such that when derivatives of any two non constant non entire meromorphic functions share them with finite weight yield $f^{(k)} \equiv g^{(k)}$?

3 Lemmas

Throughout this paper, by H and Φ, we shall mean the following two functions:

$$H := \left(\frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1} \right) \quad \text{and} \quad \Phi := \frac{F'}{F - 1} - \frac{G'}{G - 1},$$

where $F := -\frac{(f^{(k)})^{n-1}(f^{(k)}+a)}{-b}$ and $G := -\frac{(g^{(k)})^{n-1}(g^{(k)}+a)}{-b}$, for $n, k \in \mathbb{N}$. Also we define $T(r) := \max\{T(r, f^{(k)}), \ T(r, g^{(k)})\}$ and $S(r) = o(T(r))$.

LEMMA 1. If $f^{(k)}$ and $g^{(k)}$ share $(0, 0)$, then $F \equiv G$ gives $f^{(k)} \equiv g^{(k)}$ where $k(\geq 1)$ and $n(\geq 1)$ are two integers.

PROOF. The inequality $\overline{N}(r, \infty; f^{(k)}) \leq \frac{1}{1+k} N(r, \infty; f^{(k)}) \leq \frac{1}{2} T(r, f^{(k)}) + O(1)$ implies $\Theta(\infty, f^{(k)}) \geq \frac{1}{2}$ for $k \geq 1$. Again $F \equiv G$ implies $f^{(k)}$ and $g^{(k)}$ share (∞, ∞). Now rest of the proof is similar as *Lemma 1* of ([14]). So we omit the details.

LEMMA 2. Suppose $f^{(k)}$ and $g^{(k)}$ share $(0,0)$, then $FG \not\equiv 1$ for $k \geq 1$ and $n \geq 3$.

PROOF. On contrary, we assume that $FG \equiv 1$. Then

$$(f^{(k)})^{n-1} \left(f^{(k)} + a \right) (g^{(k)})^{n-1} \left(g^{(k)} + a \right) \equiv b^2. \tag{3}$$

If z_0 be a $(-a)$-point of $f^{(k)}$ of order p, then z_0 is a pole of g of order q such that $p = n(q + k) \geq n$. So

$$\overline{N}(r, -a; f^{(k)}) \leq \frac{1}{n} N(r, -a; f^{(k)}).$$

Again from (3), we have $\overline{N}(r, 0; f^{(k)}) = \overline{N}(r, 0; g^{(k)}) = S(r)$ as $E_{f^{(k)}}(0,0) = E_{g^{(k)}}(0,0)$. Now using Mokhon'ko's Lemma ([10]) and the Second Fundamental Theorem, we get

$$\begin{aligned}
T(r, f^{(k)}) &\leq \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, 0; f^{(k)}) + \overline{N}(r, -a; f^{(k)}) + S(r, f^{(k)}) \quad (4) \\
&\leq \overline{N}(r, -a; g^{(k)}) + \overline{N}(r, -a; f^{(k)}) + S(r) \\
&\leq \frac{2}{n} T(r, f^{(k)}) + S(r),
\end{aligned}$$

which is a contradiction as $n \geq 3$. This proves the lemma.

LEMMA 3. ([2]) If F and G share $(1, l)$ where $0 \leq l < \infty$, then

$$\overline{N}(r, 1; F) + \overline{N}(r, 1; G) - N_E^{1)}(r, 1, F) + (l - \frac{1}{2}) \overline{N}_*(r, 1; F, G) \leq \frac{1}{2} (N(r, 1; F) + N(r, 1; G)).$$

LEMMA 4. (Lemma 3.6, [6]) Let F, G and Φ be defined as previously and $F \not\equiv G$. If $f^{(k)}$ and $g^{(k)}$ share $(0, q)$ where $0 \leq q < \infty$ and F, G share $(1, l)$, then

$$\begin{aligned}
&\{(n-1)q + n - 2\} \, \overline{N}(r, 0; f^{(k)} \, | \geq q + 1) \\
&\leq \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + \overline{N}_*(r, 1; F, G) + S(r),
\end{aligned}$$

for $n(\geq 3) \in \mathbb{N}$. Similar expressions hold for $g^{(k)}$ also.

LEMMA 5. If $f^{(k)}$ and $g^{(k)}$ share $(0, 0)$; F and G share $(1, 2)$, where $n(\geq 3)$ is an integer, then

$$\overline{N}_*(r, 1; F, G) \leq \frac{n}{4n - 10} \left\{ \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) \right\} + S(r).$$

PROOF. Using Lemma 4, we have

$$\begin{aligned}
2\overline{N}_*(r, 1; F, G) &\leq 2\overline{N}(r, 1; F| \geq 3) \leq N \left(r, 0; f^{(k+1)} \mid f^{(k)} \neq 0 \right) + S(r) \\
&\leq \overline{N} \left(r, 0; f^{(k)} \right) + \overline{N} \left(r, \infty; f^{(k)} \right) + S(r)
\end{aligned}$$

$$\leq \quad \frac{1}{n-2}\left\{\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})+\overline{N}_*(r,1;F,G)\right\}$$
$$+\overline{N}(r,\infty;f^{(k)})+S(r). \tag{5}$$

Again,

$$2\overline{N}_*(r,1;F,G) \quad \leq \quad 2\overline{N}(r,1;G|\geq 3)$$
$$\leq \quad \frac{1}{n-2}\left(\overline{N}(r,\infty;g^{(k)})+\overline{N}(r,\infty;f^{(k)})+\overline{N}_*(r,1;F,G)\right)$$
$$+\overline{N}(r,\infty;g^{(k)})+S(r). \tag{6}$$

Adding (5) and (6), we have

$$\overline{N}_*(r,1;F,G)\leq \frac{n}{4n-10}\left\{\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\right\}+S(r).$$

Hence the proof of the lemma is completed.

LEMMA 6. Suppose f and g are two non constant meromorphic functions. If $f^{(k)}$ and $g^{(k)}$ share (S,l) and $(0,q)$, where $k\geq 1$, $l\geq 2$, $n\geq 5$ and $q\geq 1$, then $H\equiv 0$.

PROOF. On contrary, we assume that $H\not\equiv 0$. Then clearly $F\not\equiv G$ and

$$\overline{N}(r,1;F|=1) \quad = \quad \overline{N}(r,1;G|=1)\leq N(r,\infty;H)$$
$$\leq \quad \overline{N}\left(r,\infty;f^{(k)}\right)+\overline{N}\left(r,\infty;g^{(k)}\right)+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)$$
$$+\overline{N}\left(r,-a\frac{n-1}{n};g^{(k)}\right)+\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}_*(r,1;F,G)$$
$$+\overline{N}_0\left(r,0;f^{(k+1)}\right)+\overline{N}_0\left(r,0;g^{(k+1)}\right), \tag{7}$$

where $\overline{N}_0\left(r,0;f^{(k+1)}\right)$ is the reduced counting function of zeros of $f^{(k+1)}$, which is not zeros of $f^{(k)}\left(f^{(k)}+a\frac{n-1}{n}\right)$ and $(F-1)$. Similarly $\overline{N}_0\left(r,0;g^{(k+1)}\right)$ is defined. Using the Second Fundamental Theorem, Lemma 3 and inequality (7), we get

$$(n+1)\left(T(r,f^{(k)})+T(r,g^{(k)})\right)$$
$$\leq \quad \overline{N}\left(r,\infty;f^{(k)}\right)+\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}\left(r,\infty;g^{(k)}\right)+\overline{N}\left(r,0;g^{(k)}\right)$$
$$+\overline{N}(r,1;F)+\overline{N}(r,1;G)+\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right)+\overline{N}\left(r,-a\frac{n-1}{n};g^{(k)}\right)$$
$$-N_0\left(r,0,f^{(k+1)}\right)-N_0\left(r,0,g^{(k+1)}\right)+S\left(r,f^{(k)}\right)+S\left(r,g^{(k)}\right)$$
$$\leq \quad 2\left\{\overline{N}(r,\infty;f^{(k)})+\overline{N}(r,\infty;g^{(k)})\right\}+\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}\left(r,0;g^{(k)}\right)$$
$$+\overline{N}_*\left(r,0;f^{(k)},g^{(k)}\right)+\overline{N}(r,1;F)+\overline{N}(r,1;G)-\overline{N}(r,1;F|=1)$$

$$+\overline{N}_*(r,1;F,G) + 2\left\{\overline{N}\left(r,-a\frac{n-1}{n};f^{(k)}\right) + \overline{N}\left(r,-a\frac{n-1}{n};g^{(k)}\right)\right\}$$

$$+S\left(r,f^{(k)}\right) + S\left(r,g^{(k)}\right).$$

That is,

$$(n+1)\left(T(r,f^{(k)}) + T(r,g^{(k)})\right)$$

$$\leq \quad 2\left\{T(r,f^{(k)}) + T(r,g^{(k)})\right\} + 2\left\{\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)})\right\}$$

$$+ \quad \overline{N}\left(r,0;f^{(k)}\right) + \overline{N}\left(r,0;g^{(k)}\right) + \overline{N}\left(r,0;f^{(k)} \mid \ \geq q+1\right)$$

$$+ \quad \frac{1}{2}\left(N(r,1;F) + N(r,1;G)\right) + \left(\frac{3}{2} - l\right)\overline{N}_*(r,1;F,G)$$

$$+ \quad S\left(r,f^{(k)}\right) + S\left(r,g^{(k)}\right)$$

$$\leq \quad \left(2 + \frac{n}{2}\right)\left\{T(r,f^{(k)}) + T(r,g^{(k)})\right\} + 2\left\{\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)})\right\}$$

$$+ \quad 2\overline{N}\left(r,0;f^{(k)}\right) + \overline{N}\left(r,0;f^{(k)} \mid \ \geq q+1\right) + \left(\frac{3}{2} - l\right)\overline{N}_*(r,1;F,G)$$

$$+ \quad S\left(r,f^{(k)}\right) + S\left(r,g^{(k)}\right).$$

Using Lemma 4 and Lemma 5 in the above inequality, we obtain

$$\left(\frac{n}{2} - 1\right)(T(r,f^{(k)}) + T(r,g^{(k)})) \qquad\qquad\qquad (8)$$

$$\leq \quad 2\{\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)})\}$$

$$+ \left(\frac{2}{n-2} + \frac{1}{(n-1)q + (n-2)}\right)\{\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)})$$

$$+ \overline{N}_*(r,1;F,G)\} + \left(\frac{3}{2} - l\right)\overline{N}_*(r,1;F,G) + S(r,f^{(k)}) + S(r,g^{(k)})$$

$$\leq \quad \left(2 + \frac{2}{n-2} + \frac{1}{(n-1)q + (n-2)}\right)\{\overline{N}(r,\infty;f^{(k)}) + \overline{N}(r,\infty;g^{(k)})\}$$

$$+ \left(\frac{n}{4n-10}\right)\left(\frac{3}{2} - l + \frac{2}{n-2} + \frac{1}{(n-1)q + (n-2)}\right)\{\overline{N}(r,\infty;f^{(k)})$$

$$+ \overline{N}(r,\infty;g^{(k)})\} + S(r,f^{(k)}) + S(r,g^{(k)}).$$

So from (8), we get

$$\left(\frac{n}{2} - 1\right)(T(r,f^{(k)}) + T(r,g^{(k)}))$$

$$\leq \quad \left(1 + \frac{1}{n-2} + \frac{1}{2(n-1)q + 2(n-2)}\right)\{N(r,\infty;f^{(k)}) + N(r,\infty;g^{(k)})\}$$

$$+ \left(\frac{n}{4n-10}\right)\left(\frac{1}{n-2} + \frac{1}{2(n-1)q + 2(n-2)} - \frac{1}{4}\right)\{N(r,\infty;f^{(k)})$$

$$+ N(r, \infty; g^{(k)}) \} + S(r, f^{(k)}) + S(r, g^{(k)}),$$

which leads to a contradiction when $n \geq 5$ and $q \geq 1$. Thus $H \equiv 0$. Hence the proof of the lemma is completed.

REMARK 2. It is easy to see that if we take f and g as two non constant entire functions in Lemma 6 and if $f^{(k)}$ and $g^{(k)}$ share (S, l) and $(0, 0)$, where $k \geq 1$, $l \geq 2$, $n \geq 3$, then $H \equiv 0$.

LEMMA 7. Suppose $f^{(k)}$ and $g^{(k)}$ share $(0, 0)$ and $H \equiv 0$. If $n \geq 4$ and $k \geq 1$, then $f^{(k)} \equiv g^{(k)}$.

PROOF. Given $H \equiv 0$. On integration, we have

$$F \equiv \frac{AG + B}{CG + D}, \tag{9}$$

where A, B, C, D are constant satisfying $AD - BC \neq 0$. So $A = C = 0$ never occur. Thus clearly F and G share $(1, \infty)$. Now by Mokhon'ko's Lemma ([10]), we have

$$T\left(r, f^{(k)}\right) = T\left(r, g^{(k)}\right) + S(r). \tag{10}$$

Next we consider the following two cases:

CASE-1. Assume $AC \neq 0$. In this case (9) can be written as

$$F - \frac{A}{C} \equiv \frac{BC - AD}{C(CG + D)}. \tag{11}$$

Now by using the Second Fundamental Theorem, equations (10) and (11), we get

$$
\begin{aligned}
nT(r, f^{(k)}) + O(1) &= T(r, F) \\
&\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}\left(r, \frac{A}{C}; F\right) + S(r, F) \\
&\leq \overline{N}(r, \infty; f^{(k)}) + \overline{N}(r, 0; f^{(k)}) + T(r, f^{(k)}) + \overline{N}(r, \infty; g^{(k)}) + S(r, f^{(k)}) \\
&\leq \frac{1}{2}\left(N(r, \infty; f^{(k)}) + N(r, \infty; g^{(k)})\right) + \overline{N}(r, 0; f^{(k)}) + T(r, f^{(k)}) + S(r, f^{(k)}) \\
&\leq 3T(r, f^{(k)}) + S(r, f^{(k)}),
\end{aligned}
$$

which is a contradiction as $n \geq 4$.

CASE-2. Next we assume $AC = 0$. Now we consider the following subcases:

SUBCASE-2.1. Let $A = 0$ and $C \neq 0$. Hence $B \neq 0$. Thus equation (9) becomes

$$F \equiv \frac{1}{\gamma G + \delta}, \quad \text{where} \quad \gamma = \frac{C}{B} \quad \text{and} \quad \delta = \frac{D}{B}. \tag{12}$$

If F has no 1-point, then in view of the Second Fundamental Theorem, we get

$$nT(r, f^{(k)}) + O(1) = T(r, F)$$

$$\begin{aligned}
&\leq\ \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1;F)+S(r,F)\\
&\leq\ \overline{N}(r,\infty;f^{(k)})+\overline{N}(r,0;f^{(k)})+T(r,f^{(k)})+S(r,f^{(k)})\\
&\leq\ \frac{5}{2}T(r,f^{(k)})+S(r,f^{(k)}),
\end{aligned}$$

which is a contradiction as $n\geq 4$.

Thus there exist atleast one z_0 such that $F(z_0)=G(z_0)=1$. Hence from equation (12), we get $\gamma+\delta=1$, $\gamma\neq 0$ and hence

$$F\equiv\frac{1}{\gamma G+1-\gamma}.$$

If $\gamma\neq 1$, then the Second Fundamental Theorem yields

$$\begin{aligned}
T(r,G)\ &\leq\ \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\left(r,0;G+\frac{1-\gamma}{\gamma}\right)+S(r,G)\\
&\leq\ \overline{N}(r,\infty;g^{(k)})+\overline{N}(r,0;g^{(k)})+T(r,g^{(k)})+\overline{N}(r,\infty;f^{(k)})+S(r,g^{(k)})\\
&\leq\ \frac{3}{n}T(r,F)+S(r,F),
\end{aligned}$$

which is impossible as $n\geq 4$. Thus $\gamma=1$, i.e., $FG\equiv 1$, which is again impossible by Lemma 2.

SUBCASE-2.2. Let $A\neq 0$ and $C=0$. Hence $D\neq 0$. So equation (9) becomes

$$F\equiv\lambda G+\mu,\quad\text{where}\ \ \lambda=\frac{A}{D}\ \ \text{and}\ \ \mu=\frac{B}{D}.$$

If F has no 1-point, then proceeding as above in *Subcase-2.1*, we arrive at a contradiction. Thus $\lambda+\mu=1$ with $\lambda\neq 0$. If $\lambda\neq 1$, then equation (9) yields

$$\overline{N}(r,0,f^{(k)})=\overline{N}(r,0,g^{(k)})=S(r).$$

Now applying the Second Fundamental Theorem, we obtain

$$\begin{aligned}
T(r,G)\ &\leq\ \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}\left(r,0;G+\frac{1-\lambda}{\lambda}\right)+S(r,G)\\
&\leq\ \overline{N}(r,\infty;g^{(k)})+\overline{N}(r,0;g^{(k)})+T(r,g^{(k)})+\overline{N}(r,0;f^{(k)})+T(r,f^{(k)})\\
&+\ S(r,g^{(k)})\\
&\leq\ \frac{5}{2n}T(r,G)+S(r,G),
\end{aligned}$$

which is a contradiction as $n\geq 4$. Thus $\lambda=1$ and hence $F\equiv G$. Consequently Lemma 1 gives $f^{(k)}\equiv g^{(k)}$. Hence the proof of the lemma is completed.

REMARK 3. It is easy to see that if we take f and g as two non constant entire functions in Lemma 7 and if $f^{(k)}$ and $g^{(k)}$ share $(0,0)$ and $H\equiv 0$, then $f^{(k)}\equiv g^{(k)}$, where $n\geq 3$ and $k\geq 1$.

4 Proof of the Theorem

PROOF OF THE THEOREM 1. Given that $f^{(k)}$ and $g^{(k)}$ share $(S, 2)$ and $(0, 1)$. Since $l = 2$ and $q = 1$, so in view of Lemma 6, we get $H \equiv 0$. Next we apply the Lemma 7, and we obtain our desired result $f^{(k)} \equiv g^{(k)}$ when $n \geq 5$. Hence the theorem is proved.

References

[1] A. Banerjee, On the uniqueness of meromorphic functions that share two sets, Georgian Math. J., 15(2008), 1–18.

[2] A. Banerjee, Uniqueness of meromorphic functions sharing two sets with finite weight II, Tamkang J. Math., 41(2010), 379–392.

[3] A. Banerjee and P. Bhattacharjee, Uniqueness of derivatives of meromorphic functions sharing two or three sets, Turk. J. Math., 34(2010), 21–34.

[4] A. Banerjee and P. Bhattacharjee, Uniqueness and set sharing of derivatives of meromorphic functions, Math. Slovaca, 61(2011), 197–214.

[5] A. Banerjee and B. Chakraborty, A new type of unique range set with deficient values, Afr. Mat., 26(2015), 1561–1572.

[6] A. Banerjee and B. Chakraborty, Further results on the uniqueness of meromorphic functions and their derivative counterpart sharing one or two sets, Jordan J. Math. Stat., 9(2016), 117–139.

[7] F. Gross, Factorization of meromorphic functions and some open problems, Proc. Conf. Univ. Kentucky, Leixngton, Ky (1976); Lecture Notes in Math., 599(1977), 51–69, Springer(Berlin).

[8] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).

[9] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J., 161(2001), 193–206.

[10] A. Z. Mokhon'ko, On the Nevanlinna characteristics of some meromorphic functions, in Theory of Functions, functional analysis and their applications, Izd-vo Khar'kovsk, Un-ta, 14(1971), 83–87.

[11] H. X. Yi, On a question of Gross concerning uniqueness of entire functions, Bull. Austral. Math. Soc., 57(1998), 343–349.

[12] H. X. Yi, Meromorphic functions that share two sets. Acta Math Sinica, 45(2002), 75–82.

[13] H. X. Yi and W. R. Lü, Meromorphic functions that share two sets II, Acta Math. Sci. Ser.B Engl. Ed., 24(2004), 83–90.

[14] H. X. Yi and W. C. Lin, Uniqueness theorems concerning a question of Gross, Proc. Japan Acad., 80(2004), 136–140.

[15] H. X. Yi and W. C. Lin, Uniqueness of meromorphic functions and a question of gross, Kyungpook Math. J., 46(2006), 437–444.