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Characterizations Of (p, α)-Convex Sequences*

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Abstract

The class of convex sequences has important applications in several branches of mathematics as well as their generalizations. In this paper, we have introduced two new classes of convex sequences, the so-called (p, α)-convex sequences and p-starshaped sequences. Moreover, the characterizations of sequences belonging to these class are discussed.

1 Introduction

The set of convex sequences is one of proper and important subset of the set of real sequences. This class is raised as a result of efforts to solve several problems in mathematics. Since the beginning of time, the sequences that belong to that class, have considerable applications in some branches of mathematics, in particular in mathematical analysis. For example, such sequences are widely used in theory of inequalities (see [12, 7, 8]), in absolute summability of infinite series (see [1, 2]), and in theory of Fourier series, related to their uniform convergence and the integrability of their sum functions (see as example [6], page 587).

Let $(a_n)_{n=0}^{\infty}$ be a real sequence and let the differences of orders $0, 1, 2$ of the sequence $(a_n)_{n=0}^{\infty}$ be defined by

$$\triangle^0 a_n = a_n, \quad \triangle^1 a_n = a_{n+1} - a_n, \quad \triangle^2 a_n = a_{n+2} - 2a_{n+1} + a_n, \quad n = 0, 1, \dots,$$

and throughout the paper we shall write $\triangle a_n$ instead of $\triangle^1 a_n$.

Next definition presents the well-known notion of a convex sequence of order 2.

DEFINITION 1. A sequence $(a_n)_{n=0}^{\infty}$ is said to be convex of order 2 (or just convex) if

$$\triangle^2 a_n \geq 0,$$

for all $n \geq 0$.

Various generalizations of convexity were studied by many authors. For instance, in [5] was introduced next:

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DEFINITION 2. A sequence $(a_n)_{n=0}^{\infty}$ is said to be p-convex for a positive real number p if

$$L_p(a_n) \geq 0,$$

for all $n = 0, 1, \ldots$, where the difference operator L_p is defined by

$$L_p(a_n) = a_{n+2} - (1+p)a_{n+1} + pa_n.$$

Another generalization of the concept of convexity can be found in [4] and [3]. In [4] is given the following definition:

DEFINITION 3. If for a sequence $(a_n)_{n=0}^{\infty}$ the inequality

$$a_n - pa_{n+1} \geq 0,$$

holds true for every $n \geq 0$, then it is said that $(a_n)_{n=0}^{\infty}$ is a p-monotone sequence.

Here, we will say that $(a_n)_{n=0}^{\infty}$ is a p-increasing sequence if the inequality

$$a_{n+1} - pa_n \geq 0,$$

holds true for every $n \geq 0$.

Two other classes of sequences, the so-called, starshaped sequences and α-convex sequences have been introduced in [9] and [10]. Indeed, let $\alpha \in [0,1]$.

DEFINITION 4. A sequence $(a_n)_{n=0}^{\infty}$ is called α-convex if the sequence

$$\left(\alpha(a_{n+1} - a_n) + (1-\alpha)\frac{a_n - a_0}{n}\right)_{n=1}^{\infty}$$

is increasing.

DEFINITION 5. A sequence $(a_n)_{n=0}^{\infty}$ is called starshaped if

$$\frac{a_{n+1} - a_0}{n+1} \geq \frac{a_n - a_0}{n} \text{ for } n \geq 1.$$

Let p be a real positive number.

Now, we introduce two new classes of sequences as follows:

DEFINITION 6. A sequence $(a_n)_{n=0}^{\infty}$ is called p-starshaped if

$$\frac{a_{n+1} - a_0}{n+1} \geq p\frac{a_n - a_0}{n} \text{ for } n \geq 1. \tag{1}$$

DEFINITION 7. A sequence $(a_n)_{n=0}^{\infty}$ is called (p,α)-convex if the sequence

$$\left(\alpha(a_{n+1}-a_n)+(1-\alpha)\frac{a_n-a_0}{n}\right)_{n=1}^{\infty}$$

is p-increasing.

REMARK 8. We note that: $(1,\alpha)$-convexity is the same with α-convexity, $(p,1)$-convexity is the same with p-convexity, $(p,0)$-convexity is the same with p-star-shapedness, $(1,1)$-convexity is the same with convexity, and $(1,0)$-convexity is the same with star-shapedness.

REMARK 9. Note also that: 1-star-shapedness of a sequence is the same with its star-shapedness.

Characterizing (p,α)-convex sequences as well as p-starshaped sequences, we are going to accomplish the main aim of this paper.

2 Main Results

We begin first with:

THEOREM 10. The sequence $(a_n)_{n=0}^{\infty}$ is (p,α)-convex if and only if

$$\alpha L_p(a_n)+(1-\alpha)\left(\frac{a_{n+1}-a_0}{n+1}-p\frac{a_n-a_0}{n}\right)\geq 0,$$

for all $n\in\{0,1,\dots\}$.

PROOF. The proof of this statement is an immediate result of the Definition 1. The proof is complete.

For $p=1$ we obtain Corollary 11.

COROLLARY 11 ([10]). The sequence $(a_n)_{n=0}^{\infty}$ is α-convex if and only if

$$\alpha\triangle^2(a_n)+(1-\alpha)\left(\frac{a_{n+1}-a_0}{n+1}-\frac{a_n-a_0}{n}\right)\geq 0,$$

for all $n\in\{0,1,\dots\}$.

THEOREM 12. The sequence $(a_n)_{n=0}^{\infty}$ is (p,α)-convex if and only if

$$(a_n-a_0+\alpha[n(a_{n+1}-a_n)-(a_n-a_0)])_{n=1}^{\infty},$$

is a p-starshaped sequence.

PROOF. First let us write

$$A_n := a_n - a_0 + \alpha[n(a_{n+1} - a_n) - (a_n - a_0)], \quad n \in \{1, 2, \dots\},$$

which can be rewritten as

$$A_n = \alpha n(a_{n+1} - a_n) + (1-\alpha)(a_n - a_0), \quad n \in \{1, 2, \dots\}.$$

The proof of this Lemma follows as a direct result of Lemma 2, and the following obvious equivalences ($A_0 = 0$)

$$\begin{aligned}
\frac{A_{n+1}}{n+1} \geq p\frac{A_n}{n} \quad &\Longleftrightarrow \quad nA_{n+1} \geq p(n+1)A_n \\
&\Longleftrightarrow \quad n[\alpha(n+1)(a_{n+2} - a_{n+1}) + (1-\alpha)(a_{n+1} - a_0)] \\
&\qquad \geq p[\alpha n(a_{n+1} - a_n) + (1-\alpha)(a_n - a_0)] \\
&\Longleftrightarrow \quad \alpha n(n+1)L_p(a_n) \\
&\qquad +(1-\alpha)[n(a_{n+1} - a_0) - p(n+1)(a_{n+1} - a_0)] \geq 0 \\
&\Longleftrightarrow \quad \alpha L_p(a_n) + (1-\alpha)\left(\frac{a_{n+1} - a_0}{n+1} - p\frac{a_n - a_0}{n}\right) \geq 0.
\end{aligned}$$

The proof is complete.

For $p = 1$ we obtain Corollary 13.

COROLLARY 13 ([10]). The sequence $(a_n)_{n=0}^{\infty}$ is α-convex if and only if

$$(a_n - a_0 + \alpha[n(a_{n+1} - a_n) - (a_n - a_0)])_{n=1}^{\infty},$$

is a starshaped sequence.

THEOREM 14. The sequence $(a_n)_{n=0}^{\infty}$ is p-starshaped if and only if it may be represented by

$$a_n = np^{n-1}\sum_{k=1}^{n}\frac{c_k}{k} - (np^{n-1} - 1)c_0, \tag{2}$$

with $c_k \geq 0$, $k \geq 2$.

PROOF. Our reasoning is similar to the proof of Lemma 3 in [11], page 3. Namely, let $a_0 = c_0$ and $a_1 = c_1$ and take $n = 2$ in (1) we obtain

$$a_2 \geq 2pc_1 - (2p-1)c_0,$$

which means that there exists a number $c_2 \geq 0$ such that

$$a_2 = 2p\left(c_1 + \frac{c_2}{2}\right) - (2p-1)c_0.$$

Now we assume that

$$a_n = np^{n-1}\sum_{k=1}^{n}\frac{c_k}{k} - (np^{n-1}-1)c_0.$$

Then for $n+1$ there exists $c_{n+1}p^n \geq 0$ so that we will have

$$\begin{aligned}
& \frac{a_{n+1}-a_0}{n+1} \geq p\frac{a_n-a_0}{n} \\
\iff\quad & a_{n+1} \geq p\frac{n+1}{n}(a_n-c_0)+c_0 \\
\iff\quad & a_{n+1} = c_{n+1}p^n + p\frac{n+1}{n}\left[np^{n-1}\sum_{k=1}^{n}\frac{c_k}{k} - (np^{n-1}-1)c_0 - c_0\right] + c_0 \\
\iff\quad & a_{n+1} = (n+1)p^n\sum_{k=1}^{n+1}\frac{c_k}{k} - [(n+1)p^n-1)c_0,
\end{aligned}$$

which by mathematical induction we obtain the representation (2). The proof is complete.

COROLLARY 15. If the sequence $(a_n)_{n=0}^{\infty}$ is represented by (2), then

$$L_p(a_n) = p^n\left[(p-1)\left(\sum_{k=1}^{n}\frac{c_k}{k}-1\right) + \left(\frac{p}{n+1}-1\right)c_{n+1} + pc_{n+2}\right].$$

Taking $p=1$ in Theorem 2 and Corollary 2 we get the following:

COROLLARY 16 ([11]). The sequence $(a_n)_{n=0}^{\infty}$ is starshaped if and only if it may be represented by

$$a_n = n\sum_{k=1}^{n}\frac{c_k}{k} - (n-1)c_0 \text{ with } c_k \geq 0,\ k \geq 2. \tag{3}$$

COROLLARY 17 ([11]). If the sequence $(a_n)_{n=0}^{\infty}$ is represented by (3), then

$$\triangle^2(a_n) = c_{n+2} - \frac{n}{n+1}c_{n+1}.$$

THEOREM 18. The sequence $(a_n)_{n=0}^{\infty}$ is (p,α)-convex if and only if it may be represented by

$$a_n = np^{n-1}\sum_{k=1}^{n}\frac{c_k}{k} - (np^{n-1}-1)c_0, \tag{4}$$

with $0 < p \le 1$,

$$\sum_{k=1}^{n} \frac{c_k}{k} \le 1, \tag{5}$$

$$c_{n+2} \ge \left[\frac{1}{p}\left(1 - \frac{1}{\alpha(n+1)}\right) + \left(\frac{1}{p} - 1\right)\frac{1}{n+1}\right] c_{n+1},$$

and $c_n \ge 0$, $n \ge 2$.

PROOF. On one hand, taking into account (4), we have

$$\begin{aligned}
\alpha L_p(a_n) &= \alpha[a_{n+2} - (1+p)a_{n+1} + pa_n] \\
&= \alpha p^n \left[(p-1)\left(\sum_{k=1}^{n} \frac{c_k}{k} - 1\right) + \left(\frac{p}{n+1} - 1\right) c_{n+1} + pc_{n+2}\right].
\end{aligned} \tag{6}$$

On the other hand, using (4), we also have

$$\begin{aligned}
&(1-\alpha)\left(\frac{a_{n+1} - a_0}{n+1} - p\frac{a_n - a_0}{n}\right) \\
&\quad = (1-\alpha)\Bigg(p^n \sum_{k=1}^{n+1} \frac{c_k}{k} - \frac{[(n+1)p^n - 1]c_0}{n+1} - \frac{c_0}{n+1} \\
&\qquad - p^n \sum_{k=1}^{n} \frac{c_k}{k} + \frac{(np^n - p)c_0}{n} + \frac{pc_0}{n}\Bigg) = (1-\alpha)p^n \frac{c_{n+1}}{n+1}.
\end{aligned} \tag{7}$$

From (6) and (7) we obtain

$$\begin{aligned}
&\alpha L_p(a_n) + (1-\alpha)\left(\frac{a_{n+1} - a_0}{n+1} - p\frac{a_n - a_0}{n}\right) \\
= \; & p^n \left[\alpha(p-1)\left(\sum_{k=1}^{n} \frac{c_k}{k} - 1\right) + \alpha\left(\frac{p}{n+1} - 1\right) c_{n+1} + \alpha p c_{n+2} + (1-\alpha)p^n \frac{c_{n+1}}{n+1}\right].
\end{aligned}$$

Subsequently, it follows that

$$\alpha L_p(a_n) + (1-\alpha)\left(\frac{a_{n+1} - a_0}{n+1} - p\frac{a_n - a_0}{n}\right) \ge 0$$

if and only if $0 < p \le 1$,

$$\sum_{k=1}^{n} \frac{c_k}{k} \le 1,$$

and

$$c_{n+2} \ge \left[\frac{1}{p}\left(1 - \frac{1}{\alpha(n+1)}\right) + \left(\frac{1}{p} - 1\right)\frac{1}{n+1}\right] c_{n+1}.$$

The proof is complete.

COROLLARY 19 ([10]). The sequence $(a_n)_{n=0}^{\infty}$ is α-convex if and only if it may be represented by

$$a_n = n\sum_{k=1}^{n}\frac{c_k}{k} - (n-1)c_0,$$

with

$$c_{n+2} \geq \left(1 - \frac{1}{\alpha(n+1)}\right)c_{n+1} \text{ and } c_n \geq 0,\ n \geq 2.$$

THEOREM 20. If the sequence $(a_n)_{n=0}^{\infty}$ is (p, α)-convex, then it is (p, β)-convex for $0 \leq \beta \leq \alpha$ and $0 < p \leq 1$.

PROOF. The proof follows from Theorem 2. Indeed, let the sequence $(a_n)_{n=0}^{\infty}$ be (p, α)-convex. Then, it may be represented by (4) with (5),

$$c_{n+2} \geq \left[\frac{1}{p}\left(1 - \frac{1}{\alpha(n+1)}\right) + \left(\frac{1}{p} - 1\right)\frac{1}{n+1}\right]c_{n+1},$$

and $c_n \geq 0$, $n \geq 2$. However, since $0 \leq \beta \leq \alpha$ then we also have

$$c_{n+2} \geq \left[\frac{1}{p}\left(1 - \frac{1}{\beta(n+1)}\right) + \left(\frac{1}{p} - 1\right)\frac{1}{n+1}\right]c_{n+1},$$

with $c_n \geq 0$, $n \geq 2$, which shows that the sequence $(a_n)_{n=0}^{\infty}$ is (p, β)-convex as well. The proof is complete.

For $p = 1$, as a particular case, we obtain Corollary 21.

COROLLARY 21 ([10]). If the sequence $(a_n)_{n=0}^{\infty}$ is α-convex, then it is β-convex, for $0 \leq \beta \leq \alpha$.

References

[1] H. Bor, A new application of convex sequences, J. Class. Anal., 1(2012), 31–34.

[2] H. Bor and Xh. Z. Krasniqi, A note on absolute Cesào summability factors, Adv. Pure Appl. Math., 3(2012), 259–264.

[3] Xh. Z. Krasniqi, Some properties of $(p, q; r)$-convex sequences, Appl. Math. E-Notes, 15(2015), 38–45.

[4] L. M. Kocić, I. Z. Milovanović, A property of (p, q)-convex sequences, Period. Math. Hungar., 17(1986), 25–26.

[5] I. B. Lacković and M. R. Jovanović, On a class of real sequences which satisfy a difference inequality, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 678(1980), 99–104.

[6] B. Makarov, A. Podkorytov, Real Analysis: Measures, Integrals and Applications, Springer–Verlag London, 2013.

[7] J. E. Pečarić, On some inequalities for convex sequences. Publ. Inst. Math., 33(1983), 173–178.

[8] F. Qi and B.-N. Guo, Monotonicity of sequences involving convex function and sequence, Math. Inequal. Appl., 9(2006), 247–254.

[9] Gh. Toader, A hierarchy of convexity for sequences. Anal. Numér. Théor. Approx., 12(1983), 187–192.

[10] Gh. Toader, α-convex sequences. Itinerant seminar on functional equations, approximation and convexity (Cluj-Napoca, 1983), 167–168, Preprint, 83-2, Univ. "Babes-Bolyai", Cluj–Napoca, 1983.

[11] Gh. Toader, A hierarchy of convexity for sequences. Anal. Numér. Thér. Approx., 12(1983), 187–192.

[12] S. Wu and L. Debnath, Inequalities for convex sequences and their applications, Comput. Math. Appl., 54(2007), 525–534.