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On Some Comparative Growth Analysis Of Meromorphic Functions In The Light Of Their Generalized Relative L^*-orders*

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Abstract

In the paper we wish to establish some comparative growth properties of composite entire or meromorphic functions on the basis of generalized relative L^*-order and generalized relative L^*-lower order.

1 Introduction, Definitions and Notations

Let f be an entire function defined in the finite complex plane $\mathbb{C}$. The maximum modulus function corresponding to entire f is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as maximum modulus function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)$ $\left(\bar{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a)\log r\right),$$

moreover we denote by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \le r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

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Also the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the proximity function of f is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

where

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

If f is a non-constant entire function then $T_f(r)$ is strictly increasing and continuous and its inverse $T_f^{-1} : (T_f(0), \infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

However, the study of comparative growth properties of entire and meromorphic functions which is one of the prominent branch of the value distribution theory of entire and meromorphic functions is the prime concern of the paper. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [3, 9]. In the sequel the following two notations are used:

$$\begin{cases} \log^{[k]} x = \log\left(\log^{[k-1]} x\right) & \text{for } k = 1,2,3,..., \\ \log^{[0]} x = x \end{cases}$$

and

$$\begin{cases} \exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) & \text{for } k = 1,2,3,..., \\ \exp^{[0]} x = x. \end{cases}$$

Taking this into account the *generalized order* (respectively, *generalized lower order*) of a meromorphic function f as introduced by Sato [8] is given by:

$$\rho_f^{[k]} = \limsup_{r\to\infty} \frac{\log^{[k-1]} T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[k-1]} T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log^{[k-1]} T_f(r)}{\log r + O(1)}$$

$$\left(\text{respectively } \lambda_f^{[k]} = \liminf_{r\to\infty} \frac{\log^{[k-1]} T_f(r)}{\log T_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[k-1]} T_f(r)}{\log r + O(1)}\right)$$

where $k \geq 1$.

These definitions extend the definitions of *order* ρ_f and *lower order* λ_f of an entire or meromorphic function f since for $k = 2$, these correspond to the particular case $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.

Lahiri and Banerjee [5] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

DEFINITION 1 ([5]). Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [5] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

Debnath et. al. [2] gave a more generalized concept of relative order a meromorphic function with respect to an entire function in the following way :

DEFINITION 2 ([2]). Let f be any meromorphic function and g be any entire function with index-pairs (m_1, q) and (m_2, p) respectively where $m_1 = m_2 = m$ and p, q, m are all positive integers such that $m \geq p$ and $m \geq q$. Then the relative (p, q) th order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) == \limsup_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

For details about index-pair of meromorphic function, one may see [2].

When $p = k \geq 1$ and $q = 1$, the above definition reduces to the definition of *generalized relative order* of a meromorphic function f with respect to an entire function g, denoted by $\rho_g^{[k]}(f)$ which is as follows

$$\rho_g^{[k]}(f) = \limsup_{r\to\infty} \frac{\log^{[k]} T_g^{-1} T_f(r)}{\log r}.$$

Likewise one can define the *generalized relative lower order* of a meromorphic function f with respect to an entire function g denoted by $\lambda_g^{[k]}(f)$ as

$$\lambda_g^{[k]}(f) = \liminf_{r\to\infty} \frac{\log^{[k]} T_g^{-1} T_f(r)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [6] defined it in the following way:

DEFINITION 3 ([6]). A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon > 0$,

$$\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \text{ for } r \geq r(\varepsilon) \text{ and uniformly for } k \geq 1.$$

Somasundaram and Thamizharasi [7] introduced the notions of L-order and L-lower order for entire functions. The more generalised concept for L-order and L-lower order for entire and meromorphic functions are L^*-order and L^*-lower order respectively. Their definitions are as follows:

DEFINITION 4 ([7]). The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]}.$$

In the line of Somasundaram and Thamizharasi [7] and Banerjee and Jana [2], one may define the *generalized* relative L^*-order and *generalized* relative L^*-lower order of a meromorphic function f with respect to an entire function g in the following manner:

DEFINITION 5. The generalized relative L^*-order $\rho_g^{[k]L^*}(f)$ and the generalized relative L^*-lower order $\lambda_g^{[k]L^*}(f)$ of a meromorphic function f with respect to an entire function g are defined by

$$\rho_g^{[k]L^*}(f) = \limsup_{r\to\infty} \frac{\log^{[k]} T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{[k]L^*}(f) = \liminf_{r\to\infty} \frac{\log^{[k]} T_g^{-1} T_f(r)}{\log\left[re^{L(r)}\right]},$$

where $k \geq 1$.

In this paper we study some growth properties of composition of entire and meromorphic functions with respect to their generalized relative L^*-orders and generalized relative L^*-lower orders as compared to the corresponding left and right factors.

2 Lemma

In this section we present a lemma which will be needed in the sequel.

LEMMA 1 ([1]). Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,

$$T_{f\circ g}(r) \geq T_f\left(\exp\left(r^\mu\right)\right).$$

3 Theorems

In this section we present the main results of the paper.

THEOREM 1. Let f be a meromorphic function and g, h be any two entire functions such that

$$\liminf_{r\to\infty} \frac{\log^{[k]} T_h^{-1}(r)}{\left(\log\left[re^{L(r)}\right]\right)^\alpha} = A \text{ is a positive number} \tag{1}$$

and

$$\liminf_{r\to\infty}\frac{\log^{[k]} T_h^{-1} T_f\left(\exp r^{\mu}\right)}{\left(\log^{[k]} T_h^{-1}(r)\right)^{\beta+1}} = B \text{ is a positive number,} \tag{2}$$

for any α, β, μ satisfying $0 < \alpha < 1,\ \beta > 0,\ \alpha(\beta+1) > 1$ and $0 < \mu < \rho_g \le \infty$. Then

$$\rho_h^{[k]L^*}(f\circ g) = \infty \text{ for } k = 2, 3, 4, ...$$

PROOF. From (1), we have for all sufficiently large values of r that

$$\log^{[k]} T_h^{-1}(r) \ge (A-\varepsilon)\left(\log\left[re^{L(r)}\right]\right)^{\alpha} \tag{3}$$

and from (2) we obtain for all sufficiently large values of r that

$$\log^{[k]} T_h^{-1} T_f\left(\exp r^{\mu}\right) \ge (B-\varepsilon)\left(\log^{[k]} T_h^{-1}(r)\right)^{\beta+1}. \tag{4}$$

Also $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, (3) and (4) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log^{[k]} T_h^{-1} T_{f\circ g}(r) &\ge \log^{[k]} T_h^{-1} T_f\left(\exp\left(r^{\mu}\right)\right)\\
\text{i.e., } \log^{[k]} T_h^{-1} T_{f\circ g}(r) &\ge (B-\varepsilon)\left(\log^{[k]} T_h^{-1}(r)\right)^{\beta+1},\\
\text{i.e., } \log^{[k]} T_h^{-1} T_{f\circ g}(r) &\ge (B-\varepsilon)\left[(A-\varepsilon)\left(\log\left[re^{L(r)}\right]\right)^{\alpha}\right]^{\beta+1},\\
\text{i.e., } \log^{[k]} T_h^{-1} T_{f\circ g}(r) &\ge (B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left[re^{L(r)}\right]\right)^{\alpha(\beta+1)},\\
\text{i.e., } \frac{\log^{[k]} T_h^{-1} T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} &\ge \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left[re^{L(r)}\right]\right)^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]},
\end{aligned}$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1} T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \ge \liminf_{r\to\infty}\frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left[re^{L(r)}\right]\right)^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary and $\alpha(\beta+1) > 1$, it follows from above that

$$\rho_h^{[k]L^*}(f\circ g) = \infty,$$

which proves the theorem.

THEOREM 2. Let f be a meromorphic function and g, h be any two entire functions such that

$$\liminf_{r\to\infty}\frac{\log^{[k]} T_h^{-1}\left(\exp\left(r^{\mu}\right)\right)}{(\log r)^{\alpha}} = A \text{ is a positive number} \tag{5}$$

and

$$\liminf_{r\to\infty}\frac{\log\left[\frac{\log^{[k]}T_h^{-1}(T_f(\exp r^{\mu}))}{\log^{[k]}T_h^{-1}(\exp r^{\mu})}\right]}{\left[\log^{[k]}T_h^{-1}(\exp r^{\mu})\right]^{\beta}}=B \text{ is a positive number} \tag{6}$$

for any α,β satisfying $\alpha>1,\ 0<\beta<1,\ \alpha\beta>1$ and $0<\mu<\rho_g\leq\infty$. Then

$$\rho_h^{[k]L^*}(f\circ g)=\infty \text{ for } k=2,3,4,....$$

PROOF. From (5) we have for all sufficiently large values of r that

$$\log^{[k]}T_h^{-1}(\exp(r^{\mu}))\geq(A-\varepsilon)(\log r)^{\alpha} \tag{7}$$

and from (6) we obtain for all sufficiently large values of r that

$$\log\left[\frac{\log^{[k]}T_h^{-1}(T_f(\exp r^{\mu}))}{\log^{[k]}T_h^{-1}(\exp r^{\mu})}\right]\geq(B-\varepsilon)\left[\log^{[k]}T_h^{-1}(\exp r^{\mu})\right]^{\beta}$$

$$\text{i.e., } \frac{\log^{[k]}T_h^{-1}(T_f(\exp r^{\mu}))}{\log^{[k]}T_h^{-1}(\exp r^{\mu})}\geq\exp\left[(B-\varepsilon)\left[\log^{[k]}T_h^{-1}(\exp r^{\mu})\right]^{\beta}\right]. \tag{8}$$

Also $T_h^{-1}(r)$ is increasing function of r, it follows from Lemma 1, equations (7) and (8) for a sequence of values of r tending to infinity that

$$\frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\frac{\log^{[k]}T_h^{-1}T_f(\exp(r^{\mu}))}{\log\left[re^{L(r)}\right]}$$

$$\text{i.e., } \frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\frac{\log^{[k]}T_h^{-1}T_f(\exp(r^{\mu}))}{\log^{[k]}T_h^{-1}(\exp(r^{\mu}))}\cdot\frac{\log^{[k]}T_h^{-1}(\exp(r^{\mu}))}{\log\left[re^{L(r)}\right]}$$

$$\text{i.e., } \frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\exp\left[(B-\varepsilon)\left[\log^{[k]}T_h^{-1}(\exp r^{\mu})\right]^{\beta}\right]\cdot\frac{(A-\varepsilon)(\log r)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$\text{i.e., } \frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}(\log r)^{\alpha\beta}\right]\cdot\frac{(A-\varepsilon)(\log r)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$\text{i.e., } \frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}(\log r)^{\alpha\beta-1}\log r\right]\cdot\frac{(A-\varepsilon)(\log r)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$\text{i.e., } \frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq[r]^{(B-\varepsilon)(A-\varepsilon)^{\beta}(\log r)^{\alpha\beta-1}}\cdot\frac{(A-\varepsilon)(\log r)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[k]}T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\liminf_{r\to\infty}[r]^{(B-\varepsilon)(A-\varepsilon)^{\beta}(\log r)^{\alpha\beta-1}}\cdot\frac{(A-\varepsilon)(\log r)^{\alpha}}{\log\left[re^{L(r)}\right]},$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \liminf_{r\to\infty}[r]^{(B-\varepsilon)(A-\varepsilon)^{\beta}(\log r)^{\alpha\beta-1}}\cdot\liminf_{r\to\infty}\frac{(A-\varepsilon)(\log r)^{\alpha}}{\log\left[re^{L(r)}\right]}.$$

As $\varepsilon\,(>0)$ is arbitrary and $\alpha>1$, $\alpha\beta>1$, $\liminf\limits_{r\to\infty}[r]^{(B-\varepsilon)(A-\varepsilon)^{\beta}(\log r)^{\alpha\beta-1}}$ exits. Therefore theorem follows from above.

THEOREM 3. Let f be a meromorphic function and g, h be any two entire functions such that g is of non zero order and $\lambda_h^{[k]L^*}(f)>0$ where $k=2,3,4,....$ Then

$$\rho_h^{[k]L^*}(f\circ g)=\infty\ .$$

PROOF. Suppose $0<\mu<\rho_g\leq\infty$. As $T_h^{-1}(r)$ is an increasing function of r, we get from Lemma 1, for a sequence of values of r tending to infinity that

$$\log^{[k]} T_h^{-1}T_{f\circ g}(r)\geq\log^{[k]} T_h^{-1}T_f\left(\exp\left(r^{\mu}\right)\right)$$

$$\text{i.e., }\ \log^{[k]} T_h^{-1}T_{f\circ g}(r)\geq\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)\log\left[\exp\left(r^{\mu}\right)e^{L^{(\exp(r^{\mu}))}}\right],$$

$$\text{i.e., }\ \frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\frac{\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)\left[r^{\mu}+L^{(\exp(r^{\mu}))}\right]}{\log\left[re^{L(r)}\right]},$$

$$\text{i.e., }\ \limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\liminf_{r\to\infty}\frac{\left(\lambda_h^{[k]L^*}(f)-\varepsilon\right)\left[r^{\mu}+L^{(\exp(r^{\mu}))}\right]}{\log r+L(r)},$$

$$\text{i.e., }\ \rho_h^{[k]L^*}(f\circ g)=\infty.$$

Thus the theorem follows.

THEOREM 4. Let f be a meromorphic function and g, h be any two entire functions such that g is of non zero order and $0<\lambda_h^{[k]L^*}(f)\leq\rho_h^{[k]L^*}(f)<\infty$ where $k=2,3,4,....$ Then

$$\limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log^{[k]} T_h^{-1}T_f(r)}=\infty.$$

PROOF. In view of Theorem 3, we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log^{[k]} T_h^{-1}T_f(r)}\geq\limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\cdot\liminf_{r\to\infty}\frac{\log\left[re^{L(r)}\right]}{\log^{[k]} T_h^{-1}T_f(r)}$$

$$\text{i.e., }\ \limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log^{[k]} T_h^{-1}T_f(r)}\geq\rho_h^{[k]L^*}(f\circ g)\cdot\frac{1}{\rho_h^{[k]L^*}(f)},$$

$$\text{i.e., }\ \limsup_{r\to\infty}\frac{\log^{[k]} T_h^{-1}T_{f\circ g}(r)}{\log^{[k]} T_h^{-1}T_f(r)}=\infty.$$

Thus the theorem follows.

THEOREM 5. Let f be a meromorphic function and h be an entire function such that $0 < \lambda_h^{[k]L^*}(f) \leq \rho_h^{[k]L^*}(f) < \infty$ where $k = 2, 3, 4, ...$ Also let g be an entire function with non zero order. Then for every positive constant A and every real number α,

$$\limsup_{r \to \infty} \frac{\log^{[k]} T_h^{-1} T_{f \circ g}(r)}{\left\{\log^{[k]} T_h^{-1} T_f(r^A)\right\}^{1+\alpha}} = \infty.$$

PROOF. If α be such that $1 + \alpha \leq 0$, then the theorem is trivial. So we suppose that $1 + \alpha > 0$. Since $T_h^{-1}(r)$ is an increasing function of r, we get from Lemma 1 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[k]} T_h^{-1} T_{f \circ g}(r) &\geq \log^{[k]} T_h^{-1} T_f(\exp(r^\mu)) \\ \text{i.e., } \log^{[k]} T_h^{-1} T_{f \circ g}(r) &\geq \left(\lambda_h^{[k]L^*}(f) - \varepsilon\right)[r^\mu + L(\exp(r^\mu))]. \end{aligned} \quad (9)$$

where we choose $0 < \mu < \rho_g \leq \infty$. Again from the definition of $\rho_h^{[k]L^*}(f)$, it follows for all sufficiently large values of r that

$$\log^{[k]} T_h^{-1} T_f(r^A) \leq \left(\rho_h^{[k]L^*}(f) + \varepsilon\right)\left(A \log r + L\left(r^A\right)\right).$$

$$\textit{i.e.,} \quad \left\{\log^{[k]} T_h^{-1} T_f(r^A)\right\}^{1+\alpha} \leq \left(\rho_h^{[k]L^*}(f) + \varepsilon\right)^{1+\alpha}\left(A \log r + L\left(r^A\right)\right)^{1+\alpha}. \quad (10)$$

Now from (9) and (10), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[k]} T_h^{-1} T_{f \circ g}(r)}{\left\{\log^{[k]} T_h^{-1} T_f(r^A)\right\}^{1+\alpha}} \geqslant \frac{\left(\lambda_h^{[k]L^*}(f) - \varepsilon\right)[r^\mu + L(\exp(r^\mu))]}{\left(\rho_h^{[k]L^*}(f) + \varepsilon\right)^{1+\alpha}(A \log r + L(r^A))^{1+\alpha}}.$$

Since $\frac{r^\mu}{(\log r)^{1+\alpha}} \to \infty$ as $r \to \infty$, the theorem follows from above.

THEOREM 6. Let f be a meromorphic function and g be an entire function with non zero order. Also let h and l be any two entire functions such that $0 < \lambda_h^{[k]L^*}(f)$ and $\rho_l^{L^*}(g) < \infty$ where $k = 2, 3, 4, ...$ Then for every positive constant A and every real number α,

$$\limsup_{r \to \infty} \frac{\log^{[k]} T_h^{-1} T_{f \circ g}(r)}{\left\{\log T_l^{-1} T_g(r^A)\right\}^{1+\alpha}} = \infty.$$

We omit the proof of Theorem 6 since it can be carried out in the line of Theorem 5.

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