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Reconstruction Of Bivariate Cardinal Splines Of Polynomial Growth From Their Local Average Samples*

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Abstract

We analyse the following local average sampling problem for two variables: Let h be a nonnegative function supported in the rectangle $\left[-\frac{1}{2},\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]$. Given a sequence of samples $\{y_{ij}\}_{i,j\in\mathbb{Z}}$, find a bivariate spline $f(x,y)$ such that $(f\star h)(i,j)=y_{ij}$. It is shown that this problem has infinitely many solutions. Further, under some realistic conditions on h it is shown that the above said problem has unique solution when both samples $\{y_{ij}\}$ and the spline f are of polynomial growth.

1 Introduction

The extension of classical Whittaker-Shannon-Kotel'nikov sampling formula for k-dimension may be stated as follows [8, 6, 7]: Any function f bandlimited to the k-dimensional cube $[-\frac{1}{2},\frac{1}{2}]^k$ can be reconstructed from its sequence of samples $\{f(n)\}_{n\in\mathbb{Z}^k}$ using the formula

$$f(t_1,t_2,\ldots,t_k)=\sum_{\alpha\in\mathbb{Z}^k} f(\alpha)sinc(t_1-\alpha_1)sinc(t_2-\alpha_2)\ldots sinc(t_k-\alpha_k),$$

where the sinc function is defined by $\operatorname{sinc}(x)=\dfrac{\sin(x)}{x}$. Although the bandlimited condition is eminently useful, it is not always realistic, since a bandlimited signal is of infinite duration. It is natural to investigate other signal classes for which a sampling theorem holds. The reconstruction has been investigated for non-bandlimited functions in [1, 2, 5, 6, 7, 8, 9, 10]. In this paper, we consider the class of bivariate splines of polynomial growth.

The B-splines with equally spaced knots in two variables is defined [3, 4] as

$$\beta_{d_1d_2}(x,y)=\beta_{d_1}(x)\beta_{d_2}(y),$$

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where β_d is the cardinal central B-spline of degree d in single variable which is given by

$$\beta_d := \chi_{[-\frac{1}{2},\frac{1}{2}]} \star \ldots \star \chi_{[-\frac{1}{2},\frac{1}{2}]}, (d+1 \text{ terms})$$

and $\star$ denotes the convolution. $\beta_{d_1 d_2}$ is a tensor-product of two B-splines and its piecewise pieces are separated by a rectangular partition. Let S_{d_1,d_2} be the class of functions $f(x,y)$ satisfying the following properties:

1. The $d_1 d_2$ partial derivatives $\dfrac{\partial^{u+v}}{\partial x^u \partial y^v} f(x,y)$, $0 \le u \le d_1 - 1, 0 \le v \le d_2 - 1$ are continuous in the entire plane $\mathbb{R}^2$.
2. Let $\Pi_{x,y}$ denote set of all polynomials in x and y of degree $\le (d_1 + d_2)$, i.e.,

$$\Pi_{x,y} = \left\{ \sum_{u=0}^{d_1} \sum_{v=0}^{d_2} a_{uv} x^u y^v : a_{uv} \in \mathbb{R} \right\}.$$

In each square $[i-1,i] \times [j-1,j]$, $f(x,y) \in \Pi_{x,y}$ for both d_1 and d_2 odd. If d_1 is odd and d_2 is even, then in each square $[i-1,i] \times [j+\frac{1}{2}-1, j+\frac{1}{2}]$, $f(x,y) \in \Pi_{x,y}$. In each square $[i+\frac{1}{2}-1, i+\frac{1}{2}] \times [j-1,j]$, $f(x,y) \in \Pi_{x,y}$ for d_1 even and d_2 odd.

Also, when d_1 and d_2 are even, then in each square $[i+\frac{1}{2}-1, i+\frac{1}{2}] \times [j+\frac{1}{2}-1, j+\frac{1}{2}]$, $f(x,y) \in \Pi_{x,y}$.

We note that $\Pi_{x,y}$ depends on $(d_1+1)(d_2+1)$ parameters and we can write

$$S_{d_1,d_2} = \left\{ f(x,y) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} a_{ij} \beta_{d_1}(x-i) \beta_{d_2}(y-j) : a_{ij} \in \mathbb{R} \right\}.$$

2 Bivariate Cardinal Spline Interpolation

The bivariate cardinal spline interpolation problem defined in [10] is as follows: Given a double sequence $\{y_{ij}\}_{i,j \in \mathbb{Z}}$ of real numbers, find a bivariate spline $f \in S_{d_1,d_2}$ such that

$$f(i,j) = y_{ij}, \ i,j \in \mathbb{Z}. \tag{1}$$

It can be easily checked that for $d_1 = d_2 = 1$, this problem has a unique solution.

LEMMA 1. Let $d_1, d_2 > 1$. Then given a double sequence $\{y_{ij}\}_{i,j \in \mathbb{Z}}$ of real numbers, there are infinitely many bivariate splines $f \in S_{d_1,d_2}$ such that $f(i,j) = y_{ij}$, for $i,j \in \mathbb{Z}$. Moreover, the set of all such solutions in S_{d_1,d_2} form a linear manifold of dimension $(d_1+1)(d_2+1) - 4$ when both d_1, d_2 are odd or d_1 is odd and d_2 is even or d_1 is even and d_2 is odd and of dimension $(d_1+1)(d_2+1) - 3$ when both d_1, d_2 are even.

PROOF. Case (i): d_1, d_2 are odd. In this case every $f(x,y) \in S_{d_1,d_2}$ can be uniquely represented in the form

$$f(x,y) = P(x,y) + \sum_{u>0} \sum_{v>0} a_{uv} (x-u)_+^{d_1} (y-v)_+^{d_2} + \sum_{u \ge 0} \sum_{v \ge 0} a_{-u-v} (-x-u)_+^{d_1} (-y-v)_+^{d_2},$$

where $P(x,y) \in \Pi_{x,y}$, a_{uv} are constants, and $x_+ := max(0,x)$. Since $f(x,y) = P(x,y)$ in $[0,1]\times[0,1]$, we have the relations, $P(0,0) = y_{00}$, $P(1,0) = y_{10}$, $P(0,1) = y_{01}$ and $P(1,1) = y_{11}$. The coefficients a_{uv} are uniquely determined by the interpolation conditions $f(i,j) = y_{ij},\ i,j \in \mathbb{Z}$. Therefore $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-4$ parameters in $P(x,y)$.

Case(ii): d_1 is odd and d_2 is even. Every $f(x,y) \in S_{d_1,d_2}$ has a unique representation of the form

$$\begin{aligned} f(x,y) &= P(x,y) + \sum_{u>0}\sum_{v>0} a_{uv}(x-u)_+^{d_1}\left(\left(y+\frac{1}{2}\right)-v\right)_+^{d_2} \\ &\quad + \sum_{u\geq 0}\sum_{v\geq 0} a_{-u-v}(-x-u)_+^{d_1}\left(-\left(y+\frac{1}{2}\right)-v\right)_+^{d_2}. \end{aligned}$$

Therefore if $f(x,y)$ satisfies equation (1), then $P(0,0) = y_{00}$, $P(1,0) = y_{10}$, $P(0,1) = y_{01}$ and $P(1,1) = y_{11}$. As in the previous case the coefficients a_{uv} are calculated from the interpolation conditions 1. Hence $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-4$ parameters.

Case(iii): d_1 is even and d_2 is odd. In this case every $f(x,y) \in S_{d_1,d_2}$ can be uniquely written in the form

$$\begin{aligned} f(x,y) &= P(x,y) + \sum_{u>0}\sum_{v>0} a_{uv}\left(\left(x+\frac{1}{2}\right)-u\right)_+^{d_1}(y-v)_+^{d_2} \\ &\quad + \sum_{u\geq 0}\sum_{v\geq 0} a_{-u-v}\left(-\left(x+\frac{1}{2}\right)-u\right)_+^{d_1}(-y-v)_+^{d_2}. \end{aligned}$$

In this case also $P(0,0) = y_{00}$, $P(1,0) = y_{10}$, $P(0,1) = y_{01}$ and $P(1,1) = y_{11}$. As in the previous cases, $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-4$ parameters.

Case(iv): Both d_1 and d_2 are even. Every $f(x,y) \in S_{d_1,d_2}$ has unique representation of the form

$$\begin{aligned} f(x,y) &= P(x,y) + \sum_{u>0}\sum_{v>0} b_{uv}\left((x+\frac{1}{2})-u\right)_+^{d_1}\left((y+\frac{1}{2})-v\right)_+^{d_2} \\ &\quad + \sum_{u\geq 0}\sum_{v\geq 0} b_{-u-v}\left(-\left(x+\frac{1}{2}\right)-u\right)_+^{d_1}\left(-\left(y+\frac{1}{2}\right)-v\right)_+^{d_2}, \end{aligned}$$

where $P(x,y) \in \Pi_{x,y}$. Then $P(x,y)$ satisfies the relations $P(0,0) = y_{00}$, $P(1,0) = y_{10}$ and $P(0,1) = y_{01}$. Thus three coefficients in $P(x,y)$ can be uniquely found from these relations. The coefficients b_{uv} are uniquely determined using the conditions $f(i,j) = y_{ij},\ i,j \in \mathbb{Z}$. Therefore, in this case $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-3$ parameters.

In order to obtain uniqueness of solution Schoenberg [10] has applied the following power growth condition on the bivariate cardinal spline and the bi-infinite samples:

$$S_\gamma = \{f(x,y) \in S_{d_1,d_2} : f(x,y) = O(|x|+|y|+1)^\gamma\},$$

$$D_\gamma = \{\{y_{ij}\}_{i,j\in\mathbb{Z}} : y_{ij} = O(|i| + |j| + 1)^\gamma\}.$$

Further, he has shown in [10] that for $\gamma \geq 0$ and for a given double sequence of real numbers $\{y_{ij}\}_{i,j\in\mathbb{Z}} \in D_\gamma$, there exists a unique bivariate spline $f \in S_\gamma$ such that $f(i,j) = y_{ij},\ i,j \in \mathbb{Z}$.

In practice, the available samples are not always exact. The samples of f are the local averages of the function f at (m,n). i.e.,

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \int_{m-\frac{1}{2}}^{m+\frac{1}{2}} f(x,y)h(n-x,m-y)dxdy,$$

where h is suitable weight function which reflects the characteristic of the measurement process.

Average sampling problem:

Given a sequence of real numbers$\{y_{ij}\}_{i,j\in\mathbb{Z}}$, find a bivariate spline $f \in S_{d_1,d_3}$ such that $f \star h(i,j) = y_{ij},\ i,j \in \mathbb{Z}$, where the averaging function h satisfies

$$supp(h) \subseteq \left[-\frac{1}{2},\frac{1}{2}\right] \times \left[-\frac{1}{2},\frac{1}{2}\right] \text{ and } h(x,y) \geq 0, \tag{2}$$

$$0 < \int_{-\frac{1}{2}}^{0} \int_{-\frac{1}{2}}^{0} h(x,y)dxdy < \infty \text{ and } 0 < \int_{0}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}} h(x,y)dxdy < \infty. \tag{3}$$

We show that this average sampling problem has infinitely many solutions for every d. Further, by applying the polynomial growth conditions as that of [10], the uniqueness is obtained.

LEMMA 2. If the averaging function h satisfies (2) and (3), then for a given double sequence of real numbers $\{y_{ij}\}_{i,j\in\mathbb{Z}}$, there are infinitely many bivariate splines $f \in S_{d_1,d_2}$ such that

$$f \star h(i,j) = y_{ij},\ i,j \in \mathbb{Z}. \tag{4}$$

The set of such solutions in S_{d_1,d_2} form a linear manifold of dimension $(d_1+1)(d_2+1)$ if d_1, d_2 are odd and of dimension $(d_1+1)(d_2+1)-1$ for the other three cases.

PROOF. Case(i): Both d_1 and d_2 are odd. In this case, the functions $f \in S_{d_1,d_2}$ can be uniquely represented in the form

$$f(x,y) = P(x,y) + \sum_{u>0}\sum_{v>0} a_{uv}(x-u)_+^{d_1}(y-v)_+^{d_2} + \sum_{u\geq 0}\sum_{v\geq 0} a_{-u-v}(-x-u)_+^{d_1}(-y-v)_+^{d_2}$$

with appropriate coefficients a_{uv} and $P(x,y) = f(x,y)$ in $[0,1]\times[0,1]$. If $f(x,y)$ satisfies (4), then $f \star h(1,1) = y_{1,1}$. i.e.,

$$h \star f(1,1) = h \star P(1,1) + a_{11}\int_{\frac{1}{2}}^{\frac{3}{2}} \int_{\frac{1}{2}}^{\frac{3}{2}} h(1-x,1-y)(x-1)_+^{d}(y-1)_+^{d}dxdy$$

and $\int_{\frac{1}{2}}^{\frac{3}{2}}\int_{\frac{1}{2}}^{\frac{3}{2}} h(1-x,1-y)(x-1)_+^{d_1}(y-1)_+^{d_2}dxdy > 0$. From this the coefficient a_{11} can be uniquely determined such that $f \star h(1,1) = y_{1,1}$. Similarly the other coefficients a_{ij} can be uniquely determined using the other conditions of $f \star h(i,j) = y_{i,j}$. Thus the solutions linearly depends on $(d_1+1)(d_2+1)$ parameters.

Case(ii): d_1 is odd and d_2 is even. Then every $f(x,y) \in S_{d_1,d_2}$ can be uniquely written in the form

$$\begin{aligned} f(x,y) &= P(x,y) + \sum_{u>0}\sum_{v>0} a_{uv}(x-u)_+^{d_1}\left((y+\frac{1}{2})-v\right)_+^{d_2} \\ &+ \sum_{u\geq 0}\sum_{v\geq 0} a_{-u-v}(-x-u)_+^{d_1}\left(-(y+\frac{1}{2})-v\right)_+^{d_2}. \end{aligned}$$

If $f(x,y)$ satisfies equation (4), then in this case

$$h \star f(0,0) = \int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{\frac{-1}{2}}^{\frac{1}{2}} h(-x,-y)P(x,y)dxdy.$$

The coefficients a_{uv} can be found from the conditions (4). Hence $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-1$ parameters.

Case(iii): d_1 is even and d_2 is odd. In this case every function $f \in S_{d_1,d_2}$ has a unique representation of the form

$$\begin{aligned} f(x,y) &= P(x,y) + \sum_{u>0}\sum_{v>0} a_{uv}\left((x+\frac{1}{2})-u\right)_+^{d_1}(y-v)_+^{d_2} \\ &+ \sum_{u\geq 0}\sum_{v\geq 0} a_{-u-v}\left(-(x+\frac{1}{2})-u\right)_+^{d_1}(-y-v)_+^{d_2}. \end{aligned}$$

In this case the condition (4) implies $P(x,y)$ satisfies

$$h \star f(0,0) = \int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{\frac{-1}{2}}^{\frac{1}{2}} h(-x,-y)P(x,y)dxdy.$$

Therefore $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-1$ coefficients.

Case(iv): Suppose that both d_1 and d_2 are even. Then every function $f \in S_{d_1,d_2}$ can be uniquely represented in the form

$$\begin{aligned} f(x,y) &= P(x,y) + \sum_{u>0}\sum_{v>0} b_{uv}\left((x+\frac{1}{2})-u\right)_+^{d_1}\left((y+\frac{1}{2})-v\right)_+^{d_2} \\ &+ \sum_{u\geq 0}\sum_{v\geq 0} b_{-u-v}\left(-(x+\frac{1}{2})-u\right)_+^{d_1}\left(-(y+\frac{1}{2})-v\right)_+^{d_2} \qquad (5) \end{aligned}$$

with appropriate coefficients b_{uv}. Since

$$\begin{aligned} h \star f(0,0) &= \int_{\frac{-1}{2}}^{\frac{1}{2}} \int_{\frac{-1}{2}}^{\frac{1}{2}} h(-x,-y) f(x,y) dx dy \\ &= \int_{\frac{-1}{2}}^{\frac{1}{2}} \int_{\frac{-1}{2}}^{\frac{1}{2}} h(-x,-y) P(x,y) dx dy \end{aligned}$$

and $f(x,y)$ satisfies $h \star f(0,0) = y_{00}$, the solutions $f(x,y)$ linearly depends on $(d_1+1)(d_2+1)-1$ parameters.

3 Average Sampling Theorem

THEOREM 1. [Main Theorem] Let $d_1, d_2 \in \mathbb{N}$ and $h(x,y) = h_1(x)h_2(y)$ satisfy the conditions (2) and (3). Then for a given double sequence of real numbers $\{y_{ij}\}_{i,j\in\mathbb{Z}} \in D_\gamma$, there exists a unique bivariate spline $f \in S_\gamma$ such that

$$f \star h(i,j) = y_{ij} \;\; i,j \in \mathbb{Z}.$$

In order to prove this theorem, we introduce the vector

$$G_h(z,w) = (G_{h_1,d_1}(z), G_{h_2,d_2}(w)),$$

where $G_{h_1,d_1}(z)$ and $G_{h_2,d_2}(w)$ are Laurent polynomials defined by

$$\begin{aligned} G_{h_1,d_1}(z) &= \int_{\frac{-1}{2}}^{\frac{1}{2}} h_1(x) \Upsilon_{z,d_1}(x) dx, \\ G_{h_2,d_2}(w) &= \int_{\frac{-1}{2}}^{\frac{1}{2}} h_2(y) \Upsilon_{w,d_2}(y) dy, \\ \Upsilon_{z,d_1}(x) &= \sum_{i\in\mathbb{Z}} z^{-i} \beta_{d_1}(i-x), \\ \Upsilon_{w,d_2}(y) &= \sum_{j\in\mathbb{Z}} w^{-j} \beta_{d_2}(j-y). \end{aligned}$$

For the proof we also need the following properties [9]:

LEMMA 3 ([9]). For $d_1 \in \mathbb{N}, n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$ we have

(i) $\Upsilon_{z^{-1},d_1}(-x) = \Upsilon_{z,d_1}(x)$.

(ii) $\Upsilon_{z,d_1}(x+n) = z^{-n} \Upsilon_{z,d_1}(x)$.

(iii) $\Upsilon'_{z,d_1+1}(x) = (1-z) \Upsilon_{z,d_1}(x + \frac{1}{2})$.

(iv) $\Upsilon_{-1,d_1}(x)$ is even, $\Upsilon_{-1,d_1}(\frac{1}{2}) = 0$ and $\Upsilon_{-1,d_1}(x) > 0$ for $x \in (\frac{-1}{2}, \frac{1}{2})$.

THEOREM 2. Consider the linear space

$$\bigwedge := \{f(x,y) \in S_{d_1,d_2} : h \star f(i,j) = 0 \; \forall \; i,j \in \mathbb{Z}\}.$$

If $z_1, z_2, \ldots, z_p$ are the simple roots of $G_{h_1,d_1}(z)$ and $w_1, w_2, \ldots, w_q$ are the simple roots of $G_{h_2,d_2}(w)$ then the set of functions

$$\{\Upsilon_{z_r^{-1},d_1} \cdot \Upsilon_{w_s^{-1},d_2} : 1 \leq r \leq p, 1 \leq s \leq q\}$$

forms a basis of $\bigwedge$.

PROOF. We have to find l linearly independent functions in $\bigwedge$ related to the roots of $G_h(z,w)$. The dimension of $\bigwedge$ is

$$l := \begin{cases} (d_1+1)(d_2+1) & \text{if both } d_1 \text{ and } d_2 \text{ are odd,} \\ (d_1+1)(d_2) & \text{if } d_1 \text{ is odd and } d_2 \text{ are even,} \\ d_1\,(d_2+1) & \text{if } d_1 \text{ is even and } d_2 \text{ are odd,} \\ d_1 d_2 & \text{if both } d_1 \text{ and } d_2 \text{ are even,} \end{cases}$$

Now

$$\begin{aligned} & \Upsilon_{z_r^{-1},d_1} \cdot \Upsilon_{w_s^{-1},d_2} \star h(i,j) \\ = \; & \Upsilon_{z_r^{-1},d_1} \star h_1(i) \cdot \Upsilon_{w_s^{-1},d_2} \star h_2(j) \\ = \; & \int_{-\frac{1}{2}}^{\frac{1}{2}} \Upsilon_{z_r^{-1},d_1}(i-u) h_1(u) du \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} \Upsilon_{w_s^{-1},d_2}(i-v) h_2(v) dv \\ = \; & z_r^i \int_{-\frac{1}{2}}^{\frac{1}{2}} \Upsilon_{z_r^{-1},d_1}(-u) h_1(u) du \cdot w_s^j \int_{-\frac{1}{2}}^{\frac{1}{2}} \Upsilon_{w_s^{-1},d_2}(-v) h_2(v) dv \\ = \; & z_r^i \int_{-\frac{1}{2}}^{\frac{1}{2}} \Upsilon_{z_r^{-1},d_1}(u) h_1(u) du \cdot w_s^j \int_{-\frac{1}{2}}^{\frac{1}{2}} \Upsilon_{w_s^{-1},d_2}(v) h_2(v) dv \\ = \; & z_r^i G_{h_1,d_1}(z_r) \cdot w_s^j G_{h_2,d_2}(w_s) \\ = \; & 0 \text{ for } r = 1,2,\ldots,p \text{ and } s = 1,2,\ldots,q. \end{aligned}$$

Now we prove $\Upsilon_{z_r^{-1},d_1} \cdot \Upsilon_{w_s^{-1},d_2}$ are linearly independent. For, suppose that

$$\sum_{r=1}^{p} \sum_{s=1}^{q} c_{rs} \left[\Upsilon_{z_r^{-1},d_1}(x) \cdot \Upsilon_{w_s^{-1},d_2}(y) \right] = 0.$$

Then

$$\sum_{r=1}^{p} \sum_{s=1}^{q} c_{rs} \left[\sum_{i \in \mathbb{Z}} z_r^i \beta_{d_1}(i-x) \sum_{j \in \mathbb{Z}} w_s^j \beta_{d_2}(j-y) \right] = 0.$$

i.e.,

$$\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \left[\sum_{r=1}^{p} \sum_{s=1}^{q} c_{rs} z_r^i w_s^j \right] \beta_{d_1}(i-x) \beta_{d_2}(j-y) = 0.$$

As $\beta_{d_1}(i-x)\beta_{d_2}(j-y)$ are linearly independent, $\sum_{r=1}^{p}\sum_{s=1}^{q} c_{rs} z_r^i w_s^j = 0 \ \forall \ i,j \in \mathbb{Z}$. The above system is a set of linear equations in c_{ij} with coefficient matrix, the Vandermonde's matrix. Since the Vandermonde determinant is not zero, $c_{rs} = 0$. Therefore the functions $\Upsilon_{z_r^{-1},d_1} \cdot \Upsilon_{w_s^{-1},d_2}$ form a basis of $\bigwedge$.

THEOREM 3. Suppose that $d_1, d_2 \in \mathbb{N}, \gamma \geq 0$ and h is in the separable form satisfying conditions (2) and (3). If the roots of $G_{h_1,d_1}(z)$,$G_{h_2,d_2}(w)$ are simple and no roots on the unit circles $|z| = 1$,$|w| = 1$ respectively, then for a given double sequence of real numbers $\{y_{ij}\}_{i,j\in\mathbb{Z}} \in D_\gamma$ the problem, of finding a bivariate spline $f \in S_\gamma$ satisfying

$$f \star h(i,j) = y_{ij}, \ i,j \in \mathbb{Z}$$

has a unique solution. The solution is of the form

$$f(x,y) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} L_{h_1,d_1}(x-i) L_{h_2,d_2}(y-j),$$

where $L_{h_1,d_1}(x) = \sum_{i\in\mathbb{Z}} c_i \beta_{d_1}(x-i)$, $L_{h_2,d_2}(y) = \sum_{j\in\mathbb{Z}} d_j \beta_{d_2}(y-j)$, c_i and d_j are coefficients of the Laurent expansion of $G^{-1}_{h_1,d_1}(z)$ and $G^{-1}_{h_2,d_2}(w)$ respectively. The spline L_{h_1,d_1} and L_{h_2,d_2}have exponential decay.

PROOF. The coefficients c_i, d_j are given by $C(z) = G^{-1}_{h_1,d_1}(z) = \sum_{i\in\mathbb{Z}} c_i z^{-i}$ and $D(w) = G^{-1}_{h_2,d_2}(w) = \sum_{j\in\mathbb{Z}} d_j w^{-j}$. These coefficients have exponential decay. Therefore $c_i = O(\phi^{|i|}_{h_1,d_1}), d_j = O(\phi^{|j|}_{h_2,d_2})$, where $\phi_{h_1,d_1}, \phi_{h_2,d_2} \in (0,1)$. Hence $L_{h_1,d_1} = O(\phi^{|x|}_{h_1,d_1})$ and $L_{h_2,d_2} = O(\phi^{|y|}_{h_2,d_2})$. For $|x| > 2, |y| > 2$ consider,

$$\begin{aligned} & \frac{\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(|i|+|j|+1)^\gamma \phi^{|x-i|}_{h_1,d_1}\phi^{|y-j|}_{h_2,d_2}}{(|x|+|y|+1)^\gamma} \\ \leq & \frac{\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(|x-i+1|+|y-j+1|+1)^\gamma \phi^{|i|-1}_{h_1,d_1}\phi^{|j|-1}_{h_2,d_2}}{(|x|+|y|+1)^\gamma} \\ \leq & \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(1+|i|+|j|)^\gamma \phi^{|i|-1}_{h_1,d_1}\phi^{|j|-1}_{h_2,d_2} < \infty. \end{aligned}$$

Therefore from the order of y_{ij} we get

$$\begin{aligned} f(x,y) &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} L_{h_1,d_1}(x-i) L_{h_2,d_2}(y-j) \\ &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}(|i|+|j|+1)^\gamma \phi^{|x-i|}_{h_1,d_1}\phi^{|y-j|}_{h_2,d_2} \\ &\leq K(|x|+|y|+1)^\gamma \ \forall \ (x,y) \in \mathbb{R}^2. \end{aligned}$$

Hence

$$f(x,y) = O(|x|+|y|+1)^\gamma \ \forall (x,y) \in \mathbb{R}^2.$$

Now

$$\begin{aligned} f(x,y) &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} L_{h_1,d_1}(x-i) L_{h_2,d_2}(y-j) \\ &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} \left[\sum_{u\in\mathbb{Z}} c_u \beta_{d_1}(x-u-i)\right]\left[\sum_{v\in\mathbb{Z}} d_v \beta_{d_2}(y-v-j)\right] \\ &= \sum_{u\in\mathbb{Z}}\sum_{v\in\mathbb{Z}} \left[\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} c_{u-i} d_{v-j}\right] \beta_{d_1}(x-u)\beta_{d_2}(y-v). \end{aligned}$$

From this we conclude that $f \in S_\gamma$. As $C(z)G_{h_1,d_1}(z) = 1$ and $D(w)G_{h_2,d_2}(w) = 1$ we obtain

$$L_{h_1,d_1} \star h_1(i) = \sum_{u\in\mathbb{Z}} c_u [h_1 \star \beta_{d_1}](i-u) = \delta_i,$$

$$L_{h_2,d_2} \star h_2(j) = \sum_{v\in\mathbb{Z}} d_v [h_2 \star \beta_{d_2}](j-v) = \delta_j.$$

Hence we get

$$\begin{aligned} f \star h(i,j) &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} L_{h_1,d_1}(x-i) L_{h_2,d_2}(y-j) \star h(i,j) \\ &= \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} y_{ij} \left[[L_{h_1,d_1}(x-i) \star h_1(i)][L_{h_2,d_2}(y-j) \star h_2(j)]\right]. \end{aligned}$$

Clearly $f \star h(i,j) = y_{ij}$ $i,j \in \mathbb{Z}$. We conclude that $f(x,y)$ is a solution. Now we shall show the uniqueness. If the bivariate spline $f, g \in S_\gamma$ are two solutions, then by the Theorem 2 we have

$$f(x,y) - g(x,y) = \sum_{r=1}^{p}\sum_{s=1}^{q} c_{rs} \left[\Upsilon_{z_r^{-1},d_1}(x) \cdot \Upsilon_{w_s^{-1},d_2}(y)\right],$$

for some constants c_{rs}. Using the behaviour of $\Upsilon_{z_r^{-1},d_1}(x) \cdot \Upsilon_{w_s^{-1},d_2}(y)$ at $x \to \pm\infty$ and $y \to \pm\infty$ we get that $c_{rs} = 0$. Therefore $f = g$.

PROOF OF THEOREM 1. In view of Theorem 3, it is sufficient to prove that the roots of $G_{h_1,d_1}(z)$ and $G_{h_2,d_2}(w)$ are simple and none of them is on the unit circles $|z| = 1$ and $|w| = 1$ respectively. We can write

$$\begin{aligned} P(z) &= z^{\frac{p}{2}} G_{h_1,d_1}(z) \\ &= h_1 \star \beta_{d_1}\left(\frac{p}{2}\right) + h_1 \star \beta_{d_1}\left(\frac{p}{2}-1\right) z + h_1 \star \beta_{d_1}\left(\frac{p}{2}-2\right) z^2 \\ &\quad + \ldots + h_1 \star \beta_{d_1}\left(-\frac{p}{2}\right) z^p, \end{aligned}$$

where

$$p := \begin{cases} d_1 + 1 & \text{if } d_1 \text{ is odd,} \\ d_1 & \text{if } d_1 \text{ is even.} \end{cases}$$

Also

$$\begin{aligned} Q(w) &= w^{\frac{q}{2}} G_{h_2,d_2}(w) \\ &= h_2 \star \beta_{d_2}\left(\frac{q}{2}\right) + h_2 \star \beta_{d_2}\left(\frac{q}{2}-1\right) w + h_2 \star \beta_{d_2}\left(\frac{q}{2}-2\right) w^2 \\ &\quad + \ldots + h_2 \star \beta_{d_2}\left(-\frac{q}{2}\right) w^q, \end{aligned}$$

where

$$q := \begin{cases} d_2+1 & \text{if } d_2 \text{ is odd,} \\ d_2 & \text{if } d_2 \text{ is odd.} \end{cases}$$

It is shown in [5] that the roots of $G_{h_1,d_1}(z)$ and $G_{h_2,d_2}(w)$ are simple and none of them is on the unit circles $|z| = 1$ and $|w| = 1$ respectively.

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