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Uniqueness Of Entire Functions Of Certain Difference Polynomials Sharing A Small Function *

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Abstract

In this paper, we study the uniqueness problems of difference polynomials of entire functions sharing a small function α, using the concept of weakly weighted sharing and relaxed weighted sharing. Our results extend and generalise the results due to Pulak Sahoo and Himadri Karmakar [12].

1 Introduction and Main Results

In this paper, we mainly study the uniqueness of entire functions of certain difference polynomials sharing a small function. It is assumed that the reader is familiar with the standard notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $S(r, f)$ and so on (see [4, 7, 14]). A meromorphic function f means meromorphic in the whole complex plane. If no poles occur, then f is called an entire function. We say that the meromorphic function $\alpha(\not\equiv 0, \infty)$ is a small function of f, if $T(r, \alpha) = S(r, f)$.

Let k be a positive integer. Set $E(a, f) = \{z : f(z)-a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zero points are counted only once, then we denote the set by $\overline{E}(a, f)$. Let f and g be two non-constant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM; if $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a, f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a, f)$ the set of distinct a-points of f with multiplicities not greater than k. We denote order of f by $\rho(f)$ (see [7, 14]). We now explain the following definitions.

DEFINITION 1 ([6]). Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer k, we denote by $N(r, a; f \mid\leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f \mid\leq k)$ we denote the corresponding reduced counting function. Analogously, we can define $N(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\geq k)$.

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DEFINITION 2 ([5]). Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \cdots + \overline{N}(r,a;f \mid\geq k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Let $N_E(r,a;f,g)$ ($\overline{N}_E(r,a;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a;f,g)$ ($\overline{N}_0(r,a;f,g)$) the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. If

$$\overline{N}\,(r,a;f) + \overline{N}\,(r,a;g) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share a "CM". On the other hand, if

$$\overline{N}\,(r,a;f) + \overline{N}\,(r,a;g) - 2\overline{N}_0(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that f and g share a "IM".

DEFINITION 3 ([8]). Let f and g share a "IM" and k be a positive integer or infinity. $\overline{N}^{E}_{k)}(r,a;f,g)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g and both of their multiplicities are not greater than k. $\overline{N}^{0}_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of f which are a-points of g and both of their multiplicities are not less than k.

The following is the definition of weakly weighted sharing which is a scaling between sharing IM and sharing CM.

DEFINITION 4 ([8]). For $a \in \mathbb{C} \cup \{\infty\}$, if k is a positive integer or infinity and

$$\overline{N}(r,a;f \mid\leq k) - \overline{N}^{E}_{k)}(r,a;f,g) = S(r,f),$$

$$\overline{N}(r,a;g \mid\leq k) - \overline{N}^{E}_{k)}(r,a;f,g) = S(r,g),$$

$$\overline{N}(r,a;f \mid\geq k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) = S(r,f),$$

$$\overline{N}(r,a;g \mid\geq k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) = S(r,g),$$

or if $k=0$ and

$$\overline{N}(r,a;f) - \overline{N}_0(r,a;f,g) = S(r,f), \quad \overline{N}(r,a;g) - \overline{N}_0(r,a;f,g) = S(r,g),$$

then we say that f and g weakly share a with weight k. Here, we write $f,\ g$ share "(a,k)" to mean that $f,\ g$ weakly share a with weight k.

The following is the definition of relaxed weighted sharing, weaker than weakly weighted sharing.

DEFINITION 5 ([1]). We denote by $\overline{N}(r,a;f\mid=p;g\mid=q)$ the reduced counting function of common a-points of f and g with multiplicities p and q respectively.

DEFINITION 6 ([1]). Let f, g share a "IM". Also let k be a positive integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. If for $p\neq q$,

$$\sum_{p,q\leq k}\overline{N}(r,a;f\mid=p;g\mid=q)=S(r),$$

then we say that f and g share a with weight k in a relaxed manner. Here we write f and g share $(a,k)^*$ to mean that f and g share a with weight k in a relaxed manner.

In recent years, there has been an increasing interest in studying difference equations in the complex plane.

In 2014, C. Meng [10] proved the following results using the concept of weakly weighted sharing and relaxed weighted sharing.

THEOREM A. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0,\infty)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant and $n\geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha,2)$", then $f=g$.

THEOREM B. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0,\infty)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant and $n\geq 10$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $(\alpha,2)^*$, then $f=g$.

THEOREM C. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0,\infty)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant and $n\geq 16$ is an integer. If

$$\overline{E}_{2)}(\alpha(z),f^n(z)(f(z)-1)f(z+c))=\overline{E}_{2)}(\alpha(z),g^n(z)(g(z)-1)g(z+c)),$$

then $f=g$.

Recently, P. Sahoo [11] generalised the above theorems and obtained the following results.

THEOREM D. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0,\infty)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant, n and m (≥ 2) are integers satisfying $n+m\geq 10$. If

$f^n(z)(f(z)-1)^m f(z+c)$ and $g^n(z)(g(z)-1)^m g(z+c)$ share "$(\alpha, 2)$", then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m w_1(z + c) - w_2^n(w_2 - 1)^m w_2(z + c).$$

THEOREM E. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0, \infty)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant, n and m (≥ 2) are integers satisfying $n + m \geq 13$. If $f^n(z)(f(z)-1)^m f(z+c)$ and $g^n(z)(g(z)-1)^m g(z+c)$ share $(\alpha, 2)^*$, then the conclusions of Theorem D hold.

THEOREM F. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0, \infty)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant, n and m (≥ 2) are integers satisfying $n + m \geq 19$. If $\overline{E}_{2)}(\alpha(z), f^n(z)(f(z) - 1)^m f(z + c)) = \overline{E}_{2)}(\alpha(z), g^n(z)(g(z) - 1)^m g(z + c))$, then the conclusions of Theorem D hold.

Recently, P. Sahoo and H. Karmakar [12] extended the above theorems and proved the following results.

THEOREM G. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)$ be a small function of both f and g. Suppose that c is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and k (≥ 0) are integers satisfying $n \geq 2k + m + 6$ when $m \leq k+1$ and $n \geq 4k - m + 10$ when $m > k + 1$. If $(f^n(z)(f(z) - 1)^m f(z + c))^{(k)}$ and $(g^n(z)(g(z) - 1)^m g(z + c))^{(k)}$ share "$(\alpha, 2)$", then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m w_1(z + c) - w_2^n(w_2 - 1)^m w_2(z + c).$$

THEOREM H. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)$ be a small function of both f and g. Suppose that c is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and k (≥ 0) are integers satisfying $n \geq 3k + 2m + 8$ when $m \leq k+1$ and $n \geq 6k - m + 13$ when $m > k + 1$. If $(f^n(z)(f(z) - 1)^m f(z + c))^{(k)}$ and $(g^n(z)(g(z) - 1)^m g(z + c))^{(k)}$ share $(\alpha, 2)^*$, then the conclusions of Theorem G hold.

THEOREM I. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)$ be a small function of both f and g. Suppose that c is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and k (≥ 0) are integers satisfying $n \geq 5k + 4m + 12$ when $m \leq k+1$ and $n \geq 10k - m + 19$ when $m > k + 1$. If $\overline{E}_{2)}(\alpha(z), (f^n(z)(f(z) - 1)^m f(z + c))^{(k)}) = \overline{E}_{2)}(\alpha(z), (g^n(z)(g(z) - 1)^m g(z + c))^{(k)})$, then the conclusions of Theorem G hold.

In this paper, we assume $c_j \in \mathbb{C}\backslash\{0\}$ $(j = 1, 2, ..., d)$ are constants, $n(\geq 1)$, $m(\geq 1)$ and k (≥ 0) are integers, $s_j(j = 1, 2, ..., d)$ are non-negative integers, $\lambda = \sum_{j=1}^{d} s_j =$

$s_1 + s_2 + \cdots + s_d$. With these assumptions, we study the uniqueness problems of difference polynomials sharing a small function of more general form

$$(f(z)^n(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}$$

and hence obtain the following theorems which extends and generalises the results obtained by P. Sahoo and H. Karmakar [12].

THEOREM 1. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)$ be a small function of both f and g. Let c_j $(j = 1, 2, ..., d)$ be complex constants, s_j $(j = 1, 2, ..., d)$ be non-negative integers. Suppose $n\,(\geq 1)$, $m\,(\geq 1)$ and k (≥ 0) are integers satisfying $n \geq 2k+m+\lambda+5$ when $m \leq k+1$ and $n \geq 4k-m+\lambda+9$ when $m > k+1$. If

$$(f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)} \text{ and } (g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)}$$

share "$(\alpha, 2)$", then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{d} w_1(z+c_j)^{s_j} - w_2^n(w_2-1)^m \prod_{j=1}^{d} w_2(z+c_j)^{s_j}.$$

THEOREM 2. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)$ be a small function of both f and g. Let c_j $(j = 1, 2, ..., d)$ be complex constants, $s_j(j = 1, 2, ..., d)$ be non-negative integers. Suppose $n\,(\geq 1)$, $m\,(\geq 1)$ and k (≥ 0) are integers satisfying $n \geq 3k+2m+2\lambda+6$ when $m \leq k+1$ and $n \geq 6k-m+2\lambda+11$ when $m > k+1$. If

$$(f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)} \text{ and } (g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)}$$

share $(\alpha, 2)^*$, then the conclusions of Theorem 1 hold.

THEOREM 3. Let f and g be two transcendental entire functions of finite order and $\alpha(\not\equiv 0)$ be a small function of both f and g. Let c_j $(j = 1, 2, ..., d)$ be complex constants, $s_j(j = 1, 2, ..., d)$ be non-negative integers. Suppose $n\,(\geq 1)$, $m\,(\geq 1)$ and k (≥ 0) are integers satisfying $n \geq 5k + 4m + 4\lambda + 8$ when $m \leq k + 1$ and $n \geq 10k - m + 4\lambda + 15$ when $m > k + 1$. If

$$\begin{aligned} & \overline{E}_{2)}(\alpha(z), (f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) \\ = \quad & \overline{E}_{2)}(\alpha(z), (g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)}), \end{aligned}$$

then the conclusions of Theorem 1 hold.

REMARK 1. For $j = 1, 2, ..., d$, if ($s_j = 0$ for $j \neq 1$) and ($c_j = c,\ s_j = 1$ for $j = 1$) (i.e., $\lambda = 1$) in Theorems $1 - 3$, we obtain Theorems $G - I$ respectively.

REMARK 2. For $j = 1, 2, ..., d$, if ($s_j = 0$ for $j \neq 1$) and ($c_j = c,\ s_j = 1$ for $j = 1$) (i.e., $\lambda = 1$) also $k = 0$ in Theorems $1 - 3$, we obtain Theorems $D - F$ respectively.

REMARK 3. For $j = 1, 2, ..., d$, if ($s_j = 0$ for $j \neq 1$) and ($c_j = c,\ s_j = 1$ for $j = 1$) (i.e., $\lambda = 1$) also $m = 1$, $k = 0$ in Theorems $1 - 3$, we obtain Theorems $A - C$ respectively.

2 Preliminary Lemmas

In this section, we present some necessary lemmas. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are non-constant meromorphic functions defined in the complex plane.

LEMMA 1 ([15]). Let f be a non-constant meromorphic function and p, k be positive integers. Then

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{1}$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2}$$

LEMMA 2 ([2]). Let f be meromorphic function of order $\rho(f) < \infty$, and let c be a non-zero complex constant. Then, for each $\varepsilon > 0$, we have

$$T(r, f(z + c)) = T(r, f) + O\{r^{\rho(f)-1+\varepsilon}\} + O\{logr\}.$$

LEMMA 3 ([3]). Let f be meromorphic function of finite order and c be a non-zero complex constant. Then,

$$m\left(r, \frac{f(z + c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z + c)}\right) = O\{r^{\rho(f)-1+\varepsilon}\}.$$

LEMMA 4. Let f be an entire function of order $\rho(f) < \infty$ and $F(z) = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}$ where $n\,(\geq 1)$, $m\,(\geq 1)$ and $k\,(\geq 0)$ are integers. Then,

$$T(r, F) = (n + m + \lambda)T(r, f) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r, f),$$

for all r outside of a set of finite linear measure where $\lambda = s_1 + s_2 + ... + s_d = \sum_{j=1}^{d} s_j$.

PROOF. Since f is an entire function of finite order, from Lemma 3 and standard Valiron-Mohon'ko theorem [13], we have

$$\begin{aligned}(n+m+\lambda)T(r,f(z)) &= T(r,f^{n+\lambda}(z)(f(z)-1)^m) + S(r,f)\\ &= m\left(r,f^{n+\lambda}(z)(f(z)-1)^m\right) + S(r,f)\\ &\le m\left(r,\frac{f^{n+\lambda}(z)(f(z)-1)^m}{F(z)}\right) + m(r,F(z)) + S(r,f)\\ &\le m\left(r,\frac{f^{\lambda}(z)}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + m(r,F(z)) + S(r,f)\\ &\le T(r,F(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r,f). \qquad (3)\end{aligned}$$

On the other hand, from Lemma 2, we have

$$\begin{aligned}T(r,F(z)) &\le m(r,f^n(z)) + m(r,(f(z)-1)^m) + m\left(r,f^{\lambda}(z)\cdot\prod_{j=1}^{d}\frac{f(z+c_j)^{s_j}}{f(z)^{s_j}}\right) + S(r,f)\\ &\le (n+m)\,m(r,f(z)) + \lambda m(r,f(z)) + \sum_{j=1}^{d} s_j\cdot m\left(r,\frac{f(z+c_j)}{f(z)}\right) + S(r,f)\\ &\le (n+m+\lambda)\,m(r,f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r,f)\\ &\le (n+m+\lambda)\,T(r,f(z)) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r,f). \qquad (4)\end{aligned}$$

From (3) and (4), we can prove this lemma easily.

LEMMA 5. Let f and g be entire functions, $n(\ge 1)$, $m(\ge 1)$ and $k(\ge 0)$ be integers and let

$$F(z) = \left(f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}$$

and

$$G(z) = \left(g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}.$$

If there exists non-zero constants b_1 and b_2 such that $\overline{N}(r,b_1;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,b_2;G) = \overline{N}(r,0;F)$, then $n \le 2k+m+\lambda+2$ when $m \le k+1$ and $n \le 4k-m+\lambda+4$ when $m > k+1$.

PROOF. Let $F_1(z) = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $G_1(z) = g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{s_j}$. From Lemma 4, we have

$$T(r,F_1) = (n+m+\lambda)\,T(r,f) + O\{r^{\rho(f)-1+\varepsilon}\} + S(r,f), \tag{5}$$
$$T(r,G_1) = (n+m+\lambda)\,T(r,g) + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,g). \tag{6}$$

By second fundamental theorem and by the hypothesis, we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F) \\ &\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F). \end{aligned} \tag{7}$$

Using (1), (2), (5) and (7), we have

$$\begin{aligned} (n+m+\lambda)\,T(r,f) &\le T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\le \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\le N_{k+1}(r,0;F_1) + \overline{N}_{k+1}(r,0;G_1) + S(r,f) + S(r,g). \end{aligned} \tag{8}$$

When $m \le k+1$, using (8) and Lemma 2, we see that

$$\begin{aligned} (n+m+\lambda)\,T(r,f) &\le (k+m+\lambda+1)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned} (n+m+\lambda)\,T(r,g) &\le (k+m+\lambda+1)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \end{aligned} \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} (n-2k-m-\lambda-2)\,(T(r,f)+T(r,g)) &\le O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} \\ &\quad + S(r,f) + S(r,g), \end{aligned}$$

which gives $n \le 2k+m+\lambda+2$. When $m > k+1$, using (8) and Lemma 2, we have

$$\begin{aligned} (n+m+\lambda)\,T(r,f) &\le (2k+\lambda+2)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} (n+m+\lambda)\,T(r,g) &\le (2k+\lambda+2)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\} \\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \end{aligned} \tag{12}$$

From (11) and (12), we have

$$(n-4k+m-\lambda-4)\,(T(r,f)+T(r,g)) \le O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f)+S(r,g),$$

which gives $n \le 4k-m+\lambda+4$. This proves the lemma.

LEMMA 6 ([1]). Let F and G be non-constant meromorphic functions that share "(1,2)" and $H \not\equiv 0$. Then

$$T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)-\sum_{p=3}^{\infty}\overline{N}\left(r,0;\frac{G'}{G}\mid\ge p\right) + S(r,F)+S(r,G)$$

and the same inequality holds for $T(r,G)$.

LEMMA 7 ([1]). Let F and G be non-constant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then

$$T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\overline{N}(r,0;F) + \overline{N}(r,\infty;F)-m(r,1;G)+S(r,F)+S(r,G)$$

and the same inequality holds for $T(r,G)$.

LEMMA 8 ([9]). Let F and G be non-constant entire functions and $p \ge 2$ be an integer. If $\overline{E}_{p)}(1,F)=\overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then

$$T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+2\,\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G),$$

and the same inequality holds for $T(r,G)$.

3 Proofs of the Theorems

PROOF OF THEOREM 1. Let $F=\dfrac{F_1^{(k)}}{\alpha}$ and $G=\dfrac{G_1^{(k)}}{\alpha}$ where

$$F_1(z)=f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j} \text{ and } G_1(z)=g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j}.$$

Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$. Suppose that $H \not\equiv 0$. Using (1), (5) and Lemma 4, we have

$$\begin{aligned} N_2(r,0;F) &\le N_2(r,0;F_1^{(k)})+S(r,f)\\ &\le T(r,F_1^{(k)})-(n+m+\lambda)T(r,f)+N_{k+2}(r,0;F_1)+S(r,f)\\ &\le T(r,F)-(n+m+\lambda)T(r,f)+N_{k+2}(r,0;F_1)+S(r,f). \end{aligned}$$

From this, we get

$$(n+m+\lambda)T(r,f) \le T(r,F) - N_2(r,0;F) + N_{k+2}(r,0;F_1) + S(r,f). \tag{13}$$

Also by (2), we obtain

$$N_2(r,0;F) \le N_2(r,0;F_1^{(k)}) + S(r,f) \le N_{k+2}(r,0;F_1) + S(r,f).$$

Similarly,

$$N_2(r,0;G) \le N_{k+2}(r,0;G_1) + S(r,g). \tag{14}$$

Using (14) and Lemma 6 in (13), we have

$$\begin{aligned}(n+m+\lambda)\,T(r,f) &\le N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1)\\ &\quad + S(r,f) + S(r,g)\\ &\le N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g). \end{aligned}\tag{15}$$

Suppose that $m \le k+1$, then from (15), we have

$$\begin{aligned}(n+m+\lambda)\,T(r,f) &\le (k+m+\lambda+2)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\}\\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g).\end{aligned}\tag{16}$$

Similarly,

$$\begin{aligned}(n+m+\lambda)\,T(r,g) &\le (k+m+\lambda+2)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\}\\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g).\end{aligned}\tag{17}$$

From (16) and (17), we have

$$(n-2k-m-\lambda-4)\,(T(r,f)+T(r,g)) \le O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g),$$

which contradicts the assumption that $n \ge 2k+m+\lambda+5$. Next, assume that $m > k+1$. From (15), we have

$$\begin{aligned}(n+m+\lambda)\,T(r,f) &\le (2k+\lambda+4)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\}\\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g).\end{aligned}\tag{18}$$

Similarly,

$$\begin{aligned}(n+m+\lambda)\,T(r,g) &\le (2k+\lambda+4)\,(T(r,f)+T(r,g)) + O\{r^{\rho(f)-1+\varepsilon}\}\\ &\quad + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g).\end{aligned}\tag{19}$$

From (18) and (19), we have

$$(n+m-4k-\lambda-8)\,(T(r,f)+T(r,g)) \le O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g),$$

a contradiction, since $n \ge 4k-m+\lambda+9$. Therefore, we have $H=0$. It implies that

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating twice, we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{20}$$

From (20), F and G share 1 CM and hence they share "(1, 2)". Therefore $n \geq 2k+m+\lambda+5$ if $m \leq k+1$ and $n \geq 4k-m+\lambda+9$ if $m > k+1$.

Next, we discuss the following three cases.

Case 1. Suppose that $B \neq 0$ and $A = B$. Then from (20), we have

$$\frac{1}{F-1}=\frac{BG}{G-1}. \tag{21}$$

If $B=-1$, then from (21), we have $FG=1$. Then

$$(f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}\cdot(g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)}=\alpha^2.$$

It follows that $N(r,0;f)=S(r,f)$ and $N(r,1;f)=S(r,f)$. Thus, we have

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3,$$

which is not possible. If $B \neq -1$, then from (21), we have $\frac{1}{F}=\frac{BG}{(1+B)G-1}$. So $\overline{N}\left(r,\frac{1}{1+B};G\right)=\overline{N}(r,0;F)$. Using (1), (2), (6) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{1+B};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G)\\ &\leq N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)\\ &-(n+m+\lambda)T(r,g)+S(r,g). \end{aligned} \tag{22}$$

If $m \leq k+1$, then from (22) we have

$$\begin{aligned} (n+m+\lambda)\,T(r,g) \quad &\leq \quad (k+m+\lambda+1)\,(T(r,f)+T(r,g))\\ &+O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g). \end{aligned}$$

Hence,

$$\begin{aligned} (n-2k-m-\lambda-2)\,(T(r,f)+T(r,g)) \quad &\leq \quad O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}\\ &+S(r,f)+S(r,g), \end{aligned}$$

a contradiction since $n \geq 2k+m+\lambda+5$. If $m > k+1$, from (22), we have

$$\begin{aligned} (n+m+\lambda)T(r,g) \quad &\leq \quad (2k+\lambda+2)\,(T(r,f)+T(r,g))+O\{r^{\rho(f)-1+\varepsilon}\}\\ &+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g). \end{aligned}$$

Hence,

$$(n-4k+m-\lambda-4)\,(T(r,f)+T(r,g)) \leq O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\} +S(r,f)+S(r,g),$$

which is a contradiction since $n \geq 4k-m+\lambda+9$.

Case 2. Let $B \neq 0$ and $A \neq B$. From (20), we have

$$F=\frac{(B+1)G-(B-A+1)}{BG+(A-B)}$$

and hence

$$\overline{N}\left(r,\frac{B-A+1}{B+1};G\right)=\overline{N}(r,0;F).$$

Proceeding as in case 1, we get a contradiction.

Case 3. Let $B=0$ and $A \neq 0$. From (20), we have $F=\dfrac{G+A-1}{A}$ and $G=AF-(A-1)$. If $A \neq 1$, then it follows that

$$\overline{N}\left(r,\frac{A-1}{A};F\right)=\overline{N}(r,0;G) \text{ and } \overline{N}(r,1-A;G)=\overline{N}(r,0;F).$$

By applying Lemma 5, we arrive at a contradiction. Therefore $A=1$ and hence $F=G$. It implies that

$$(f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j})^{(k)}=(g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j})^{(k)}.$$

By integration, we obtain

$$(f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j})^{(k-1)}=(g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j})^{(k-1)}+c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, then by Lemma 5, we get $n \leq 2k+m+\lambda$ when $m \leq k+1$ and $n \leq 4k-m+\lambda$ when $m > k+1$, which contradicts the hypothesis. Hence, $c_{k-1}=0$. Repeating the same process $k-1$ times, we get

$$f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{s_j}=g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{s_j} \tag{23}$$

Set $h=f/g$. If h is a constant, then substituting $f=gh$ in (23), we have

$$g^n\prod_{j=1}^{d}g(z+c_j)^{s_j}[g^m(h^{n+m+\lambda}-1)-mC_1g^{m-1}(h^{n+m+\lambda-1}-1)+ \cdots+(-1)^m(h^{n+\lambda}-1)]=0. \tag{24}$$

Since g is a transcendental entire function, we have $g^n \prod_{j=1}^{d} g(z+c_j)^{s_j} \not\equiv 0$. Hence, from (24), we get

$$g^m(h^{n+m+\lambda}-1) - mC_1 g^{m-1}(h^{n+m+\lambda-1}-1) + \cdots + (-1)^m(h^{n+\lambda}-1) = 0,$$

which implies $h = 1$ and hence $f = g$. If h is not constant, then from (23), we find that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(f,g)$ is given by

$$R(w_1,w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{d} w_1(z+c_j)^{s_j} - w_2^n(w_2-1)^m \prod_{j=1}^{d} w_2(z+c_j)^{s_j}.$$

Hence the proof of Theorem 1.

PROOF OF THEOREM 2. Let F, G, $F_1(z)$ and $G_1(z)$ be defined as in Theorem 1. Then, F and G are transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$. By using (2) for $p = 1$, (14) and Lemma 7 in (13), we get

$$\begin{aligned}
&(n+m+\lambda)\,T(r,f)\\
&\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G)\\
&+ \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g)\\
&\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + N_{k+1}(r,0;F_1) + S(r,f) + S(r,g) \qquad (25)
\end{aligned}$$

If $m \leq k+1$, then from (25), we obtain

$$\begin{aligned}
&(n+m+\lambda)\,T(r,f)\\
&\leq (2k+2m+2\lambda+3)\,T(r,f) + (k+m+\lambda+2)T(r,g) + O\{r^{\rho(f)-1+\varepsilon}\}\\
&+ O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \qquad (26)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(n+m+\lambda)\,T(r,g)\\
&\leq (2k+2m+2\lambda+3)\,T(r,g) + (k+m+\lambda+2)T(r,f) + O\{r^{\rho(f)-1+\varepsilon}\}\\
&+ O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \qquad (27)
\end{aligned}$$

From (26) and(27), we get

$$(n-3k-2m-2\lambda-5)\,(T(r,f)+T(r,g)) \leq O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g),$$

contradicting the fact that $n \geq 3k+2m+2\lambda+6$. If $m > k+1$, then from (25), we obtain

$$\begin{aligned}
(n+m+\lambda)\,T(r,f) &\leq (4k+2\lambda+6)T(r,f) + (2k+\lambda+4)T(r,g) + O\{r^{\rho(f)-1+\varepsilon}\}\\
&+ O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \qquad (28)
\end{aligned}$$

Similarly,

$$(n+m+\lambda)\,T(r,g) \leq (4k+2\lambda+6)T(r,g)+(2k+\lambda+4)T(r,f)+O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g). \tag{29}$$

From (28) and (29), we get

$$(n-6k+m-2\lambda-10)\,(T(r,f)+T(r,g)) \leq O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g),$$

contradicting the fact that $n \geq 6k-m+2\lambda+11$. Thus, $H \equiv 0$ and the rest of the theorem follows from the proof of Theorem 1. Hence the proof of Theorem 2.

PROOF OF THEOREM 3. Let $F,\ G,\ F_1(z)$ and $G_1(z)$ be defined as in Theorem 1. Then, F and G are transcendental meromorphic functions such that $\overline{E}_{2)}(1,F) = \overline{E}_{2)}(1,G)$ except the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$. Then, by (2), (14) and Lemma 8 in (13), we get

$$\begin{aligned}
&(n+m+\lambda)\,T(r,f)\\
&\leq N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;F_1)+S(r,f)+S(r,g)\\
&\leq N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2\,N_{k+1}(r,0;F_1)\\
&+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g).
\end{aligned} \tag{30}$$

If $m \leq k+1$, then from (30), we obtain

$$\begin{aligned}
&(n+m+\lambda)\,T(r,f)\\
&\leq (3k+3m+3\lambda+4)\,T(r,f)+(2k+2m+2\lambda+3)T(r,g)+O\{r^{\rho(f)-1+\varepsilon}\}\\
&+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g).
\end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned}
&(n+m+\lambda)\,T(r,g)\\
&\leq (3k+3m+3\lambda+4)\,T(r,g)+(2k+2m+2\lambda+3)T(r,f)+O\{r^{\rho(f)-1+\varepsilon}\}\\
&+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g).
\end{aligned} \tag{32}$$

From (31) and (32), we get

$$(n-5k-4m-4\lambda-7)\,(T(r,f)+T(r,g)) \leq O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g),$$

contradicting the fact that $n \geq 5k+4m+4\lambda+8$. If $m > k+1$, then from (30), we obtain

$$(n+m+\lambda)T(r,f) \leq (6k+3\lambda+8)\,T(r,f)+(4k+2\lambda+6)T(r,g)+O\{r^{\rho(f)-1+\varepsilon}\} + O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g). \tag{33}$$

Similarly,

$$(n+m+\lambda)T(r,g) \leq (6k+3\lambda+8)\,T(r,g) + (4k+2\lambda+6)T(r,f) + O\{r^{\rho(f)-1+\varepsilon}\} \\ + O\{r^{\rho(g)-1+\varepsilon}\} + S(r,f) + S(r,g). \quad (34)$$

From (33) and (34), we get

$$(n-10k+m-4\lambda-14)\,(T(r,f)+T(r,g)) \leq O\{r^{\rho(f)-1+\varepsilon}\}+O\{r^{\rho(g)-1+\varepsilon}\}+S(r,f)+S(r,g),$$

contradicting the fact that $n \geq 10k - m + 4\lambda + 15$. Thus $H \equiv 0$ and the rest of the theorem follows from the proof of Theorem 1. Hence the proof of Theorem 3.

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