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Some Characterization Results Based On Expected Values Of Generalized Order Statistics*

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Abstract

In this paper, we have characterized continuous probability distributions by considering the conditional expectations of functions of generalized order statistics conditioned on non-adjacent generalized order statistics. Further, some important deductions for order statistics and record values are discussed.

1 Introduction

The concept of generalized order statistics (*gos*) has been introduced and extensively studied by Kamps [10]. Let $n \geq 2$, be a given integer and $\widetilde{m} = (m_1, m_2, ..., m_{n-1}) \in \Re^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=1}^{n-1} m_i \ \text{ for } \ 1 \leq i \leq n-1.$$

Then $X(1, n, \widetilde{m}, k)$, $X(2, n, \widetilde{m}, k)$, ..., $X(r, n, \widetilde{m}, k)$ are called *gos* from continuous population with the cummulative distribution function (cdf) $F(x)$ and the probability density function (pdf) $f(x)$ if their joint pdf has the form

$$k\left(\prod_{j=1}^{n=1} \gamma_j\right)\left(\prod_{i=1}^{n=1}[1-F(x_i)]^{m_i} f(x_i)\right)[1-F(x_n)]^{k-1} f(x_n),$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1)$ of $\Re^n$. The *pdf* of r-th m-*gos* $X(r, n, m, k)$ is given by Kamps [10],

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\overline{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)] f(x),$$

and the joint *pdf* of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \leq r < s \leq n$, is given by Kamps [10],

$$f_{X(r,n,m,k),X(s,n,m,k)}(x, y)$$

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$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}[F(x)]$$
$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(x)f(y),\ \alpha \le x < y \le \beta,$$

where

$$\bar{F}(x) = 1 - F(x),\ \ \gamma_i = k + (n-i) + (m+1),\ \ C_{s-1} = \prod_{i=1}^{s}\gamma_i,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & \text{for } m \ne -1, \\ -\log(1-x) & \text{for } m = -1. \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) \ \text{ for } x \in [0,1).$$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, is given by

$$f_{x(r,n,m,k)|x(s,n,m,k)}(y \mid x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}$$
$$\times\frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)}{[1-F(x)]^{\gamma_{r+1}}}.$$

The *gos* is reduced to some well known ordered random schemes like order statistics, record values, sequential order statistics etc. with the proper choice of parameters of *gos* (Cf. Kamps [10]). The conditional moments of generalized order statistics are extensively used in characterizing the probability distributions. Various approaches are available in the literature. For a detailed survey one may refer to Khan and Alzaid [2], Khan et al. [6], Beg and Ahsanullah [9], Haque et al. [11] and Khan et al. [5, 6], Noor et al. [12] amongst others.

In this paper, we consider continuous probability distributions

$$\overline{F}(x) = [1-(m-1)x^a]^{\frac{1}{m+1}} \ \text{ and } \overline{F}(x) = [1-(m-1)e^{ax}]^{\frac{1}{m+1}},$$

and characterize these continuous distributions through conditional expectation of *gos*. Throughout the paper, we assume

$$m_1 = m_2, ...,\ m_{n-1} = m.$$

2 Characterization of Distributions

THEOREM 2.1. Let $X(r,n,m,k)$, $r = 1,2,...,n$ be the r^{th}-*gos* from a continuous with the *df* $F(x)$ and the *pdf* $f(x)$. Then, for $1 \le r < s \le n$,

$$E[X^\alpha(r,n,m,k) \mid X(s,n,m,k) = x] = a_{s|l}x^\alpha + b_{s|l}, \quad l = r, r+l,$$

if and only if

$$\overline{F}(x) = [1-(m-1)x^a]^{\frac{1}{m+1}}, \quad \alpha > 0, \quad 0 \le x \le \beta,$$

where

$$\beta = (\frac{1}{m+1})^{1/\alpha}, \;\; a^{\star}_{s|r} = \frac{\gamma_s}{\gamma_r} \;\; \text{and} \;\; b^{\star}_{s|r} = \frac{1}{m+1}[1 - a^{\star}_{s|r}].$$

PROOF. We have

$$E[X^{\alpha}(r,n,m,k)|\; X(s,n,m,k) = x]$$
$$= \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$
$$\times \int_x^{\beta} y^{\alpha} \left[1 - \left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1} \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1} \frac{f(x)}{\overline{F}(x)} dy. \tag{1}$$

Set $u = \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{m+1}$. Then the RHS of (1) reduces to

$$E[X^{\alpha}(r,n,m,k)|\; X(s,n,m,k) = x] = \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^(s-r-1)}$$
$$\times \int_0^1 [1 - u\,(1-(m+1)x^{\alpha})](1-u)^{s-r-1}[u]^{\frac{k+(m+1)(n-s)}{m+1}-1} du.$$

Thus

$$E[X^{\alpha}(r,n,m,k) \mid X(s,n,m,k) = x] = \frac{1}{m+1} - \frac{1}{m+1}[1-(m+1)x^{\alpha}]\frac{\gamma_s}{\gamma_r}, \tag{2}$$

and hence the necessary part. To prove the sufficiency part, we have from Khan et al. [6],

$$\text{if} \;\; E[X^{\alpha}(r,n,m,k) \mid X(s,n,m,k) = x] = g_{s|r}(x),$$

then

$$\frac{f(x)}{\overline{F}(x)} = -\frac{1}{\gamma_{r+1}} \frac{g'_{s|r}(x)}{[g_{s|r+1}(x) - g_{s|r}(x)]}.$$

Now,

$$g_{s|r+1}(x) - g_{s|r}(x) = (a_{s|r+1} - a_{s|r})\left(x^{\alpha} - \frac{1}{m+1}\right).$$

Thus,

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha x^{\alpha-1}}{[(m+1)x^{\alpha} - 1]}$$

implies

$$\overline{F}(x) = [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}}, \; \alpha > 0 \text{ and } 0 \le x \le \beta,$$

and hence the proof is complete.

REMARK 2.1. When $m = 0$ and $k = 1$, Theorem 2.1 reduces for order statistics: For $1 \le r < s \le n$,

$$E[X^{\alpha}_{s:n} \mid X_{r:n} = x] = a^{\star}_{s|r} x^{\alpha} + b^{\star}_{s|r},$$

if and only if

$$F(x) = x^{\alpha} \quad \text{for } 0 < x < 1 \text{ and } \ \alpha > 0,$$

where

$$a^{\star}_{s|r} = \left(\frac{n-s-1}{n-r-1}\right) \quad \text{and} \quad b^{\star}_{s|r} = \left(\frac{s-r}{n-r-1}\right)$$

as obtained by Khan and Abu-Salih [4], Franco and Ruiz [9], Dembinska and Wesolowski [1], Khan and Abouammoh [3] and Khan and Alzaid [2].

REMARK 2.2. When $m = -1,\ k = 1$, Theorem 2.1 reduces for order statistics: For $1 \le r < s \le n$,

$$E[X^{\alpha}_{U(s)} \mid X_{U(r)} = x] = x^{\alpha} \quad \text{with } a^{\star}_{s|r} = 1 \ \text{ and } b^{\star}_{s|r} = 0,$$

if and only if

$$F(x) = e^{-x^{\alpha}} \quad \text{for } 0 < x < \infty \ \text{ and } \ \alpha > 0,$$

as obtained by Franco and Ruiz [9] and Athar et al. [8].

THEOREM 2.2**.** Let $X(r,n,m,k)$, $r = 1, 2, ..., n$, be the r^{th}-*gos* from a continuous population with the *df* $F(x)$ and the $f(x)$, then, for $1 \le r < s \le t \le n$,

$$\begin{aligned} &E[X^{\alpha}(t,n,m,k)|\ X(r,n,m,k) = x] \\ &\quad = a^{\star}_{t|s} E[X^{\alpha}(s,n,m,k) \mid X(r,n,m,k) = x] + b^{\star}_{t|s}, \end{aligned} \tag{3}$$

if and only if

$$\overline{F}(x) = [1-(m+1)x^{\alpha}]^{\frac{1}{m+1}} \quad \text{for } \alpha > 0 \text{ and } 0 \le x \le \beta, \tag{4}$$

where

$$\beta = (\frac{1}{m+1})^{1/\alpha}, \ \ a^{\star}_{t|s} = \frac{\gamma_t}{\gamma_s}, \ \text{ and } \ b^{\star}_{t|s} = \frac{1}{m+1}[1 - a^{\star}_{t|s}].$$

PROOF. It is easy to see that (4) implies (3) and hence the necessary part. For the sufficiency part, we have

$$\begin{aligned} &\frac{C_{t-1}}{C_{r-1}(t-r-1)!(m+1)^{t-r-1}} \\ &\times \int_x^{\beta} \frac{1}{[\overline{F}(y)]^{\gamma_{r+1}}} y^{\alpha}[(\overline{F}(x))^{m+1} - (\overline{F}(x))^{m+1}]^{t-r-1}[\overline{F}(y)]^{\gamma_t - 1} f(x)dy \\ &= \alpha_{t|s} \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}} \\ &\times \int_x^{\beta} \frac{1}{[\overline{F}(y)]^{\gamma_r+1}} y^{\alpha}[(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}]^{s-r-1}[\overline{F}(y)]^{\gamma_s - 1} f(x)dy + b_{t|s}. \end{aligned} \tag{5}$$

Differentiating both the sides of (5) $(s-r)$ times *w.r.t.* x, we get

$$\frac{C_{t-1}}{C_{s-1}(t-r-1)!(m+1)^{t-s-1}}$$

$$
\begin{aligned}
&\times \int_x^\beta \frac{1}{[\overline{F}(y)]^{\gamma_{s+1}}} y^\alpha [(\overline{F}(x))^{m+1} - (\overline{F}(x))^{m+1}]^{t-r-1} [\overline{F}(y)]^{\gamma_t - 1} f(y) dy \\
= \quad & a_{t|s} x^\alpha + b_{t|s}
\end{aligned}
$$

i.e.

$$g_{t|s}(x) = a_{t|s} x^\alpha + b_{t|s}.$$

Using the result in Khan et al. [6], we get

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha x^{\alpha-1}}{[(m+1)x^\alpha - 1]}$$

and hence the proof is complete.

REMARK 2.3**.** When $s = r$, Theorem 2.2 reduces to Theorem 2.1.

THEOREM 2.3. Under the conditions given in Theorem 2.1 and, for $1 \le r < s \le n$,

$$E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k) = x] = a_{s|l}\ e^{\alpha x} + b_{s|l} \quad \text{for } l = r \text{ and } r+1,$$

if and only if

$$\overline{F}(x) = [1 - (m+1)e^{\alpha x}]^{\frac{1}{m+1}} \quad \text{for} \quad -\infty < x \le \ln\beta, \tag{6}$$

where

$$\beta = (\frac{1}{m+1})^{1/\alpha}, \quad a^\star_{s|r} = \frac{\gamma_s}{\gamma_r} \quad \text{and} \quad b^\star_{s|r} = \frac{1}{m+1}[1 - a^\star_{s|r}].$$

PROOF. We have

$$
\begin{aligned}
&E[e^{\alpha X(r,n,m,k)} | \, X(s,n,m,k) = x] \\
&\qquad = \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{(s-r-1)}} \\
&\qquad \times \int_x^{\ln\beta} e^{\alpha y} \left[1 - \left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{m+1}\right]^{s-r-1} \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k+(m+1)(n-s)-1} \frac{f(x)}{\overline{F}(x)} dy.
\end{aligned} \tag{7}
$$

Set $u = \left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{m+1}$. Then the RHS of (7) reduces to

$$
\begin{aligned}
&E[e^{\alpha X(r,n,m,k)} | \, X(s,n,m,k) = x] \\
&\qquad = \frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^s - r - 1} \\
&\qquad \times \int_0^1 [1 - u\,(1 - (m+1)e^{\alpha x})](1-u)^{s-r-1} [u]^{\frac{k+(m+1)(n-s)}{m+1} - 1} du.
\end{aligned}
$$

Then

$$E[e^{\alpha X(r,n,m,k)} \mid X(s,n,m,k) = x] = \frac{1}{(m+1)} - \frac{1}{(m+1)}[1 - (m+1)e^{\alpha x}]\frac{\gamma_s}{\gamma_r}.$$

and hence the proof of necessity is complete. For the sufficiency part we use the result in Khan et al. [6] to get,

$$\frac{f(x)}{\overline{F}(x)} = \frac{\alpha e^{\alpha x}}{[(m+1)e^{\alpha x} - 1]},$$

which implies

$$\overline{F}(x) = [1 - (m+1)e^{\alpha x}]^{\frac{1}{m+1}} \quad \text{for } -\infty < x \leq \ln \beta.$$

REMARK 2.4. When $m = 0,\ k = 1$, Theorem 2.3 reduces for order statistics: For $1 \leq r < s \leq n$,

$$E[e^{\alpha X_{s:n}} \mid X_{r:n} = x] = a^{\star}_{s|r} e^{\alpha x} + b^{\star}_{s|r}.$$

if and only if

$$F(x) = e^{\alpha x} \quad \text{for } -\infty < x < 0 \text{ and } \alpha > 0,$$

where

$$a^{\star}_{s|r} = \left(\frac{n-s+1}{n-r+1}\right) \quad \text{and} \quad b^{\star}_{s|r} = \left(\frac{s-r}{n-r+1}\right),$$

as obtained by Franco and Ruiz [9].

REMARK 2.5. When $m = -1$ and $k = 1$, Theorem 2.3 reduces for record statistics: For $1 \leq r < s$,

$$E[e^{\alpha X_{U(s)}} \mid X_{U(r)} = x] = e^{\alpha x} \quad \text{for } a^{\star}_{s|r} = 1 \text{ and } b^{\star}_{s|r} = 0$$

if and only if

$$\bar{F}(x) = e^{-e^{\alpha x}} \quad \text{for } -\infty < x < 0 \text{ and } \alpha > 0,$$

as obtained by Franco and Ruiz [9].

THEOREM 2.4. Under the conditions given in Theorem 2.2 and for $1 \leq r < s < t \leq n$,

$$E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k) = x] = a_{t|s} E[e^{\alpha X(s,n,m,k)} \mid X(r,n,m,k) = x] + b_{t|s},$$

if and only if

$$\bar{F}(x) = [1 - (m+1)e^{\alpha x}]^{\frac{1}{m+1}} \quad \text{for } -\infty < x \leq \ln \beta,$$

where

$$\beta = (\frac{1}{m+1})^{1/\alpha}, \ \ a^{\star}_{t|s} = \frac{\gamma_t}{\gamma_s} \ \text{ and } b^{\star}_{t|s} = \frac{1}{m+1}[1 - a^{\star}_{t|s}].$$

PROOF. The necessity is obvious. For the sufficiency part, we have

$$\frac{C_{t-1}}{C_{r-1}(t-r-1)!(m+1)^{t-r-1}}$$
$$\times \int_x^{\ln \beta} \frac{1}{[\overline{F}(y)]^{\gamma_s+1}} e^{\alpha y} [(\overline{F}(x))^{m+1} - (\overline{F}(y))^{m+1}]^{t-r-1} [\overline{F}(y)]^{\gamma_t - 1} f(y) dy$$

$$
\begin{aligned}
&= a_{t|s}\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^{\ln\beta}\frac{1}{[\overline{F}(y)]^{\gamma_s+1}}e^{\alpha y} \\
&\quad\times[(\overline{F}(x))^{m+1}-(\overline{F}(y))^{m+1}]^{s-r-1}[\overline{F}(y)]^{\gamma_s-1}f(y)dy+b_{t|s}. \qquad (8)
\end{aligned}
$$

Differentiating both sides of (8) (s-r) times *w.r.t.* x, we get

$$
\begin{aligned}
&\frac{C_{t-1}}{C_{s-1}(t-r-1)!(m+1)^{t-s-1}} \\
&\times\int_x^{\ln\beta}\frac{1}{[\overline{F}(y)]^{\gamma_s+1}}e^{\alpha y}[(\overline{F}(x))^{m+1}-(\overline{F}(y))^{m+1}]^{t-s-1}[\overline{F}(y)]^{\gamma_t-1}f(y)dy \\
&= a_{t|s}e^{\alpha x}+b_{t|s},
\end{aligned}
$$

i.e.,

$$g_{t|s}(x)=a_{t|s}e^{\alpha x}+b_{t|s}.$$

Using the result in Khan et al. [6], we get

$$\frac{f(x)}{\overline{F}(x)}=\frac{\alpha e^{\alpha x}}{[(m+1)e^{\alpha x}-1]}$$

which implies

$$\overline{F}(x)=[1-(m+1)e^{\alpha x}]^{\frac{1}{m+1}}.$$

REMARK 2.5. When $s=r$, Theorem 2.4 reduces to Theorem 2.3.

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