

# Asymptotic Attractors Of Two-dimensional Generalized Benjamin-Bona-Mahony Equations\*

Sheng-Ping Zhang<sup>†</sup>, Chao-Sheng Zhu<sup>‡</sup>

Received 13 May 2016

## Abstract

In this paper, we consider the long time behavior of solutions for two-dimensional generalized Benjamin-Bona-Mahony equations with periodic boundary conditions. By the method of orthogonal decomposition, we show that the existence of asymptotic attractor from the precision of approximate inertial manifolds. Moreover, the dimensions estimate of the asymptotic attractor is obtained.

## 1 Introduction

It is well known that the concept of an inertial manifold plays an important role in the investigation of the long-time behavior of infinite dimensional dynamical systems, see, for example, [5, 6, 8, 10]. Inertial manifold is a finite dimensional invariant manifold in the phase space  $\mathbf{H}$  of the system which attracts exponentially all orbits. It is constructed as the graph of a mapping from  $\mathbf{P}\mathbf{H}$  to  $(\mathbf{I}-\mathbf{P})\mathbf{H}$ , where  $\mathbf{P}$  is a projection of finite dimension  $\mathbf{N}$ . However, the existence usually holds under a restrictive spectral gap condition. To investigate the case when the spectral gap condition does not hold, the concepts of approximate inertial manifolds have been introduced in [7].

But the precision of approximate inertial manifolds is inextricably difficult at all times. To overcome this difficulty the concept of asymptotic attractor has been introduced [14].

Now let us recall the definition of the asymptotic attractor. We consider the solution  $u(t)$  of a differential equation

$$u_t + Au = F(u), \quad (1)$$

with initial data

$$u(0) = u_0. \quad (2)$$

The variable  $u(t)$  belongs to a linear space  $\mathbf{E}$  called the phase space, and  $F$  is a mapping of  $\mathbf{E}$  into itself. The semigroup  $S(t)_{t \geq 0}$  denotes the solution map associated to problem (1)–(2):

$$S(t) : u_0 \in \mathbf{E} \rightarrow u(t) \in \mathbf{E}. \quad (3)$$

---

\*Mathematics Subject Classifications: 20F05, 20F10, 20F55, 68Q42.

<sup>†</sup>School of Mathematics and statistics, Southwest University, Chongqing 400715, P. R. China

<sup>‡</sup>Corresponding author, School of Mathematics and statistics, Southwest University, Chongqing 400715, P. R. China

If  $B$  is a bounded absorbing set, then

$$\mathcal{A} = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s, u_0 \in B} S(t)u_0} \quad (4)$$

is a global attractor for problem (1)–(2).

DEFINITION 1.1 ([14]). Let  $\mathcal{E}$  be a finite-dimensional subspace  $\mathbf{E}$ , and let  $B$  be a bounded absorbing set in  $\mathbf{E}$ . Suppose there exists a number  $t^*(B) > 0$  such that for all  $u_0 \in B$  and all  $t > t^*(B)$ , there exists a sequence  $\{u^k(t)\}_N \subset \mathcal{E}$  such that

$$\|u^k(t) - S(t)u_0\|_{\mathbf{E}} \rightarrow 0, \quad k \rightarrow \infty. \quad (5)$$

Then the sequence of sets  $\mathcal{A}^k$  defined by

$$\mathcal{A}^k = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s, u_0 \in B} u^k(t)} \quad (6)$$

is called an **asymptotic attractor** of the problem (1)–(2).

In an asymptotic attractor  $\mathcal{A}^k$ , we know  $\|\cdot\|_{\mathbf{E}}$  is the module of phase space  $\mathbf{E}$ ,  $u^k(t)$  depends on the initial value of  $u_0$ , and  $t^*(B)$  only depends on the radius of absorbing sets. In other words,  $t^*$  is consistent with  $u_0$  of  $\mathcal{B}$ . When  $k \rightarrow +\infty$ , if the limit value exists, then the limit value is a global attractor; otherwise, there is no global attractor. we can discuss the structure of  $\mathcal{A}^k$ , since  $u^k$  is the solution of finite dimensional dynamical systems. (5) guarantees the asymptotic approximation of  $u^k(t)$  to the real solution of  $u(t)$ , and not only the approximation. In the next section, we construct a finite dimensional asymptotic solution to the generalized Benjamin-Bona-Mahony equations, and then prove the asymptotic solution to the real solution. Then we give the asymptotic attractor of the generalized Benjamin-Bon-Mahony equations.

In this paper, we will show the existence of the asymptotic attractor for the following generalized Benjamin-Bona-Mahony equations with periodic boundary conditions

$$u_t - \delta \Delta u_t - \mu \Delta u + \nabla \cdot \mathbf{F}(u) = h(x), \quad (7)$$

$$\frac{\partial^j u(x, t)}{\partial x_i^j} = \frac{\partial^j u(x + 2\pi e_i, t)}{\partial x_i^j}, \quad j = 0, 1, 2, \quad i = 1, 2. \quad (8)$$

$$u(x, 0) = u_0(x), \quad u_0(x) = u_0(x + 2\pi e_i), \quad (9)$$

$$\int_{\Omega} u(x, t) dx = 0, \quad (10)$$

where  $x = (x_1, x_2) \in \Omega$ ,  $\Omega = [0, 2\pi] \times [0, 2\pi]$ ,  $e_1 = (1, 0)$ ,  $e_2 = (0, 1)$ ,  $\delta$  and  $\mu$  are positive constants,  $\mathbf{F} = (F_1(s), F_2(s))$  is a given vector field satisfying the following properties:

(i)  $F_k(0) = 0$ ,  $k = 1, 2$ ;

(ii) the function  $F_k$ ,  $k = 1, 2$  are twice continuously differentiable in  $\mathbb{R}^1$ ;

(iii) the functions  $f_k(s) = \frac{d}{ds} F_k(s)$ ,  $k = 1, 2$ , satisfy the growth conditions

$$|f_k(s)| \leq C(1 + |s|^m), \quad k = 1, 2, \quad 0 \leq m < 2.$$

The existence and uniqueness of solutions, as well as the decay rates of solutions for this equation was studied by many authors, see, for example, [1, 2, 3]. On the other hand, the long-time behavior for this equation were considered also by many authors, see, for example, [4, 9, 11, 12, 13, 15, 16, 17].

Here, by the method of orthogonal decomposition, we show the existence of asymptotic attractor for problem (7)–(10). Furthermore, the dimensions estimate of the asymptotic attractor is obtained. Throughout this paper, we set  $\|u\|^2 = \int_{\Omega} |u|^2 dx$  and

$$\dot{H}_{per}^2(\Omega) =: \left\{ u : D^{\alpha} u \in L^2(\Omega), \quad \forall 0 \leq |\alpha| \leq 2; \right. \\ \left. \int_{\Omega} u(x, t) dx = 0; \quad u(x, t) = u(x + 2\pi e_i, t), \quad x \in \mathbb{R}^2 \right\}.$$

Applying Faedo-Galerkin method similar to [4], it is easy to prove that the problem (7)–(10) has a unique solution  $u(t) \in \dot{H}_{per}^2(\Omega)$  if  $u_0(x) \in \dot{H}_{per}^2(\Omega)$  and  $h(x) \in L^2(\Omega)$ . Moreover, there are  $t_0 > 0$  and  $\rho_0 > 0$  such that

$$B = \left\{ u(t) \in \dot{H}_{per}^2(\Omega) : \|u(x, t)\|^2 + \delta \|\nabla u(x, t)\|^2 \leq \rho_0^2, \quad t \geq t_0 \right\}$$

is a bounded absorbing set. Now we are in a position to state our main result:

**THEOREM 1.2.** If  $u_0(x) \in \dot{H}_{per}^2(\Omega)$  and  $h(x) \in L^2(\Omega)$ , the semigroup  $S(t)$  associated with problems (7)–(10) possesses an asymptotic attractor  $\mathcal{A}^k$  in  $\dot{H}_{per}^2(\Omega)$ . Moreover, the dimension of  $\mathcal{A}^k$  satisfies

$$N_{\mathcal{A}^k} = \min \left\{ N \in \mathbb{N} : \frac{(\|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m))^2}{C_2 \mu (N + 1)^2 \rho_0^2} \leq 1, \right. \\ \left. \frac{2}{C_2 \mu} \left( \frac{\sqrt{2} C_1 (1 + \rho_0^m)}{N + 1} + \frac{C_4 \rho_0}{2(N + 1)^2 \delta^{\frac{1}{2}}} \right)^2 < 1 \right\},$$

where  $C_2 = \min\{\mu(N + 1)^2, \frac{\mu}{2\delta}\}$ .

## 2 Asymptotic Attractor

In this section, we show the existence of asymptotic attractor for problem (7)–(10) by the method of the orthogonal decomposition. Let

$$\{ \cos k_1 x_1 \cos k_2 x_2, \quad \cos k_1 x_1 \sin k_2 x_2, \quad \sin k_1 x_1 \cos k_2 x_2, \\ \sin k_1 x_1 \sin k_2 x_2, \quad k_1, k_2 = 1, 2, \dots \}$$

be an orthogonal basis of  $\dot{L}_{per}^2(\Omega)$  and denote

$$H_N = \text{span} \{ \cos k_1 x_1 \cos k_2 x_2, \cos k_1 x_1 \sin k_2 x_2, \sin k_1 x_1 \cos k_2 x_2, \sin k_1 x_1 \sin k_2 x_2, k_1, k_2 = 1, 2, \dots, N \}.$$

Let  $P_N: \dot{L}_{per}^2(\Omega) \rightarrow H_N$  and  $Q_N = I - P_N$ . For any  $u(x, t) \in \dot{L}_{per}^2(\Omega)$ , we denote

$$p = P_N u \text{ and } q = Q_N u.$$

By projecting (7) on the  $H_N$ , we have

$$p_t - \delta \Delta p_t - \mu \Delta p + P_N(\nabla \cdot \mathbf{F}(u)) = P_N h \quad (11)$$

and

$$q_t - \delta \Delta q_t - \mu \Delta q + Q_N(\nabla \cdot \mathbf{F}(u)) = Q_N h. \quad (12)$$

For any  $u_0(x) \in B$ , we set  $u^k = p + q^k$  satisfying:

$$\begin{cases} q_t^0 - \delta \Delta q_t^0 - \mu \Delta q^0 + Q_N(\nabla \cdot \mathbf{F}(p)) = Q_N h, \\ q^0(x, t) = q^0(x + 2\pi e_i, t), \quad i = 1, 2, \\ q^0(x, 0) = Q_N u_0 \end{cases} \quad (13)$$

and

$$\begin{cases} q_t^k - \delta \Delta q_t^k - \mu \Delta q^k + Q_N(\nabla \cdot \mathbf{F}(u^{k-1})) = Q_N h, \\ q^k(x, t) = q^k(x + 2\pi e_i, t), \quad i = 1, 2, \\ q^k(x, 0) = Q_N^k u_0. \end{cases}$$

where  $Q_N^k = Q_N - Q_{2^{k+1}N}$ ,  $k = 1, 2, \dots$ .

Thus by (12)–(13), we can get a sequence  $\{u^k(t)\}$  for problem (7)–(10). To prove Theorem 1, it suffices to check the condition (5), that is, to prove the following Lemmas 2.2 and 2.3.

LEMMA 2.1. Under the hypotheses of  $q = Q_N u$  for  $N \in \mathbb{N}$ , we can get

$$\|\nabla q\|^2 \geq 2(N+1)^2 \|q\|^2.$$

PROOF. Here we have

$$\begin{aligned} u = & \sum_{k, k_1, k_2=1}^{\infty} \left( u_k^1 \cos k_1 x_1 \cos k_2 x_2 + u_k^2 \cos k_1 x_1 \sin k_2 x_2 \right. \\ & \left. + u_k^3 \sin k_1 x_1 \cos k_2 x_2 + u_k^4 \sin k_1 x_1 \sin k_2 x_2 \right), \end{aligned}$$

where  $u_k^1, u_k^2, u_k^3, u_k^4$  are constants. Noting that

$$q = Q_N u \text{ for } N \in \mathbb{N},$$

it follows that

$$\begin{aligned} q = & \sum_{k, k_1, k_2=N+1}^{\infty} \left( u_k^1 \cos k_1 x_1 \cos k_2 x_2 + u_k^2 \cos k_1 x_1 \sin k_2 x_2 \right. \\ & \left. + u_k^3 \sin k_1 x_1 \cos k_2 x_2 + u_k^4 \sin k_1 x_1 \sin k_2 x_2 \right) \end{aligned}$$

and

$$\|q\|^2 = \sum_{k=N+1}^{\infty} \left( |u_k^1|^2 + |u_k^2|^2 + |u_k^3|^2 + |u_k^4|^2 \right).$$

Therefore

$$\begin{aligned} \nabla q &= \left( \frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2} \right) \\ &= \left( \sum_{k,k_1,k_2=N+1}^{\infty} \left[ -k_1 u_k^1 \sin k_1 x_1 \cos k_2 x_2 - k_1 u_k^2 \sin k_1 x_1 \sin k_2 x_2 \right. \right. \\ &\quad \left. \left. + k_1 u_k^3 \cos k_1 x_1 \cos k_2 x_2 + k_1 u_k^4 \sin k_1 x_1 \sin k_2 x_2 \right], \right. \\ &\quad \left. \sum_{k,k_1,k_2=N+1}^{\infty} \left[ -k_2 u_k^1 \cos k_1 x_1 \sin k_2 x_2 + k_2 u_k^2 \cos k_1 x_1 \sin k_2 x_2 \right. \right. \\ &\quad \left. \left. - k_2 u_k^3 \sin k_1 x_1 \sin k_2 x_2 + k_2 u_k^4 \sin k_1 x_1 \cos k_2 x_2 \right] \right), \end{aligned}$$

and

$$\begin{aligned} \|\nabla q\|^2 &= \left\| \frac{\partial q}{\partial x_1} \right\|^2 + \left\| \frac{\partial q}{\partial x_2} \right\|^2 \\ &= \sum_{k,k_1,k_2=N+1}^{\infty} (k_1^2 + k_2^2) (|u_k^1|^2 + |u_k^2|^2 + |u_k^3|^2 + |u_k^4|^2) \\ &\geq 2(N+1)^2 \sum_{k=N+1}^{\infty} (|u_k^1|^2 + |u_k^2|^2 + |u_k^3|^2 + |u_k^4|^2) \\ &= 2(N+1)^2 \|q\|^2. \end{aligned}$$

LEMMA 2.2. Assume that  $u(x, t)$  is a solution for problem (7)–(10) with  $u_0(x) \in B$ , and  $q^k (k = 0, 1, 2, \dots)$  satisfy (12)–(13). Then there are  $N_0 \in \mathbb{N}$  and  $t_1^*(B) > 0$  such that for  $N \geq N_0$ ,

$$\|u^k\|^2 + \delta \|\nabla u^k\|^2 \leq 2\rho_0^2, \quad t \geq t_1^*(B), \quad k = 0, 1, 2, \dots \quad (14)$$

PROOF. We only need to prove the following inequality:

$$\|q^k\|^2 + \delta \|\nabla q^k\|^2 \leq \rho_0^2. \quad (15)$$

Here we verify (15) by the inductive method. Firstly, multiplying (12) by  $q^0$ , we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|q^0\|^2 + \delta \|\nabla q^0\|^2) + \mu \|\nabla q^0\|^2 \leq \|q^0\| (\|h\| + \|\nabla \cdot \mathbf{F}(p)\|) \\
& \leq \frac{\|\nabla q^0\|}{\sqrt{2(N+1)}} (\|h\| + \|f_1(p)p_{x_1} + f_2(p)p_{x_2}\|) \\
& \leq \frac{\|\nabla q^0\|}{\sqrt{2(N+1)}} (\|h\| + 2C_1(1 + |p|^m) \|\nabla p\|) \\
& \leq \frac{\|\nabla q^0\|}{\sqrt{2(N+1)}} \left( \|h\| + 2C_1(1 + \rho_0^m) \delta^{-\frac{1}{2}} \rho_0 \right) \\
& \leq \frac{\mu}{2} \|\nabla q^0\|^2 + \frac{1}{4\mu(N+1)^2} \left( \|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m) \right)^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt} (\|q^0\|^2 + \delta \|\nabla q^0\|^2) + \mu \|\nabla q^0\|^2 \leq \frac{1}{2\mu(N+1)^2} \left( \|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m) \right)^2.$$

Note that

$$\begin{aligned}
\mu \|\nabla q^0\|^2 &= \frac{\mu}{2} \|\nabla q^0\|^2 + \frac{\mu}{2} \|\nabla q^0\|^2 \\
&\geq \mu(N+1)^2 \|q^0\|^2 + \frac{\mu}{2\delta} \delta \|\nabla q^0\|^2 \geq C_2 (\|q^0\|^2 + \delta \|\nabla q^0\|^2),
\end{aligned}$$

where  $C_2 = \min\{\mu(N+1)^2, \frac{\mu}{2\delta}\}$ . Then we have

$$\frac{d}{dt} (\|q^0\|^2 + \delta \|\nabla q^0\|^2) + C_2 (\|q^0\|^2 + \delta \|\nabla q^0\|^2) \leq \frac{\left( \|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m) \right)^2}{2\mu(N+1)^2}.$$

By Gronwall's Lemma, we have

$$\begin{aligned}
& \|q^0(t)\|^2 + \delta \|\nabla q^0(t)\|^2 \\
& \leq (\|q^0(0)\|^2 + \delta \|\nabla q^0(0)\|^2) e^{-C_2 t} + \frac{\left( \|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m) \right)^2}{2C_2 \mu(N+1)^2} (1 - e^{-C_2 t}).
\end{aligned}$$

There exists a  $t_{11}^*(B) > 0$ , such that for  $\forall t \geq t_{11}^*(B)$ , we have

$$\|q^0(t)\|^2 + \delta \|\nabla q^0\|^2 \leq \frac{\left( \|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m) \right)^2}{C_2 \mu(N+1)^2}.$$

Let  $N$  be so large that

$$\frac{\left( \|h\| + 2C_1 \delta^{-\frac{1}{2}} \rho_0 (1 + \rho_0^m) \right)^2}{C_2 \mu(N+1)^2 \rho_0^2} \leq 1, \tag{16}$$

we have

$$\|q^0\|^2 + \delta\|\nabla q^0\|^2 \leq \rho_0^2, \quad t \geq t_{11}^*(B). \quad (17)$$

Now assume that  $\|q^{k-1}\|^2 + \delta\|\nabla q^{k-1}\|^2 \leq \rho_0^2$  holds, we shall prove that for  $\forall k$  (15) holds. Multiplying (13) by  $q^k$ , we have

$$\frac{1}{2} \frac{d}{dt} (\|q^k\|^2 + \delta\|\nabla q^k\|^2) + \mu\|\nabla q^k\|^2 \leq \left( \|h\| + 2C_1\delta^{-\frac{1}{2}}\rho_0(1 + \rho_0^m)^2 \right) \|q^k\|.$$

By using similar argument as above, we can obtain

$$\frac{d}{dt} (\|q^k\|^2 + \delta\|\nabla q^k\|^2) + C_2(\|q^k\|^2 + \delta\|\nabla q^k\|^2) \leq \frac{\left( \|h\| + 2C_1\delta^{-\frac{1}{2}}\rho_0(1 + \rho_0^m)^2 \right)^2}{2\mu(N+1)^2}.$$

By Gronwall's Lemma, there exists a  $t_{12}^*(B) > 0$ , such that for  $\forall t \geq t_{12}^*(B)$ , we have

$$\|q^k(t)\|^2 + \delta\|\nabla q^k\|^2 \leq \frac{\left( \|h\| + 2C_1\delta^{-\frac{1}{2}}\rho_0(1 + \rho_0^m)^2 \right)^2}{C_2\mu(N+1)^2}.$$

Let  $N$  be so large that

$$\frac{\left( \|h\| + 2C_1\delta^{-\frac{1}{2}}\rho_0(1 + \rho_0^m)^2 \right)^2}{C_2\mu(N+1)^2\rho_0^2} \leq 1, \quad (18)$$

we have

$$\|q^k\|^2 + \delta\|\nabla q^k\|^2 \leq \rho_0^2, \quad t \geq t_{11}^*(B). \quad (19)$$

Let  $t_1^*(B) = \max\{t_{11}^*(B), t_{12}^*(B)\}$ . Then (15) follows from (17) and (19). The proof of Lemma 2.2 is completed.

LEMMA 2.3. Under the hypotheses of Lemma 2.2, there are  $N_1 \in \mathbb{N}$  and  $t_2^*(B) > 0$  such that for  $N > N_1$  we have

$$\|q^k - q\|^2 + \delta\|\nabla q^k - \nabla q\|^2 \rightarrow 0, \quad k \rightarrow \infty, \quad t \geq t_2^*(B). \quad (20)$$

PROOF. Here we verify (20) by the inductive method. Firstly, set  $w^0 = q^0 - q$ , by (11) and (12) we have

$$w_t^0 - \delta\Delta w_t^0 - \mu\Delta w^0 + Q_N(\nabla \cdot \mathbf{F}(p) - \nabla \cdot \mathbf{F}(u)) = 0. \quad (21)$$

Multiplying (21) by  $w^0$  we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w^0\|^2 + \delta\|\nabla w^0\|^2) + \mu\|\nabla w^0\|^2 \\ & \leq \|\nabla \cdot \mathbf{F}(p) - \nabla \cdot \mathbf{F}(u)\| \|w^0\| \leq 4C_1\delta^{-\frac{1}{2}}\rho_0(1 + \rho_0^m)\|w^0\|. \end{aligned}$$

By using similar argument as above, we can obtain

$$\frac{d}{dt} (\|w^0\|^2 + \delta\|\nabla w^0\|^2) + C_2(\|w^0\|^2 + \delta\|\nabla w^0\|^2) \leq \frac{8C_1^2\rho_0^2(1 + \rho_0^m)^2}{\delta\mu(N+1)^2}.$$

By Gronwall's Lemma, there exists a  $t_{20}^*(B) > 0$  such that

$$\|w^0(t)\|^2 + \delta\|\nabla w^0(t)\|^2 \leq \frac{16C_1^2\rho_0^2(1+\rho_0^m)^2}{C_2\delta\mu(N+1)^2}, \quad t \geq t_{20}^*(B). \quad (22)$$

Denote  $w^k = q^k - q$ , by (11) and (13), we have

$$w_t^k - \delta\Delta w_t^k - \mu\Delta w^k + Q_N(\nabla \cdot \mathbf{F}(u^{k-1}) - \nabla \cdot \mathbf{F}(u)) = 0, \quad (23)$$

where  $k = 1, 2, \dots$ . Multiplying (23) by  $w^k$ , we have

$$\frac{1}{2} \frac{d}{dt} (\|w^k\|^2 + \delta\|\nabla w^k\|^2) + \mu\|\nabla w^k\|^2 + (Q_N(\nabla \cdot \mathbf{F}(u^{k-1}) - \nabla \cdot \mathbf{F}(u)), w^k) = 0,$$

Now let us consider the last term

$$\begin{aligned} & (Q_N(\nabla \cdot \mathbf{F}(u^{k-1}) - \nabla \cdot \mathbf{F}(u)), w^k) \\ &= \left( \sum_{i=1}^2 [f_i(u^{k-1})u_{x_i}^{k-1} - f_i(u)u_{x_i}], Q_N w^k \right) \\ &= \left( \sum_{i=1}^2 [f_i(u^{k-1})w_{x_i}^{k-1} + (f_i(u^{k-1}) - f_i(u))u_{x_i}], Q_N w^k \right) \\ &= \left( \sum_{i=1}^2 f_i(u^{k-1})w_{x_i}^{k-1}, Q_N w^k \right) \\ &\quad + \left( \sum_{i=1}^2 \int_0^1 f_i'(\theta u^{k-1} + (1-\theta)u) d\theta w^{k-1} u_{x_i}, Q_N w^k \right). \end{aligned}$$

So we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w^k\|^2 + \delta\|\nabla w^k\|^2) + \mu\|\nabla w^k\|^2 \\ &\leq 2C_1(1 + |u^{k-1}|^m)\|\nabla w^{k-1}\|\|w^k\| + C_3\|w^{k-1}\|\|\nabla u\|\|w^k\| \\ &\leq 2C_1(1 + \rho^m)\|\nabla w^{k-1}\| \cdot \frac{1}{\sqrt{2}(N+1)}\|\nabla w^k\| \\ &\quad + C_3\delta^{-\frac{1}{2}}\rho_0 \cdot \frac{1}{2(N+1)^2}\|\nabla w^{k-1}\|\|\nabla w^k\| \\ &\leq \frac{\mu}{2}\|\nabla w^k\|^2 + \frac{1}{2\mu} \cdot \left( \frac{\sqrt{2}C_1(1 + \rho_0^m)}{(N+1)} + \frac{C_3\rho_0}{2(N+1)^2\delta^{\frac{1}{2}}} \right)^2 \|\nabla w^{k-1}\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{d}{dt} (\|w^k\|^2 + \delta\|\nabla w^k\|^2) + C_2(\|w^k\|^2 + \delta\|\nabla w^k\|^2) \\ &\leq \frac{1}{\mu} \left( \frac{\sqrt{2}C_1(1 + \rho_0^m)}{N+1} + \frac{C_3\rho_0}{2(N+1)^2\delta^{\frac{1}{2}}} \right)^2 \|\nabla w^{k-1}\|^2, \quad (24) \end{aligned}$$



where ( $k = 1, 2, \dots$ ). Let  $k = 1$  in (24), we have

$$\begin{aligned} & \frac{d}{dt} (\|w^1\|^2 + \delta \|\nabla w^1\|^2) + C_2 (\|w^1\|^2 + \delta \|\nabla w^1\|^2) \\ & \leq \frac{1}{\mu} \left( \frac{\sqrt{2}C_1(1 + \rho_0^m)}{N+1} + \frac{C_3\rho_0}{2(N+1)^2\delta^{\frac{1}{2}}} \right)^2 \|\nabla w^0(t)\|^2. \end{aligned} \quad (25)$$

By Gronwall's lemma, there is a  $t_{21}^*(B) > 0$  such that

$$\begin{aligned} & \|w^1\|^2 + \delta \|\nabla w^1\|^2 \\ & \leq \frac{2}{C_2\mu} \left( \frac{\sqrt{2}C_1(1 + \rho_0^m)}{(N+1)} + \frac{C_3\rho_0}{2(N+1)^2\delta^{\frac{1}{2}}} \right)^2 \|\nabla w^0(t)\|^2, \quad t \geq t_{21}^*(B). \end{aligned} \quad (26)$$

By the inductive method, there is a  $t_{2k}^*(B) > 0$  such that

$$\begin{aligned} & \|w^k\|^2 + \delta \|\nabla w^k\|^2 \\ & \leq \frac{2^k}{C_2^k\mu^k} \left( \frac{\sqrt{2}C_1(1 + \rho_0^m)}{N+1} + \frac{C_3\rho_0}{2(N+1)^2\delta^{\frac{1}{2}}} \right)^{2k} \|\nabla w^0(t)\|^2, \quad t \geq t_{2k}^*(B), \end{aligned} \quad (27)$$

where  $k = 1, 2, \dots$ . If  $N$  is large enough, such that

$$\frac{2}{C_2\mu} \left( \frac{\sqrt{2}C_1(1 + \rho_0^m)}{N+1} + \frac{C_4\rho_0}{2(N+1)^2\delta^{\frac{1}{2}}} \right)^2 < 1, \quad (28)$$

then (20) follows from (22) and (27). The proof of Lemma 2.3 is completed.

In the above lemma, the asymptotic approximation of the real solution is proved, and the dimension of the asymptotic solution is estimated. Now we explain the compactness of  $\mathcal{A}^k$  in  $\dot{H}_{per}^2(\Omega)$ . It is indispensable. First we use the characterisation

$$\mathcal{A}^k = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s, u_0 \in B} u^k(t)}.$$

Since for  $t \geq s$ , the sets  $\overline{\bigcup_{t \geq s, u_0 \in B} u^k(t)}$  form a sequence of nonempty compact sets decreasing as  $t$  increases, their intersection  $\mathcal{A}^k$  is nonempty and compact. Next, to show invariance, suppose that

$$x \in \mathcal{A}^k = \{y : \exists t_n \rightarrow \infty, S^k(t_n)u_0 \rightarrow y\},$$

we find that there exist sequences  $\{t_n\}$  with  $t_n \rightarrow \infty$  such that  $S^k(t_n)u_0 \rightarrow x$  and

$$S^k(t)S^k(t_n)u_0 = S^k(t + t_n)u_0 \rightarrow S^k(t)u_0$$

since  $S(t)$  is continuous. So  $S(t)\mathcal{A}^k \subset \mathcal{A}^k$ . To show equality, for  $t_n \geq t + s$ , the sequence  $S^k(t_n - t)u_0$  is in the set

$$\overline{\bigcup_{t \geq s, u_0 \in B} u^k(t)}$$

and so possesses a convergent subsequence  $S^k(t_{n_j} - t)u_0 \rightarrow y$ , and so  $y \in \mathcal{A}^k$ . But since  $S(t)$  is continuous,

$$x = \lim_{j \rightarrow \infty} S(t)S(t_{n_j} - t)x_{n_j} = S(t)y,$$

and so  $\mathcal{A}^K \subset S(t)\mathcal{A}^k$ . Thus  $S(t)\mathcal{A}^k = \mathcal{A}^k$  for all  $t \geq s$ . The proof of invariance is completed. Therefore,  $\mathcal{A}^k$  is the asymptotic attractor.

## References

- [1] J. Albert, Dispersion of low-energy waves for the generalized Benjamin-Bona-Mahony equation, *J. Differential Equations*, 63(1986), 117–134.
- [2] J. Avrin, The generalized Benjamin-Bona-Mahony equation in  $\mathbb{R}^n$  with singular initial data, *Nonlinear Anal.*, 11(1987), 139–147.
- [3] P. Biler, Long time behavior of solutions of the generalized Benjamin-Bona-Mahony equation in two space dimensions, *Differential Integral Equations*, 5(1992), 891–901.
- [4] A. O. Celebi, V. K. Kalantarov and M. Polat, Attractors for the generalized Benjamin-Bona-Mahony equation, *J. Differential Equations*, 157(1999): 439–451.
- [5] S. N. Chow and K. Lu, Inertial manifolds for flows in Banach spaces, *J. Differential Equations*, 74(1988), 285–317.
- [6] A. Chen, L. Guo and X. Deng, Existence of solitary waves and periodic waves for a perturbed generalized BBM equation, *J. Differential Equations*, 261(2016), 5289–5922.
- [7] C. Foias, O. Manley and R. Temam, Sur l'interaction des petits et grands tourbillons dans des écoulements turbulents, *C. R. Acad. Sci. Paris Sér. I Math.*, 305(1987), 497–500.
- [8] C. Foias, G. Sell and R. Temam, Inertial manifolds for nonlinear evolutionary equations, *J. Differential Equations.*, 73(1988), 309–353.
- [9] Y. Guo, M. Wang and Y. Tang, Higher regularity of global attractor for a damped Benjamin-Bona-Mahony equation on  $R$ , *Appl. Anal.*, 94(2015), 1766–1783.
- [10] M. A. Hamed, Y. Guo and E. S. Titi, Inertial Manifolds for Certain Subgrid-Scale  $\alpha$ -Models of Turbulence, *J. Appl. Dyn. Syst.*, 14(2015), 1308–1325.
- [11] M. Stanislavova, On the global attractor for the damped Benjamin-Bona-Mahony equation, *Proceedings of the fifth international conference on dynamical systems and differential equations*, Pomona, CA, USA, June 16-19, 2004.

- [12] M. Stanislavova, A. Stefanov and B. Wang, Asymptotic Smoothing and attractors for the generalized Benjamin-Bona-Mahony equation on  $\mathbb{R}^3$ , *J. Differential Equations*, 219(2005), 451–483.
- [13] B. Wang, Strong Attractors for the Benjamin-Bona-Mahony equation, *Appl. Math. Lett.*, 10(1997), 23–28.
- [14] G X. Wang and Z. R. Liu, The asymptotic attractors of Kuramoto-Sivashinsky equation, *Acta Math. Appl. Sinica*(in Chinese), 23(2000), 329–336
- [15] B. Wang and W. Yang, Finite dimensional behaviour for the Benjamin-Bona-Mahony equation, *J. Phys. A, Math. Gen.*, 30(1997), 4877–4885.
- [16] C. S. Zhu, Global attractor for the damped Benjamin-Bona-Mahony equations on  $\mathbb{R}^1$ , *Appl. Anal.*, 86(2007), 59–65.
- [17] C. S. Zhu and C. L. Mu, Exponential decay estimates for time-delayed Benjamin-Bona-Mahony equations, *Appl. Anal.*, 87(2008), 401–07.