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$(j,\ k)$-Symmetric Points With Bounded Boundary Rotation*

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Abstract

The main objective of this paper is to derive the integral representation for the classes involving $(j,\ k)$-symmetrical functions with bounded boundary rotation.

1 Introduction

Let $\mathcal{A}_p$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \geq 1)$$

and let $\mathcal{A} = \mathcal{A}_1$. We denote by $\mathcal{S}^*$ and $\mathcal{C}$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike and convex in $\mathbb{U}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f \prec g$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Let k be a positive integer and $j = 0, 1, 2, \ldots, (k-1)$. A domain D is said to be $(j,\ k)$-fold symmetric if a rotation of D about the origin through an angle $2\pi j/k$ carries D onto itself. A function $f \in \mathcal{A}$ is said to be $(j,\ k)$-symmetrical if for each $z \in \mathbb{U}$

$$f(\varepsilon z) = \varepsilon^j f(z),$$

where $\varepsilon = \exp(2\pi i/k)$. The family of $(j,\ k)$-symmetrical functions will be denoted by $\mathcal{F}_k^j$. For every function f defined on a symmetrical subset $\mathbb{U}$ of $\mathbb{C}$, there exits a unique sequence of $(j,\ k)$-symmetrical functions $f_{j,\,k}(z)$, $j = 0, 1, \ldots,\ k-1$ such that

$$f = \sum_{j=0}^{k-1} f_{j,\,k}.$$

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Moreover,

$$f_{j,\,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{f(\varepsilon^{\nu} z)}{\varepsilon^{\nu pj}}, \quad (f \in \mathcal{A}_p;\ k = 1,\ 2,\ \ldots;\ j = 0,\ 1,\ 2,\ \ldots(k-1)). \tag{1}$$

If ν is an integer, then the following identities follow directly from (1):

$$f'_{j,\,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu pj+\nu} f'(\varepsilon^{\nu} z), \qquad f''_{j,\,k}(z) = \frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu pj+2\nu} f''(\varepsilon^{\nu} z), \tag{2}$$

and

$$\begin{aligned} f_{j,\,k}(\varepsilon^{\nu} z) &= \varepsilon^{\nu pj} f_{j,\,k}(z), \quad f_{j,\,k}(z) = \overline{f_{j,\,k}(\overline{z})} \\ f'_{j,\,k}(\varepsilon^{\nu} z) &= \varepsilon^{\nu pj-\nu} f'_{j,\,k}(z), \quad f'_{j,\,k}(\overline{z}) = \overline{f'_{j,\,k}(z)}. \end{aligned} \tag{3}$$

This decomposition is a generalization of the well known fact that each function defined on a symmetrical subset $\mathbb{U}$ of $\mathbb{C}$ can be uniquely represented as the sum of an even function and an odd function (see Theorem 1 of [4]). We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively. Further, it is obvious that $f_{j,\,k}(z)$ is a linear operator from $\mathbb{U}$ into $\mathbb{U}$. The notion of $(j,\,k)$-symmetrical functions was first introduced and studied by P. Liczberski and J. Połubiński in [4].

A function $f(z)$ is said to be in the class $\mathcal{U}_\kappa(p)$ if

$$f(z) = z^p \exp\left\{\int_0^{2\pi} -p\log\left(1-e^{-it}z\right) d\mu(t)\right\}$$

for $\mu(t) \in M_\kappa$. Geometrically the condition is that the total variation of the angle which the radius vector $f(re^{i\theta})$ makes with the positive real axis is bounded above by $\kappa p\pi$ as z describes the circle $|z| = r$ for $|z| < 1$. From [9], let $V_\kappa(p)$ denotes the class of functions f defined on $\mathbb{U}$ which map conformally onto an image domain of boundary rotation almost $\kappa p\pi$. Hence $f(z) \in V_\kappa(p)$, if and only if

$$f'(z) = pz^{p-1}\exp\left\{-p\int_0^{2\pi}\log\left(1-e^{-it}z\right) d\mu(t)\right\}$$

for some $\mu(t) \in M_\kappa$.

For an integer κ, $\kappa \geq 2$, let M_κ denote the class of real valued functions μ of bounded variation on $[0, 2\pi]$ which satisfy

$$\int_0^{2\pi} d\mu(t) = 2 \ \text{ and } \ \int_0^{2\pi} |d\mu(t)| \leq \kappa. \tag{4}$$

The class M_κ was used by Paatero [6]. Let $\mathcal{P}_\kappa$ be the class of analytic functions p defined in $\mathbb{U}$ and with representation

$$p(z) = \frac{1}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}} d\mu(t), \tag{5}$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ and it satisfies the conditions (4).

We note that $\kappa \geq 2$ and $p_2 = p$ is the class of analytic functions with positive real part in $\mathbb{U}$ with $p(0) = 1$. The class $\mathcal{P}_\kappa$ was introduced in [7]. From the integral representation (5) it is immediately clear that $p \in \mathcal{P}_\kappa$, if and only if, there are analytic functions p_1, $p_2 \in P$ such that

$$p(z) = \left(\frac{\kappa}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{\kappa}{4} - \frac{1}{2}\right) p_2(z).$$

The class $\mathcal{P}'_\kappa$ is defined to be the class of all analytic functions f such that $f' \in \mathcal{P}_\kappa$.

Recently several authors Selvaraj et al. [8], Karthikeyan [3] and Alsarari et al. [1] introduced and investigated several subclasses of symmetric conjugate points. Motivated by the concept introduced by [2, 5], in this paper, we derive the integral representation for the classes involving $(j,\, k)$-symmetrical functions with bounded boundary rotation. The result is also extended to symmetric conjugate functions.

2 Definitions

DEFINITION 1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{U}_p^{j,\,k}(\mu)$ if and only if it satisfies the condition

$$\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} \in \mathcal{P}_\kappa, \qquad (z \in \mathbb{U})$$

where $f_{j,\,k}(z) \neq 0$ and is defined by the equality (1).

DEFINITION 2. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{V}_p^{j,\,k}(\mu)$ if and only if it satisfies the condition

$$\frac{1}{p}\frac{(zf'(z))'}{f'_{j,\,k}(z)} \in \mathcal{P}_\kappa, \qquad (z \in \mathbb{U})$$

where $f_{j,\,k}(z) \neq 0$ and is defined by the equality (1). It is clear that $f \in \mathcal{V}_p^{j,\,k}(\mu)$ if and only if $zf' \in \mathcal{U}_p^{j,\,k}(\mu)$.

REMARK 1. For $p = 1$, this class reduces to the class $U_k(m, n)$, which was studied by Fuad. Alsarari et al. [2]. For $j = k = 1$ and $p = 1$, we get another class introduced by [6].

3 Main Results

THEOREM 1. Suppose a function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{U}_p^{j,\,k}(\mu)$. Then

$$f_{j,\,k}(z) = z^p \exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{2\pi} \log\left(1 - ze^{-i\left(t - \frac{2\pi\nu}{k}\right)}\right) d\mu(t)\right\},$$

where $f_{j,\,k}(z)$ is defined by (1) and $\mu(t)$ is defined by (4).

PROOF. Suppose that $f \in \mathcal{U}_p^{j,\,k}(\mu)$. Then

$$\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} = p(z), \qquad (z \in \mathbb{U};\ \nu = 0,\,1,\,2,\,\ldots,\,k-1) \tag{6}$$

where

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t).$$

Substituting z by $\varepsilon^\nu z$ in (6) respectively,

$$\frac{1}{p}\frac{\varepsilon^\nu zf'(\varepsilon^\nu z)}{f_{j,\,k}(\varepsilon^\nu z)} = p(\varepsilon^\nu z). \qquad (z \in \mathbb{U};\ \nu = 0,\,1,\,2,\,\ldots,\,k-1) \tag{7}$$

Using the equality (3), (7) becomes

$$\frac{1}{p}\frac{z\varepsilon^{\nu-\nu pj}f'(\varepsilon^\nu z)}{f_{j,\,k}(z)} = \frac{1}{2}\int_0^{2\pi} \frac{1+ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}}{1-ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}}d\mu(t). \tag{8}$$

Let ($\nu = 0,\,1,\,2,\,\ldots,\,k-1$) in (8) and summing them, we get

$$\frac{1}{p}\frac{zf'_{j,\,k}(z)}{f_{j,\,k}(z)} = \frac{1}{2k}\sum_{\nu=0}^{k-1}\int_0^{2\pi} \frac{1+ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}}{1-ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}}d\mu(t),$$

equivalently,

$$\frac{zf'_{j,\,k}(z)}{f_{j,\,k}(z)} - \frac{p}{z} = \frac{1}{2kz}\sum_{\nu=0}^{k-1}\int_0^{2\pi} \frac{1+ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}}{1-ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}}d\mu(t) - \frac{p}{z}.$$

Integrating, we get

$$\log\left(\frac{f_{j,\,k}(z)}{z^p}\right) = \frac{1}{k}\sum_{\nu=0}^{k-1}\int_0^{2\pi} -\log\left(1-ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}\right)d\mu(t),$$

which gives the required assertion of Theorem 1.

THEOREM 2. Suppose a function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{U}_p^{j,\,k}(\mu)$. Then

$$f(z) = \frac{1}{2}\int_0^z \left\{p\zeta^{p-1}\exp\left[-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{2\pi}\log\left(1-\zeta e^{-i\left(t-\frac{2\pi\nu}{k}\right)}\right)d\mu(t)\right]\times \right.$$
$$\left.\int_0^{2\pi}\frac{1+\zeta e^{-i\left(t-\frac{2\pi\nu}{k}\right)}}{1-\zeta e^{-i\left(t-\frac{2\pi\nu}{k}\right)}}d\mu(t)\right\}d\zeta,$$

where $f_{j,\,k}(z)$ is defined by (1) and $\mu(t)$ is defined by (4).

PROOF. Let $f \in \mathcal{U}_p^{j,\,k}(\mu)$. Then

$$\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} = p(z), \qquad (z \in \mathbb{U};\ \nu = 0,\ 1,\ 2,\ \ldots,\ k-1)$$

which implies that

$$zf'(z) = pf_{j,\,k}(z)p(z), \qquad (z \in \mathbb{U};\ \nu = 0,\ 1,\ 2,\ \ldots,\ k-1).$$

Using Theorem 1 and (5), we have

$$f'(z) = pz^{p-1}\exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{2\pi}\log\left(1 - ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}\right)d\mu(t)\right\} \times \frac{1}{2}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t).$$

Integrating, we get the required result of this Theorem.

COROLLARY 1. Put $p = 1$ and $j = m, k = n$, in the above Theorem 1 and 2, we get the results in [2].

COROLLARY 2. Suppose a function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{V}_p^{j,\,k}(\mu)$. Then

$$f'_{j,\,k}(z) = pz^{p-1}\exp\left\{-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{2\pi}\log\left(1 - ze^{-i\left(t-\frac{2\pi\nu}{k}\right)}\right)d\mu(t)\right\}$$

and

$$f'(z) = \frac{p}{2z}\int_0^z\left\{p\zeta^{p-1}\exp\left[-\frac{p}{k}\sum_{\nu=0}^{k-1}\int_0^{2\pi}\log\left(1 - \zeta e^{-i\left(t-\frac{2\pi\nu}{k}\right)}\right)d\mu(t)\right]\right.$$
$$\left.\int_0^{2\pi}\frac{1+\zeta e^{-i\left(t-\frac{2\pi\nu}{k}\right)}}{1-\zeta e^{-i\left(t-\frac{2\pi\nu}{k}\right)}}d\mu(t)\right\}d\zeta,$$

where $f_{j,\,k}(z)$ is defined by (1) and $\mu(t)$ is defined by (4).

THEOREM 3. Suppose $f \in \mathcal{A}_p$ belongs to the class $\mathcal{U}_p^{j,\,k}(\mu)$. Then $f_{j,\,k} \in \mathcal{U}_\kappa$.

PROOF: Let $f \in \mathcal{U}_p^{j,\,k}(\mu)$. Then

$$\frac{1}{p}\frac{zf'(z)}{f_{j,\,k}(z)} = p(z). \qquad (z \in \mathbb{U};\ \nu = 0,\ 1,\ 2,\ \ldots,\ k-1)$$

Replacing z by $\varepsilon^\nu z$,

$$\frac{1}{p}\frac{\varepsilon^\nu zf'(\varepsilon^\nu z)}{f_{j,\,k}(\varepsilon^\nu z)} = p(\varepsilon^\nu z). \qquad (z \in \mathbb{U};\ \nu = 0,\ 1,\ 2,\ \ldots,\ k-1)$$

Let $(\nu = 0,\ 1,\ 2,\ \ldots,\ k-1)$ in (8) and summing them, we get

$$\frac{1}{p}\frac{zf'_{j,\,k}(z)}{f_{j,\,k}(z)} = \frac{1}{k}\sum_{\nu=0}^{k-1}p(\varepsilon^\nu z).$$

It is clear that $\frac{1}{k}\sum_{\nu=0}^{k-1}p(\varepsilon^\nu z)$ belongs to $\mathcal{P}_\kappa$. Hence the proof is complete.

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