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Convex And Starlike Functions With Respect To (j, k)-Symmetric Points*

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Abstract

The main objective of the present paper is to investigate the classes which are starlike and convex with respect to (j, k)-symmetric points. Certain interesting coefficient inequalities and distortion theorems are deduced.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \ \text{ and } \ |z| < 1\}$, and $\mathcal{S}$ denotes the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathbb{U}$. Given two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write $f \prec g$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. If g is univalent in $\mathbb{U}$ then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part [2].

DEFINITION 1 ([1]). Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathbb{U}$ and satisfying $p(0) = 1$, $\Re\{p(z)\} > 0$ and $z \in \mathbb{U}$.

Any function p in $\mathcal{P}$ has the representation $p(z) = \frac{1+w(z)}{1-w(z)}$ where $w(0) = 0$, $|w(z)| < 1$ on $\mathbb{U}$. The class $\mathcal{P}$ of functions with positive real part plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses such as the class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with

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respect to symmetric points $\mathcal{S}_s^*$ have been defined by using the concept of class of functions with positive real parts.

DEFINITION 2. Let k be a positive integer. A domain $\mathbb{D}$ is said to be k-fold symmetric if a rotation of $\mathbb{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathbb{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathbb{U}$ if for every z in $\mathbb{U}$

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get the class of odd univalent functions.

The notion of (j, k)-symmetrical functions $(j = 0, 1, 2, ..., k-1;\ k = 2, 3, ...)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function. The theory of (j, k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings [4].

DEFINITION 3. Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, .., k-1$ where $k \geq 2$ is a natural number. A function $f : \mathbb{D} \mapsto \mathbb{C}$ where $\mathbb{D}$ is a k-fold symmetric set is called (j, k)-symmetrical if

$$f(\varepsilon z) = \varepsilon^j f(z), \qquad z \in \mathbb{U}.$$

The family of all (j, k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$, and $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ the classes of even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

THEOREM 1 ([4]). For every mapping $f : \mathbb{D} \mapsto \mathbb{C}$, where $\mathbb{D}$ is a k-fold symmetric set, there exists exactly the sequence of (j, k)-symmetrical functions $f_{j,k}$ such that

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z), \tag{2}$$

$$(f \in \mathcal{A};\ j = 0, 1, 2, ..., k-1; k = 2, 3, ...).$$

From (2) we can get

$$f_{j,k}(z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z) = \frac{1}{k} \sum_{v=0}^{k-1} \varepsilon^{-vj} \left(\sum_{n=1}^{\infty} a_n (\varepsilon^v z)^n \right).$$

Then

$$f_{j,k}(z)=\sum_{n=1}^{\infty}\delta_{n,j}a_nz^n,\quad a_1=1,\quad \delta_{n,j}=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=lk+j,\\ 0, & n\neq lk+j.\end{cases}\tag{3}$$

By the uniqueness of the above decomposition, the mappings $f_{j,k}$ are called (j,k)-symmetrical parts of the mapping f. For $k=2$, the above decomposition is the well-known partition of a function f onto the sum of its even part $f_{0,2}$ and its odd part $f_{1,2}$. The following identities follow directly from (3)

$$f'_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{v-vj}f'(\varepsilon^v z),\quad f''_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{2v-vj}f''(\varepsilon^v z),\tag{4}$$

$$f_{j,k}(\varepsilon^v z)=\varepsilon^{vj}f_{j,k}(z)\ \text{ and }\ f'_{j,k}(\varepsilon^v z)=\varepsilon^{vj-v}f'_{j,k}(z).\tag{5}$$

We denote by $\mathcal{S}^*,\ \mathcal{K},\ \mathcal{C},\ \mathcal{C}^*$ the familiar subclasses consisting of functions which are starlike, convex, close-to-convex and quasi-convex in $\mathbb{U}$, respectively.

Sakaguchi [8], introduced the classes $\mathcal{S}^*_s,\mathcal{K}_s$ of functions starlike and convex with respect to symmetric points, respectively. Further, Parvatham and Radha [7], Nikola [6] and others discussed the classes with respect to n-symmetric points.

Here using the concept of (j,k)-symmetrical functions we introduce the following classes of analytic functions and derive some interesting results.

DEFINITION 5. Let $\mathcal{S}_s^{(j,k)}$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f_{j,k}(z)}\right\}\geq 0,\quad (z\in\mathbb{U}),\tag{6}$$

where $j=0,1,2,...,k-1;\ k=1,2,3,...$ and $f_{j,k}(z)$ is defined by (3).

DEFINITION 6. Let $\mathcal{K}_s^{(j,k)}$ denote the class of functions in $\mathcal{S}$ satisfying the inequality

$$\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\}\geq 0,\ (z\in\mathbb{U}),\tag{7}$$

where $j=0,1,2,...,k-1;\ k=1,2,3,...,$ and $f_{j,k}(z)$ is defined by (3).

We need the following lemmas to prove our main results.

LEMMA 1 ([3]). Suppose $\Omega\subset\mathbb{C}$ is a domain, $F:\mathbb{U}\to\Omega$ is a universal covering, $f:\mathbb{U}\to\Omega$ is holomorphic, and $f(0)=F(0)$. If $f'(0)=\cdots=f^m(0)=0$, then

$$\frac{1}{m(m-1)\cdots(1)}\,|f^m(0)|\leq|F'(0)|\tag{8}$$

and

$$f(\{z:|z|\leq r\})\subseteq F(\{z:|z|\leq r^m\}),\qquad 0<r<1.$$

Equality holds in (8) if and only if $f(z) = F(\lambda z^m)$ for some unimodular constant λ. Also if $f(z) \in \partial F(\{|z| \leq r^m\})$ for some z such that $|z| = r$ then $f(z) = F(\lambda z^m)$ for some constant λ.

LEMMA 2 ([2]). Let $p \in \mathcal{P}$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then $|p_n| \leq 2$ for $n \geq 1$.

2 Main Results

Let us begin with the following.

THEOREM 2. Let $f \in \mathcal{S}_s^{(j,k)}$. Then

$$|a_n| \leq \prod_{m=1}^{n-1} \frac{m + \delta_{m,j}}{m + 1 - \delta_{m+1,j}} \text{ for } n \geq 2, \tag{9}$$

where $\delta_{m,j}$ is defined by (3).

PROOF. By Definition 5, we have

$$\frac{zf'(z)}{f_{j,k}(z)} = p(z), \qquad p \in \mathcal{P}.$$

Then we have

$$zf'(z) = \left[1 + \sum_{n=1}^{\infty} p_n z^n\right] f_{j,k}(z).$$

By (1) and (3), we have

$$(1 - \delta_{1,j})z + \sum_{n=2}^{\infty} (n - \delta_{n,j}) a_n z^n = \left[\sum_{n=1}^{\infty} p_n z^n\right] \left[\sum_{n=1}^{\infty} \delta_{n,j} a_n z^n\right].$$

Equating coefficients of z^n on both sides, we have

$$a_n = \frac{1}{n - \delta_{n,j}} \sum_{m=1}^{n-1} p_m \delta_{n-m,j} a_{n-m}, \qquad \delta_{1,j} = 1.$$

By Lemma 2, we have

$$|a_n| \leq \frac{2}{n - \delta_{n,j}} \sum_{m=1}^{n-1} \delta_{m,j} |a_m|. \tag{10}$$

Using the induction method, we want to prove that

$$\frac{2}{n - \delta_{n,j}} \sum_{m=1}^{n-1} \delta_{m,j} \, |a_m| \leq \prod_{m=1}^{n-1} \frac{m + \delta_{m,j}}{m + 1 - \delta_{m+1,j}}. \tag{11}$$

The result is true for $n=2$ and 3. Let the hypothesis be true for $n=m$, that is

$$\frac{2}{m-\delta_{m,j}}\sum_{r=1}^{m-1}\delta_{r,j}\,|a_r|\leq\prod_{r=1}^{m-1}\frac{r+\delta_{r,j}}{r+1-\delta_{r+1,j}}.$$

Multiplying both sides by $\frac{m+\delta_{m.j}}{m+1-\delta_{m+1,j}}$, we get

$$\prod_{r=1}^{m}\frac{r+\delta_{r,j}}{r+1-\delta_{r+1,j}}\geq\frac{m+\delta_{m.j}}{m+1-\delta_{m+1,j}}.\frac{2}{m-\delta_{m,j}}\sum_{r=1}^{m-1}\delta_{r,j}|a_r|,$$

since

$$\begin{aligned}
&\frac{m+\delta_{m.j}}{m+1-\delta_{m+1,j}}\frac{2}{m-\delta_{m,j}}\sum_{r=1}^{m-1}\delta_{r,j}|a_r|\\
=\;&\frac{2}{m+1-\delta_{m+1,j}}\left[1+\frac{2\delta_{m,j}}{[m-\delta_{m,j}]}\right]\sum_{r=1}^{m-1}\delta_{r,j}|a_r|,\\
\geq\;&\frac{2}{m+1-\delta_{m+1,j}}\left[\sum_{r=1}^{m-1}\delta_{r,j}|a_r|+\delta_{m,j}|a_m|\right],\\
=\;&\frac{2}{m+1-\delta_{m+1,j}}\left[\sum_{r=1}^{m}\delta_{r,j}|a_r|\right].
\end{aligned}$$

That is

$$|a_{m+1}|\leq\frac{2}{m+1-\delta_{m+1,j}}\left[\sum_{r=1}^{m}\delta_{r,j}|a_r|\right]\leq\prod_{r=1}^{m}\frac{r+\delta_{r,j}}{r+1-\delta_{r+1,j}},$$

which shows that inequality (11) is true for $n=m+1$. This completes the proof.

By the well-know a result $f(z)\in\mathcal{K}_s^{(j,k)}\Leftrightarrow zf'(z)\in\mathcal{S}_s^{(j,k)}$, we get the following theorem.

THEOREM 3. Let $f\in\mathcal{K}_s^{(j,k)}$ for $n\geq 2$. Then

$$|a_n|\leq\frac{1}{n}\prod_{m=1}^{n-1}\frac{m+\delta_{m,j}}{m+1-\delta_{m+1,j}},\tag{12}$$

where $\delta_{m,j}$ is defined by (3).

THEOREM 4. Let $f(z)\in\mathcal{K}_s^{(j,k)}$. Then

$$\frac{1}{\left(1+r^{k+j-1}\right)^{\frac{2}{k+j-1}}}\leq\left|f'_{j,k}(z)\right|\leq\frac{1}{\left(1-r^{k+j-1}\right)^{\frac{2}{k+j-1}}},$$

where $f_{j,k}$ is defined by (3).

PROOF. Let $f(z) \in \mathcal{K}_s^{(j,k)}$. Then

$$\Re\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\} \geq 0, \ (z \in \mathbb{U}). \tag{13}$$

Substituting z by ε^v in (13), respectively, $v = 0, 1, ..., k-1$, we get

$$\Re\left\{\frac{f'(\varepsilon^v z) + \varepsilon^v z f''(\varepsilon^v z)}{f'_{j,k}(\varepsilon^v z)}\right\} \geq 0, \ (z \in \mathbb{U}), v = 0, 1, ..., k-1. \tag{14}$$

According to the definition of $f_{j,k}(\varepsilon^v z) = \varepsilon^{vj-v}$, the inequality (14) becomes

$$\Re\left\{\frac{\varepsilon^{v-vj} f'(\varepsilon^v z) + \varepsilon^{2v-vj} z f''(\varepsilon^v z)}{f'_{j,k}(z)}\right\} \geq 0, \ (z \in \mathbb{U}). \tag{15}$$

Let $v = 0, 1, ..., k-1$ in (15) respectively, we get

$$\Re\left\{\frac{\sum_{v=0}^{k-1} \varepsilon^{v-vj} f'(\varepsilon^v z) + z \sum_{v=0}^{k-1} \varepsilon^{2v-vj} f''(\varepsilon^v z)}{f'_{j,k}(z)}\right\} \geq 0, \ (z \in \mathbb{U}),$$

or equivalently

$$\Re\left\{\frac{f'_{j,k}(z) + z f''_{j,k}(z)}{f'_{j,k}(z)}\right\} \geq 0, \ (z \in \mathbb{U}),$$

that is $f_{j,k}(z) \in \mathcal{K}$. Since $f_{j,k}(z)$ is convex univalent, we have

$$\Re\left\{\frac{z f''_{j,k}(z)}{f'_{j,k}(z)}\right\} \geq -1, \ (z \in \mathbb{U}).$$

Hence the function $g(z) = \frac{z f''_{j,k}(z)}{f'_{j,k}(z)}$ is subordinate to $G(z) = \frac{1+z}{1-z} - 1 = \frac{2z}{1-z}$. Now $g(0) = 0$ and from (10) we have $\delta_{1,j} = 1$ so

$$f_{j,k}(z) = z + \sum_{n=1}^{\infty} a_{nk+j} z^{nk+j}.$$

Therefore, we get

$$g(z) = \frac{(k+j)(k+j-1)a_{k+j} z^{k+j-1} + ...}{1 + (k+j)a_{k+j} z^{k+j-1} + ...}$$

so $g'(0) = ... = g^{k+j-1}(0) = 0$. We conclude from Lemma 1 that

$$g(\{|z| \leq r\}) \leq G(\{|z| \leq r^{k+j-1}\}) \ \text{ and } \ r < 1.$$

The latter set is a disc centered on the real axis. Since

$$G(r^{k+j-1}) = \frac{2r^{k+j-1}}{1 - r^{k+j-1}}, \qquad G(-r^{k+j-1}) = \frac{-2r^{k+j-1}}{1 + r^{k+j-1}},$$

the centre of this disc is located at $\frac{2r^{2(k+j-1)}}{1-r^{2(k+j-1)}}$ and its radius is $\frac{2r^{(k+j-1)}}{1-r^{2(k+j-1)}}$. Thus for $z\in\mathbb{U}$,

$$\left|g(z)-\frac{2\left|z\right|^{2(k+j-1)}}{1-\left|z\right|^{2(k+j-1)}}\right|\leq\frac{2\left|z\right|^{2(k+j-1)}}{1-\left|z\right|^{2(k+j-1)}},$$

which gives in turn

$$\left|\frac{zf''_{j,k}(z)}{f'_{j,k}(z)}-\frac{2\left|z\right|^{2(k+j-1)}}{1-\left|z\right|^{2(k+j-1)}}\right|\leq\frac{2\left|z\right|^{(k+j-1)}}{1-\left|z\right|^{2(k+j-1)}},$$

$$\left|\frac{\left|z\right|^{2}f''_{j,k}(z)}{f'_{j,k}(z)}-\frac{2\bar{z}\left|z\right|^{2(k+j-1)}}{1-\left|z\right|^{2(k+j-1)}}\right|\leq\frac{2\left|z\right|^{(k+j)}}{1-\left|z\right|^{2(k+j-1)}}$$

and

$$\left|\frac{f''_{j,k}(z)}{f'_{j,k}(z)}-\frac{2\bar{z}\left|z\right|^{2(k+j-1)-2}}{1-\left|z\right|^{2(k+j-1)}}\right|\leq\frac{2\left|z\right|^{(k+j-2)}}{1-\left|z\right|^{2(k+j-1)}}.$$

It suffices to establish the result for $0<|z|=r<1$, in this case we have

$$\left|\frac{f''_{j,k}(r)}{f'_{j,k}(r)}-\frac{2r^{2(k+j-1)-1}}{1-r^{2(k+j-1)}}\right|\leq\frac{2r^{(k+j-2)}}{1-r^{2(k+j-1)}}.\tag{16}$$

Upon integrating (16), we readily get

$$\begin{aligned}\left|\log f'_{j,k}(r)+\frac{1}{(k+j-1)}\log\left(1-r^{2(k+j-1)}\right)\right| &\leq \int_0^r\frac{2x^{(k+j-2)}}{1-x^{2(k+j-1)}}dx\\ &= \frac{1}{(k+j-1)}\log\frac{1+r^{(k+j-1)}}{1-r^{(k+j-1)}},\end{aligned}$$

or

$$\begin{aligned}-\frac{1}{(k+j-1)}\log\frac{1+r^{(k+j-1)}}{1-r^{(k+j-1)}} &\leq \log|f'_{j,k}(r)|+\frac{1}{(k+j-1)}\log\left(1-r^{2(k+j-1)}\right)\\ &\leq \frac{1}{(k+j-1)}\log\frac{1+r^{(k+j-1)}}{1-r^{(k+j-1)}},\end{aligned}$$

or

$$\frac{1}{(k+j-1)}\log\frac{1}{[1+r^{(k+j-1)}]^2}\leq\log|f'_{j,k}(r)|\leq\frac{1}{(k+j-1)}\log\frac{1}{[1-r^{(k+j-1)}]^2},$$

or finally

$$\frac{1}{[1+r^{(k+j-1)}]^{\frac{2}{(k+j-1)}}}\leq\left|f'_{j,k}(r)\right|\leq\frac{1}{[1-r^{(k+j-1)}]^{\frac{2}{(k+j-1)}}}.$$

THEOREM 5. Let $f(z)\in\mathcal{K}_s^{(j,k)}$. Then

$$\frac{1}{r}\int_0^r\frac{1-x}{(1+x)(1+x^{k+j-1})^{\frac{2}{k+j-1}}}dx\leq|f'(z)|\leq\frac{1}{r}\int_0^r\frac{1+x}{(1-x)(1-x^{k+j-1})^{\frac{2}{k+j-1}}}dx.$$

PROOF. Since $f(z) \in \mathcal{K}_s^{(j,k)}$, we have

$$\Re\left\{\frac{(zf'(z))'}{f'_{j,k}(z)}\right\} \geq 0, \qquad |z| < 1.$$

In view a result by Libera and Livingston in [5] for $\beta = 0$, we have

$$\frac{1-r}{1+r} \leq \left|\frac{(zf'(z))'}{f'_{j,k}(z)}\right| \leq \frac{1+r}{1-r}.$$

By Theorem 4, we get

$$\frac{1-r}{[1+r][1+r^{(k+j-1)}]^{\frac{2}{(k+j-1)}}} \leq |(zf'(z))'| \leq \frac{1+r}{[1-r][1-r^{(k+j-1)}]^{\frac{2}{(k+j-1)}}},$$

or

$$\frac{1}{r}\int_0^r \frac{1-x}{[1+x][1+x^{(k+j-1)}]^{\frac{2}{(k+j-1)}}}dx \leq |f'(z)| \leq \frac{1}{r}\int_0^r \frac{1+x}{[1-x][1-x^{(k+j-1)}]^{\frac{2}{(k+j-1)}}}dx.$$

Using again the well-know result $f(z) \in \mathcal{K}_s^{(j,k)} \Leftrightarrow zf'(z) \in \mathcal{S}_s^{(j,k)}$, we get the following theorem.

THEOREM 6. Let $f(z) \in \mathcal{S}_s^{(j,k)}$. Then

$$\int_0^r \frac{1-x}{(1+x)(1+x^{k+j-1})^{\frac{2}{k+j-1}}}dx \leq |f(z)| \leq \int_0^r \frac{1+x}{(1-x)(1-x^{k+j-1})^{\frac{2}{k+j-1}}}dx.$$

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