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Three Applications In q^2-Analogue Sobolev Spaces*

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Abstract

We study three q-difference-differential operators in q^2-analogue Sobolev spaces.

1 Introduction

In this work, we are interested in the study of three q-difference-differential operators in q^2-analogue Sobolev and q^2-potential spaces introduced by the authors in their recent paper [8].

The outline of this paper is as follows. In section 2, we recall some basic facts from quantum calculus, some properties from the q-Rubin's operator and functional spaces in quantum calculus. We study in section 3 the hypoellipticity of q-Rubin operator. Section 4 is devoted to the existence and regularity of q^2-analogue wave equation. In section 5, we study the solution of q^2-analogue-Schrödinger equation.

2 Preliminary

Throughout this paper, we assume $0 < q < 1$ and we refer the reader to [3, 5] for the definitions and properties of hypergeometric functions. In this section we will fix some notations and recall some preliminary results. We put $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$ and $\tilde{\mathbb{R}}_q = \mathbb{R}_q \cup \{0\}$. For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n = 1, 2, ...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We denote also

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathcal{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

A q-analogue of the classical exponential function is given by (see [6, 7])

$$e(z;q^2) = \cos(-iz;q^2) + i\sin(-iz;q^2),$$

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where

$$\cos(z;q^2)=\sum_{n=0}^{+\infty} q^{n(n+1)}\frac{(-1)^n z^{2n}}{[2n]q!} \quad \text{and} \quad \sin(z;q^2)=\sum_{n=0}^{+\infty} q^{n(n+1)}\frac{(-1)^n z^{2n+1}}{[2n+1]q!}.$$

The q-differential-difference operator is defined as (see [6, 7])

$$\partial_q f(z)=\begin{cases}\frac{f(q^{-1}z)+f(-q^{-1}z)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z} & \text{if } z\neq 0,\\ \lim_{z\to 0}\partial_q f(z) \quad \text{in} \quad \mathbb{R}_q & \text{if } z=0.\end{cases}$$

The q-Jackson integrals are defined by (see [4])

$$\int_0^a f(x)d_qx=(1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n)$$

and

$$\int_{-\infty}^{+\infty} f(x)d_qx=(1-q)\sum_{n=-\infty}^{+\infty} q^n\left\{f(q^n)+f(-q^n)\right\},$$

provided the sums converge absolutely.

The q-Rubin-Fourier transform defined in [6], for all $x\in\tilde{\mathbb{R}}_q$ as

$$\mathcal{F}_q(f)(x):=K_q\int_{-\infty}^{+\infty} f(t)e(-itx;q^2)d_qt,$$

where

$$K_q=\frac{(1+q)^{\frac{1}{2}}}{2\Gamma_{q^2}(\frac{1}{2})}, \quad \text{and} \quad \Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}.$$

In the following we denote by

- $\mathcal{S}_q(\mathbb{R}_q)$ stands for the q-analogue Schwartz space of smooth functions over $\mathbb{R}_q$ whose q-derivatives of all order decay at infinity. $\mathcal{S}_q(\mathbb{R}_q)$ is endowed with the topology generated by the following family of semi-norms:

$$\|u\|_{n,\mathcal{S}_q}(f):=\sup_{x\in\mathbb{R}_q;k\leq n}(1+|x|)^n|\partial_q^k u(x)| \quad \text{for all} \quad u\in\mathcal{S}_q \quad \text{and} \quad n\in\mathbb{N}.$$

- $\mathcal{S'}_q(\mathbb{R}_q)$ the space of tempered distributions on $\mathbb{R}_q$, it is the topological dual of $\mathcal{S}_q(\mathbb{R}_q)$.
- $L_q^p(\mathbb{R}_q)=\left\{f:\left(\int_{-\infty}^{+\infty}|f(x)|^p d_qx\right)^{\frac{1}{p}}<\infty\right\}$.
- $L_q^\infty(\mathbb{R}_q)=\left\{f:\sup_{x\in IR_q}|f(x)|<\infty\right\}$.
- $\mathcal{E}(\mathbb{R};\mathcal{S}_q(\mathbb{R}_q))$ the space of C^∞ functions from $\mathbb{R}$ into $\mathcal{S}_q(\mathbb{R}_q)$.

- $\mathcal{E}(\mathbb{R};\mathcal{S'}_q(\mathbb{R}_q))$ the space of C^∞ functions from $\mathbb{R}$ into $\mathcal{S'}_q(\mathbb{R}_q)$.
- $\mathcal{S}(\mathbb{R};\mathcal{S}_q(\mathbb{R}_q))$ the space of Shawartz functions from $\mathbb{R}$ into $\mathcal{S}_q(\mathbb{R}_q)$.
- $C(\mathbb{R};\mathcal{W}_q^{s,p}(\mathbb{R}_q))$ the space of continuous functions on $\mathbb{R}$ into $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$.

It was shown in ([2, 6]) that $\mathcal{F}_q$ verifies the following properties for $f \in L_q^1(\mathbb{R}_q)$.

1. If $uf(u) \in L_q^1(\mathbb{R}_q)$ and $\partial_q f \in L_q^1(\mathbb{R}_q)$ then
$$\partial_q(\mathcal{F}_q)(f)(x) = \mathcal{F}_q(-iuf(u))(x) \quad \text{and} \quad \mathcal{F}_q(\partial_q(f))(x) = ix\mathcal{F}_q(f)(x).$$
2. The reciprocity formula
$$\text{for } t \in \mathbb{R}_q, \quad f(t) = K\int_{-\infty}^{+\infty} \mathcal{F}_q(f)(x)e(itx;q^2)d_qx. \tag{1}$$
3. The q-Rubin-Fourier transform $\mathcal{F}_q$ is an isomorphism from $\mathcal{S}_q(\mathbb{R}_q)$ onto itself and we have
$$\mathcal{F}_q^{-1}(f)(x) = \mathcal{F}_q(f)(-x) = \overline{\mathcal{F}_q(\overline{f})}(x) \ \text{ for } f \in \mathcal{S}_q(\mathbb{R}_q). \tag{2}$$
4. $\mathcal{F}_q$ is an isomorphism from $L_q^2(\mathbb{R}_q)$ onto itself, and we have
$$\|\mathcal{F}_q(f)\|_{L_q^2(\mathbb{R}_q)} = \|f\|_{L_q^2(\mathbb{R}_q)} \ \text{ for } f \in L_q^2(\mathbb{R}_q). \tag{3}$$

DEFINITION 1. The q-Rubin-Fourier transform of a q-distribution $\nu \in \mathcal{S}'(\mathbb{R}_q)$ is defined by
$$\langle \mathcal{F}_q(\nu), \phi\rangle = \langle \nu, \mathcal{F}_q(\phi)\rangle \ \text{ for } \phi \in \mathcal{S}_q(\mathbb{R}_q). \tag{4}$$

From the above properties, we have the following result.

PROPOSITION 1. The q-Rubin-Fourier transform is a topological isomorphism from $\mathcal{S}'(\mathbb{R}_q)$ onto itself.

Let u be in $\mathcal{S}'(\mathbb{R}_q)$. We define the distribution $\partial_q u$, by
$$\langle \partial_q u, \psi\rangle = -\langle u, \partial_q \psi\rangle \ \text{ for } \psi \in \mathcal{S}_q(\mathbb{R}_q). \tag{5}$$
Hence, if we denote the q^2-analogue Laplace operator by $\Delta_q := \partial_q^2$ we deduce
$$\langle \Delta_q u, \psi\rangle = \langle u, \Delta_q \psi\rangle \ \text{ for } \psi \in \mathcal{S}_q(\mathbb{R}_q). \tag{6}$$

These distributions satisfy the following properties
$$\mathcal{F}_q(\partial_q^n u) = (iy)^n \mathcal{F}_q(u), \quad n \in \mathbb{N}. \tag{7}$$

The q-translation operator $\tau_{q;x}, \quad x \in \mathbb{R}_q$ is defined on $L^1_q(\mathbb{R}_q)$ by (see [6])

$$\tau_{q,y}(f)(x) = K\int_{-\infty}^{+\infty} \mathcal{F}_q(f)(t)e(itx;q^2)e(ity;q^2)d_qt, \quad y \in \mathbb{R}_q,$$

$$\tau_{q,0}(f)(x) = f(x).$$

DEFINITION 2. For $f \in L^2_q(\mathbb{R}_q)$ and $g \in L^1_q(\mathbb{R}_q)$, the q-convolution product is given by

$$f * g(y) = K\int_{-\infty}^{+\infty} \tau_{q,y}f(x)g(x)d_qx.$$

PROPOSITION 2. For $f \in L^n_q(\mathbb{R}_q), \quad g \in L^p_q(\mathbb{R}_q)$ and $1 \leq n, p, r \leq \infty$ such that $\frac{1}{n} + \frac{1}{p} - \frac{1}{r} = 1$ we have

$$\|f * g\|_{L^r_q(\mathbb{R}_q)} \leq \|f\|_{L^n_q(\mathbb{R}_q)}\|g\|_{L^p_q(\mathbb{R}_q)}. \tag{8}$$

DEFINITION 3. The q^2-analogue Sobolev spaces introduced in [8] for $s \in \mathbb{R}$ and $1 \leq p < \infty$ are

$$\mathcal{W}^{s,p}_q(\mathbb{R}_q) := \left\{u \in \mathcal{S}'_q(\mathbb{R}_q) : (1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_q(u) \in L^P_q(\mathbb{R}_q)\right\},$$

provided with the norm

$$\|u\|_{\mathcal{W}^{s,p}_q(\mathbb{R}_q)} := \left(\int_{-\infty}^{+\infty}(1+\xi^2)^{\frac{sp}{2}}|\mathcal{F}_q(u)(\xi)|^p d_q\xi\right)^{\frac{1}{p}}.$$

DEFINITION 4. For $u \in \mathcal{S}'_q(\mathbb{R}_q)$ and $s \in \mathbb{R}$, the q^2-potential operator $\mathcal{P}^s_q$ of order s is defined in [1] as

$$\begin{aligned}\mathcal{P}^s_q : \mathcal{S}'_q(\mathbb{R}_q) &\longrightarrow \mathcal{S}'_q(\mathbb{R}_q)\\ u &\longmapsto (\mathcal{F}_q)^{-1}((1+\xi^2)^{-\frac{s}{2}}\mathcal{F}_q(u)).\end{aligned}$$

DEFINITION 5. For all $(s,p) \in \mathbb{R} \times [1, +\infty[$, the q^2-potential space is defined in [8] as

$$\mathcal{B}^{s,p}_q(\mathbb{R}_q) = \left\{u \in \mathcal{S}'_q(\mathbb{R}_q), \mathcal{P}^{-s}_q(u) \in L^p(\mathbb{R}_q)\right\},$$

provided with the norm

$$\|u\|_{\mathcal{B}^{s,p}_q(\mathbb{R}_q)} = \|\mathcal{P}^{-s}_q(u)\|_{L^p_q(\mathbb{R})}.$$

3 Hypoellipticity of q-Rubin Operator

In this section, we will present the hypoellipticity of q-Rubin operator.

THEOREM 1. Let $P(\partial q) = \sum_{j=0}^{n} \alpha_j (\partial_q)^j, \quad \alpha_n \neq 0$, a q-differential-difference operator with constant coefficient α_j and symbol $P(\lambda) = \sum_{j=0}^{n} \alpha_j (i\lambda)^j \neq 0, \quad \lambda \in \mathbb{R}$. If $u \in L_q^2(\mathbb{R}_q), \quad P(-\partial_q)u = f$ and $f \in L_q^2(\mathbb{R}_q)$, then $u \in \mathcal{B}_q^{n,2}(\mathbb{R}_q)$.

PROOF. One can easily show that there exists $R > 0$ and a positive constant C such that

$$|P(\xi)| \geq C|\xi|^n, \quad |\xi| \geq R. \tag{9}$$

Let $u \in \mathcal{S}_q(\mathbb{R}_q)$

$$\|u\|^2_{\mathcal{B}_q^{n,2}(\mathbb{R}_q)} = \int_{\mathbb{R}_q} (\xi^2+1)^{n/2} |\mathcal{F}_q u(\xi)|^2 d_q\xi.$$

Taking $R \geq 1$, we have

$$\|u\|^2_{\mathcal{B}_q^{n,2}(\mathbb{R}_q)} = \int_{-R}^{R} (\xi^2+1)^{n/2} |\mathcal{F}_q u(\xi)|^2 d_q\xi + \int_{|\xi|\geq R} (\xi^2+1)^{n/2} |\mathcal{F}_q u(\xi)|^2 d_q\xi.$$

Now, when $|\xi| \leq R$, we have $(\xi^2+1)^{n/2} \leq (R^2+1)^{n/2}$ and if $|\xi| \geq R$, $(\xi^2+1)^{n/2} \leq 2|\xi|^n$, we have

$$||u||^2_{\mathcal{B}_q^{n,2}(\mathbb{R}_q)} \leq C(R^2+1)^{n/2} \int_{-R}^{R} |\mathcal{F}_q u(\xi)|^2 d_q\xi + C \int_{|\xi|\geq R} |\xi|^{2n} |\mathcal{F}_q u(\xi)|^2 d_q\xi.$$

According to the relations (3), (7) and (9) we obtain

$$\begin{aligned}
\|u\|^2_{\mathcal{B}_q^{n,2}(\mathbb{R}_q)} &\leq C\left(\int_{\mathbb{R}_q} |u(x)|^2 d_q x + \int_{\mathbb{R}_q} |\xi|^{2n} |\mathcal{F}_q u(\xi)|^2 d_q\xi\right) \\
&\leq C\left(\|u\|^2_{L_q^2(\mathbb{R}_q)} + \int_{\mathbb{R}_q} |P(\xi)\mathcal{F}_q u(\xi)|^2 d_q\xi\right) \\
&\leq C\left(\|u\|^2_{L_q^2(\mathbb{R}_q)} + \int_{\mathbb{R}_q} |\mathcal{F}_q(P(\partial_q))g(\xi)|^2 d_q\xi\right) \\
&\leq C\left(\|u\|^2_{L_q^2(\mathbb{R}_q)} + \|P(\partial_q)u\|^2_{L_q^2(\mathbb{R}_q)}\right).
\end{aligned}$$

The proof is completed by using the density of $\mathcal{S}_q(\mathbb{R}_q)$ in $\mathcal{B}_q^{n,2}(\mathbb{R}_q)$.

4 q^2-Analogue Wave Equation

LEMMA 1. For all $p, n \in [1,\infty], \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{n} - 1 \geq 0, \quad s, s' \in \mathbb{R}, \quad f \in \mathcal{B}_q^{s,p}(\mathbb{R}_q)$ and $g \in \mathcal{B}_q^{s',n}(\mathbb{R}_q)$, we have

$$f * g \in \mathcal{B}_q^{s+s',r}(\mathbb{R}_q) \quad \text{and} \quad \|f * g\|_{\mathcal{B}_q^{s+s',r}(\mathbb{R}_q)} \leq C_q \|f\|_{\mathcal{B}_q^{s,p}(\mathbb{R}_q)} \|g\|_{\mathcal{B}_q^{s',n}(\mathbb{R}_q)}.$$

PROOF. The results are given by the inequality (8) and the definition of the q^2-potential spaces.

We consider the q^2-analogue wave equation where the unknown u is a real-valued function such that

$$\begin{cases} \partial_t^2 u - \Delta_q u = 0', \\ u_{|t=0} = u_0 \in \mathcal{B}_q^{s,p}(\mathbb{R}_q)', \\ \partial_t u_{|t=0} = u_1 \in \mathcal{B}_q^{s',n}(\mathbb{R}_q)', \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}_q.$$

COROLLARY 1. Let $\mathcal{C} := \{\xi \in \mathbb{R}_q,\ r \leq |\xi| \leq R\}$ for some positive reals r and R such that $r < R$. We assume that u_0 and u_1 are two functions satisfying

$$supp\ \mathcal{F}_q(u_j) \subset \mathcal{C}.$$

1. For $p = n = 2,\ u \in \mathcal{B}_q^{a+s,\infty}(\mathbb{R}_q) + \mathcal{B}_q^{b+s',\infty}(\mathbb{R}_q)$. For $a + s = b + s' = c$,

$$\|u\|_{\mathcal{B}_q^{c,\infty}(\mathbb{R}_q)} \leq C \left(\|u_0\|_{\mathcal{B}_q^{s,2}(\mathbb{R}_q)} + \|u_1\|_{\mathcal{B}_q^{s',2}(\mathbb{R}_q)} \right).$$

2. For $p \neq 2$ and $n \neq 2,\quad u \in \mathcal{B}_q^{a+s,\frac{p}{2-p}}(\mathbb{R}_q) + \mathcal{B}_q^{b+s',\frac{n}{2-n}}(\mathbb{R}_q)$. For $a + s = b + s' = c$,

$$\|u\|_{\mathcal{B}_q^{c,\frac{p}{2-p}}(\mathbb{R}_q)} \leq C \left(\|u_0\|_{\mathcal{B}_q^{s,p}(\mathbb{R}_q)} + \|u_1\|_{\mathcal{B}_q^{s',p}(\mathbb{R}_q)} \right).$$

PROOF. According to the Duhamel expression for the solution and Lemma 1 we obtain the results.

5 q^2-Analogue-Schrödinger Equation

Now we consider the following equation where the unknown u is a complex-valued function

$$\begin{cases} \partial_t u - i\Delta_q u = 0, \\ u_{t|=0} = g, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}_q.$$

THEOREM 2. Let $g \in \mathcal{S}'_q(\mathbb{R}_q)$. There exists a unique solution $u \in \mathcal{E}(\mathbb{R}; \mathcal{S}'_q(\mathbb{R}_q))$ such that

$$\begin{cases} \partial_t u - i\Delta_q u = 0, \quad \text{in } D'(\mathbb{R} \times \mathbb{R}_q), \\ u_{t|=0} = g. \end{cases}$$

PROOF. Let us prove the existence first. For $t \in \mathbb{R}$, we write

$$u_t = (\mathcal{F}_q)^{-1}(e^{-it|\xi|^2}\mathcal{F}_q(g)). \tag{10}$$

According to (4) we have

$$\langle u_t, \phi\rangle = \langle \mathcal{F}_q(g), e^{-it|\xi|^2}(\mathcal{F}_q)^{-1}(\phi)\rangle.$$

Therefore we deduce that $u_t \in \mathcal{E}(\mathbb{R}; \mathcal{S}'_q(\mathbb{R}_q))$, and $\mathcal{F}_q(u_t) \in \mathcal{E}(\mathbb{R}; \mathcal{S}'_q(\mathbb{R}_q))$. We recall that u is defined by

$$\langle u, \psi\rangle = \int_{\mathbb{R}} \langle u_t, \psi(t,.)\rangle dt, \quad \psi \in \mathcal{S}(\mathbb{R}; \mathcal{S}_q(\mathbb{R}_q)).$$

Then, using (6), we have for any ψ in $\mathcal{S}(\mathbb{R}; \mathcal{S}_q(\mathbb{R}_q))$

$$\begin{aligned}
\langle \partial_t u - i\Delta_q u, \psi\rangle &= -\langle u, \partial_t \psi(t,.) + i\Delta_q \psi(t,.)\rangle \\
&= -\int_{\mathbb{R}} \langle u_t, \partial_t \psi(t,.) + i\Delta_q \psi(t,.)\rangle dt \\
&= -\int_{\mathbb{R}} \langle \mathcal{F}_q(u_t), (\mathcal{F}_q)^{-1}(\partial_t \psi(t,.) + i\Delta_q \psi(t,.))\rangle dt \\
&= -\int_{\mathbb{R}} \langle e^{-it|.|^2}\mathcal{F}_q(g), (\partial_t - i|.|^2)(\mathcal{F}_q)^{-1}\psi(t,.)\rangle dt.
\end{aligned}$$

Since

$$\partial_t \left(e^{-it|\xi|^2}(\mathcal{F}_q)^{-1}\psi(t,\xi)\right) = \left[\partial_t - i|\xi|^2\mathcal{F}_q)^{-1}\psi(t,\xi)\right] e^{-it|\xi|^2},$$

we see that

$$\begin{aligned}
\langle \partial_t u + i\Delta_q u, \psi\rangle &= -\int_{\mathbb{R}} \langle \mathcal{F}_q(g), \partial_t \left(e^{-it|.|^2}(\mathcal{F}_q)^{-1}\psi(t,.)\right)\rangle dt \\
&= -\int_{\mathbb{R}} \partial_t \langle \mathcal{F}_q(g), e^{-it|.|^2}(\mathcal{F}_q)^{-1}\psi(t,.)\rangle dt = 0.
\end{aligned}$$

Hence the existence of a solution u is shown. Let us now prove the uniqueness, which is equivalent to show that $u \equiv 0$ is the solution of the following problem

$$\begin{cases} \partial_t u - i\Delta_q u = 0 \quad \text{in } \mathcal{E}(\mathbb{R}; \mathcal{S}'_q(\mathbb{R}_q)), \\ u_{t|=0} = 0. \end{cases}$$

In fact, for all ψ in $\mathcal{S}(\mathbb{R}; \mathcal{S}_q(\mathbb{R}_q))$ we have

$$\langle \partial_t u - i\Delta_q u, \psi\rangle = -\int_{\mathbb{R}} \langle u_t, (\partial_t + i\Delta_q)\psi(t,.)\rangle dt = 0.$$

Although

$$\frac{d}{dt}\langle u_t, \psi(t,.)\rangle = \langle u_t^{(1)}, \psi(t,.)\rangle + \langle u_t, \partial_t \psi(t,.)\rangle,$$

therefrom

$$-\int_{\mathbb{R}} \frac{d}{dt}\langle u_t, \psi(t,.)\rangle dt + \int_{\mathbb{R}} \left[\langle u_t^{(1)}, \psi(t,.)\rangle - i\langle u_t, \Delta_q \psi(t,.)\rangle\right] dt = 0. \tag{11}$$

Since $\psi(-\infty,.)=\psi(\infty,.)$, we obtain

$$\int_{\mathbb{R}}\Big[\langle u_t^{(1)},\psi(t,.)\rangle - i\langle u_t,\Delta_q\psi(t,.)\rangle\Big]\,dt=0. \tag{12}$$

Besides, using the fact that $\mathcal{F}_q(u_t^{(1)})=(\mathcal{F}_q(u_t))^{(1)}$ and the relations (7) and (12) we deduce that

$$\int_{\mathbb{R}}\Big[\langle(\mathcal{F}_q(u_t))^{(1)},(\mathcal{F}_q)^{-1}\psi(t,.)\rangle + i\langle\mathcal{F}_q(u_t),|.|^2(\mathcal{F}_q)^{-1}\psi(t,.)\rangle\Big]\,dt=0 \tag{13}$$

for $\psi\in\mathcal{S}(\mathbb{R};\mathcal{S}_q(\mathbb{R}_q))$. If we choose ψ such that $(\mathcal{F}_q)^{-1}\psi(t,\xi)=e^{it|\xi|^2}\varphi(\xi)\chi(t)$ where φ in $\mathcal{S}_q(\mathbb{R}_q)$, χ in $\mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}}\Big[\langle(\mathcal{F}_q(u_t))^{(1)},e^{it|.|^2}\varphi\rangle + i\langle\mathcal{F}_q(u_t),|.|^2e^{it|.|^2}\varphi\rangle\Big]\,\chi(t)dt=0 \tag{14}$$

for $\chi\in\mathcal{S}(\mathbb{R})$. Hence we deduce that

$$\frac{d}{dt}\langle\mathcal{F}_q(u_t),e^{it|.|^2}\varphi\rangle=\langle(\mathcal{F}_q(u_t))^{(1)},e^{it|.|^2}\varphi\rangle+i\langle\mathcal{F}_q(u_t),|.|^2e^{it|.|^2}\varphi\rangle=0 \tag{15}$$

for $\in\mathcal{S}_q(\mathbb{R}_q)$. Thus for all φ in $\mathcal{S}_q(\mathbb{R}_q)$, the function $t\mapsto\langle\mathcal{F}_q(u_t),e^{it|.|^2}\varphi\rangle$ is constant. Finally, as $u_0=0$ then

$$\langle\mathcal{F}_q(u_t),e^{it|.|^2}\varphi\rangle=\langle\mathcal{F}_q(u_0),\varphi\rangle=0\quad\text{for } t\in\mathbb{R}\text{ and }\varphi\in\mathcal{S}_q(\mathbb{R}_q).$$

Then we deduce that $u=0$.

THEOREM 3. Let g be in $\mathcal{W}_q^{s,p}(\mathbb{R}_q)$, $s\in\mathbb{R}$ and $1\le p<\infty$, the solution given by the Theorem 2 belongs to $C(\mathbb{R};\mathcal{W}_q^{s,p}(\mathbb{R}_q))$. For $m\in\mathbb{N}$, $(u_t^{(m)})\in C(\mathbb{R};\mathcal{W}_q^{s-2m,p}(\mathbb{R}_q))$ and we have for $t\in\mathbb{R}$

$$\begin{cases}\|u_t\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}=\|g\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)},\\ \|u_t^{(m)}\|_{\mathcal{W}_q^{s-2m,p}(\mathbb{R}_q)}\le C_m\|g\|_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}\quad\text{for } m\in\mathbb{N}^*.\end{cases} \tag{16}$$

PROOF. By the formula (10), we have for all t in $\mathbb{R}$

$$\mathcal{F}_q(u_t)=e^{-it|\xi|^2}\mathcal{F}_q(g),$$

so, it is easy to deduce (16). Now, we will prove that for $m=1,2,...,u_t^{(m)}$ belongs to $C(\mathbb{R};\mathcal{W}_q^{s-2m,p}(\mathbb{R}_q))$. In fact, let $(t_n)_n$ be a sequence that converge to t_0 in $\mathbb{R}$, we have

$$\|u_{t_n}-u_{t_0}\|^2_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}=\int_{\mathbb{R}_q}(1+|\xi|^2)^{\frac{sp}{2}}|e^{-it_n|\xi|^2}-e^{-it_0|\xi|^2}|^p|\mathcal{F}_q(g)(\xi)|^pd_q\xi.$$

According to the dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\|u_{t_n}-u_{t_0}\|^2_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)}=0.$$

Elsewhere, from (10) we have

$$\mathcal{F}_q(u_t^{(m)}) = (-i|\xi|^2)^m e^{-it|\xi|^2}\mathcal{F}_q(g).$$

Hence, we obtain

$$\|u_{t_n}^{(m)} - u_{t_0}^{(m)}\|^p_{\mathcal{W}_q^{s,p}(\mathbb{R}_q)} = \int_{\mathbb{R}_q} (1+|\xi|^2)^{\frac{sp}{2}} |e^{-it_n|\xi|^2} - e^{-it_0|\xi|^2}|^p |\xi|^{2mp} |\mathcal{F}_q(g)(\xi)|^p d_q\xi.$$

Finally, the dominated convergence theorem leads to the result.

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