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Independent Set Neighborhood Union And Fractional Critical Deleted Graphs*

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Abstract

A graph G is called a fractional (k, n', m)-critical deleted graph if any n' vertices are removed from G the resulting graph is a fractional (k, m)-deleted graph. In this paper, we determine that for integers $k \geq 1$, $i \geq 2$, $n', m \geq 0$, $n > 4ki + n' + 4m - 4$, and $\delta(G) \geq k(i-1) + n' + 2m$, if

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n + n'}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, n', m)-critical deleted graph. The bound for independent set neighborhood union condition is sharp.

1 Introduction

The fractional factor problem can be regarded as a relaxation problem of the cardinality matching. It has wide applications in fields such as combinatorial polyhedron, network design and scheduling. For instance, in a communication network, several large data packets were to be sent to certain destinations via multiple channels. We can improve the efficiency of this task by dividing the large data packets into small parcels. The available distribution of data packets can be considered as a fractional flow problem. Moreover, it can be looked upon as a fractional factor problem if the sources and destinations of a network are different.

In theoretical model, the whole network can be expressed as a graph in which each vertex represents a site and each edge denotes a channel between two sites. In this setting, the framework of data transmission problem is the existence of fractional factor in the graph corresponding to a network.

All graphs considered in this paper are finite, loopless, and without multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For

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$S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G-S = G[V(G)\backslash S]$. For two vertex-disjoint subsets S and T of G, we use $e_G(S,T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* $d_G(x,y)$ between two vertices x and y is defined to be the length of a shortest path connecting them. The notation and terminology used but undefined in this paper can be found in [1].

Let $k \geq 1$ be an integer. A spanning subgraph F of G is called a *k-factor* if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0,1]$ be a function. If $\sum_{x \in e} h(e) = k$ for any $x \in V(G)$, then we call $G[F_h]$ a *fractional k-factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [10] introduced the concept of a fractional (k,m)-deleted graph, that is, a graph G is called a *fractional (k,m)-deleted graph* if removing any m edges from G, the resulting graph has a fractional k-factor. A fractional (k,m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$. A graph G is called a *fractional (k,n')-critical graph* if after deleting any n' vertices from G, the resulting graph still has a fractional k-factor.

A graph G is called a *fractional (k,n',m)-critical deleted graph* if after deleting any n' vertices from G, the resulting graph is still a fractional (k,m)-deleted graph.

In what follows, we always assume that $n = |V(G)|$. Yu et al. [8] determined a degree condition for the existence of a fractional k-factor as follows.

THEOREM 1 ([8]). Let $k \geq 1$ be an integer, and let G be a connected graph with $n \geq 4k-3$ and $\delta(G) \geq k$. If

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for each pair of nonadjacent vertices x, y of G, then G has a fractional k-factor.

Liu and Zhang [7] obtained the following toughness condition for a graph to have fractional k-factors.

THEOREM 2 ([7]). Let $k \geq 2$ be an integer. A graph G of order n with $n \geq k+1$ has a fractional k-factor if $t(G) \geq k - \frac{1}{k}$.

For fractional (k,m)-deleted graphs, the following known results are stated by Zhou.

THEOREM 3 ([10]). Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph with

$$n \geq 9k - 1 - \sqrt{2(k-1)^2 + 2} + 2(2k+1)m$$

and $\delta(G) \geq k + m + \frac{(m+1)^2 - 1}{4k}$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k,m)-deleted graph.

THEOREM 4 ([9]). Let $k \geq 1$ and $m \geq 1$ be two integers. Let G be a graph with $n \geq 4k - 5 + 2(2k + 1)m$. If $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k, m)-deleted graph.

Recently, Gao et al. gave the following result on the neighborhood union condition for fractional (k, n', m)-critical deleted graphs.

THEOREM 5. Let $k \geq 2$ and $n', m \geq 0$ be three integers, and let G be a graph with $n \geq 8k + n' + 4m - 7$ and $\delta(G) \geq k + n' + m$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{n + n'}{2}$$

for each pair of non-adjacent vertices x, y of G, then G is a fractional (k, n', m)-critical deleted graph.

More sufficient conditions for graphs to have fractional factors can be found in Gao and Gao [3], Gao et al. [4], Gao and Wang [5], Zhou [10, 11, 12], and Zhou and Bian [13].

In this paper, we manifest the relationship between independent set neighborhood union condition and fractional (k, n', m)-critical deleted graph. Furthermore, we will show that the bound for independent set neighborhood union is sharp in some sense. Our main result is stated as follows.

THEOREM 6. Let G be a graph of order n. Let k, i, n', m be four integers with $i \geq 2$, $k \geq 1$ and $m, n' \geq 0$. If $\delta(G) \geq k(i - 1) + 2m + n'$, $n > 4ki + n' + 4m - 4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n + n'}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, n', m)-critical deleted graph.

Set $n' = 0$ in Theorem 6, then it becomes the following necessary independent set neighborhood union condition on fractional (k, m)-deleted graph.

COROLLARY 1. Let G be a graph of order n. Let k, i, m be three integers with $i \geq 2$, $k \geq 1$ and $m \geq 0$. If $\delta(G) \geq k(i - 1) + 2m$, $n > 4ki + 4m - 4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, m)-deleted graph.

If $m = 0$ in Theorem 6, then we deduce the following corollary on the independent set neighborhood union condition of fractional (k, n')-critical graph.

COROLLARY 2. Let G be a graph of order n. Let k, i, n' be three integers with $i \geq 2$, $k \geq 1$ and $n' \geq 0$. If $\delta(G) \geq k(i-1)+n'$, $n > 4ki+n'-4$, and

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n+n'}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, n')-critical graph.

In order to prove our main result, we need the following lemma which present the necessary and sufficient condition of fractional (k, n', m)-critical deleted graph.

LEMMA 1 ([2]). Let $k \geq 1$ and $n', m \geq 0$ be three integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k, n', m)-critical deleted graph if and only if

$$\delta_G(S,T) = k\,|S| + \sum_{x \in T} d_{G-S}(x) - k\,|T| \geq kn' + \sum_{x \in T} d_H(x) - e_H(S,T),$$

for all disjoint subsets S and T of $V(G)$ with $|S| \geq n'$.

Since our result refers to independent set neighborhood union condition, we need new tricks compare to Gao et al. [6].

2 Proof of Theorem 6

Assume to the contrary that G satisfies the conditions of the Theorem 6, but is not a fractional (k, n', m)-critical deleted graph. By Lemma 1 and noting the fact that $\sum_{x \in T} d_H(x) - e_G(T,S) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = k\,|S| + \sum_{x \in T} d_{G-S}(x) - k\,|T| \leq kn' + 2m - 1. \tag{1}$$

We choose subsets S and T such that $|T|$ is minimal. Obviously, $T \neq \emptyset$.

CLAIM 1. $d_{G-S}(x) \leq k-1$ for any $x \in T$.

PROOF. If $d_{G-S}(x) \geq k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T.

Let $d_1 = \min\{d_{G-S}(x) | x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $z \geq 2$ and $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$, let

$$d_z = \min\{d_{G-S}(x) | x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and choose $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$ such that $d_{G-S}(x_z) = d_z$. So, we get a sequence such that $0 \leq d_1 \leq d_2 \leq \cdots \leq d_\pi \leq k-1$ and an independent set $\{x_1, x_2, \ldots, x_\pi\} \subseteq T$.

CLAIM 2. $|T|\geq k(i-1)+1$.

PROOF. Assume that $|T|\leq k(i-1)$. Then $|S|+d_1\geq d_G(x_1)\geq\delta(G)\geq k(i-1)+n'+2m$. By (1) and $0\leq d_1\leq k-1$, we have

$$\begin{aligned} kn'+2m-1 &\geq k|S|+\sum_{x\in T}d_{G-S}(x)-k|T| \\ &\geq k|S|+d_1|T|-k|T| \\ &= k|S|+(d_1-k)|T| \\ &\geq k(k(i-1)-d_1+n'+2m)+(d_1-k)(i-1)k \\ &= k^2(i-1)+d_1(k(i-1)-k)-k^2(i-1)+kn'+2km \\ &\geq kn'+2m. \end{aligned}$$

This produces a contradiction.

Since $d_{G-S}(x)\leq k-1$ and $|T|\geq k(i-1)+1$, we get $\pi\geq i$. Thus, we can choose an independent set $\{x_1,x_2,\ldots,x_i\}\subseteq T$.

In view of the condition of the theorem, we deduce

$$\frac{n+n'}{2}\leq|N_G(x_1)\cup N_G(x_2)\cup\cdots\cup N_G(x_i)|\leq|S|+\sum_{j=1}^{i}d_j$$

and

$$|S|\geq\frac{n+n'}{2}-\sum_{j=1}^{i}d_j. \tag{2}$$

Noting that

$$|N_T[x_j]|-\left|N_T[x_j]\cap(\cup_{z=1}^{j-1}N_T[x_z])\right|\geq 1,\ j=2,3,\ldots,i-1$$

and

$$|\cup_{z=1}^{j}N_T[x_z]|\leq\sum_{z=1}^{j}|N_T[x_z]|\leq\sum_{z=1}^{j}(d_{G-S}(x_z)+1)=\sum_{z=1}^{j}(d_z+1),\ j=1,2,\ldots,i.$$

Hence, we infer

$$\begin{aligned} & k|S|+\sum_{x\in T}d_{G-S}(x)-k|T| \\ \geq\ & k|S|-k|T|+d_1|N_T[x_1]|+d_2(|N_T[x_2]|-|N_T[x_2]\cap N_T[x_1]|) \\ & +\cdots+d_{i-1}(|N_T[x_{i-1}]|-\left|N_T[x_{i-1}]\cap(\cup_{j=1}^{i-2}N_T[x_j])\right|) \\ & +d_i(|T|-|\cup_{j=1}^{i-1}N_T[x_j])||) \\ \geq\ & k|S|+(d_1-d_i)|N_T[x_1]|+\sum_{j=2}^{i-1}d_j+(d_i-k)|T|-d_i\sum_{j=2}^{i-1}|N_T[x_j]| \end{aligned}$$

$$= \quad k|S| + (d_1 - d_i)(d_1+1) + \sum_{j=2}^{i-1} d_j + (d_i - k)|T| - d_i \sum_{j=2}^{i-1}(d_j+1)$$

$$= \quad k|S| + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i - k)|T| - d_i \sum_{j=1}^{i-1}(d_j+1),$$

which implies

$$\begin{aligned} & (n - |S| - |T|)(k - d_i) \\ \geq \ & k|S| + \sum_{x\in T} d_{G-S}(x) - k|T| - 2m - kn' + 1 \\ \geq \ & k|S| + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i - k)|T| - d_i \sum_{j=1}^{i-1}(d_j+1) - 2m - kn' + 1. \end{aligned}$$

Equivalently,

$$0 \leq n(k-d_i) - (2k-d_i)|S| + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i-1) - d_1^2 + 2m - kn' - 1. \quad (3)$$

By (2), (3), $d_1 \leq d_2 \leq \cdots \leq d_i \leq k-1$ and $n > 4ki + n' + 4m - 4$, we yield

$$\begin{aligned} 0 \quad \leq \quad & n(k-d_i) - (2k-d_i)(\frac{n+n'}{2} - \sum_{j=1}^{i} d_j) + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\ & + d_i(i-1) - d_1^2 + 2m + kn' - 1 \\ = \quad & -\frac{n}{2} d_i + 2k \sum_{j=1}^{i} d_j - d_i \sum_{j=1}^{i} d_j + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\ & + d_i(i-1) - d_1^2 + 2m + \frac{n'd_i}{2} - 1 \\ = \quad & -\frac{n}{2} d_i + ((2k-1)d_1 - d_1^2) + (2k-1)\sum_{j=2}^{i-1} d_j \\ & + d_i(2k+i-1) - d_i^2 + 2m + \frac{n'd_i}{2} - 1 \\ \leq \quad & -\frac{n}{2} d_i + (2k-1)d_i + (2k-1)\sum_{j=2}^{i-1} d_i \\ & + d_i(2k+i-1) - d_i^2 + 2m + \frac{n'd_i}{2} - 1 \\ = \quad & -\frac{n}{2} d_i + 2kid_i - d_i^2 + 2m + \frac{n'd_i}{2} - 1. \end{aligned}$$

If $d_i > 0$, then $0 < 2d_i - d_i^2 - 1 \leq 0$ since $n > 4ki + n' + 4m - 4$ and $2m(1-d_i) \leq 0$, a contradiction.

If $d_i = 0$, then $d_1 = \cdots = d_i = 0$. By (2), we have $|S| \geq \frac{n+n'}{2}$ and $|T| \leq n - |S| \leq \frac{n-n'}{2}$. Since $d_{G-S}(T) \geq \sum_{x \in T} d_H(x) - e_G(T, S)$, we have

$$\begin{aligned} & k\,|S| + \sum_{x\in T} d_{G-S}(x) - k\,|T| - (\sum_{x\in T} d_H(x) - e_G(T,S)) - kn' \\ \geq\ & k \cdot \frac{n+n'}{2} - k \cdot \frac{n-n'}{2} + (d_{G-S}(T) - \sum_{x\in T} d_H(x) + e_G(T,S)) - kn' \\ \geq\ & 0, \end{aligned}$$

also a contradiction. This completes the proof of the theorem.

3 Sharpness

Theorem 6 is best possible, in some extent, on the conditions. Actually, we can construct some graphs such that the independent set neighborhood union condition in Theorem 6 can't be replaced by $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{n+n'}{2} - 1$.

Let $G_1 = K_{kt+n'}$ be a complete graph, $G_2 = (kt+1)K_1$ be a graph consisting of $kt+1$ isolated vertices, and $G = G_1 \vee G_2$, where t is sufficiently large (i.e., $t > \frac{2ki+2m-2}{k} + n' - \frac{1}{2k}$ for some i. Thus, $\delta(G) \geq k(i-1) + n' + 2m$, and $n > 4ki + n' + 4m - 4$). Then $n = |G_1| + |G_2| = 2kt + n' + 1$, and for any independent set $\{x_1, x_2, \ldots, x_i\} \subseteq V(G_2)$, we have

$$\frac{n+n'}{2} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| = kt + n' > \frac{n+n'}{2} - 1.$$

Let $S = V(G_1)$, and $T = V(G_2)$. Then

$$\begin{aligned} & k\,|S| - k\,|T| + d_{G-S}(T) - \left(\sum_{x\in T} d_H(x) - e_G(T,S)\right) - kn' \\ =\ & k\,|S| - k\,|T| - kn' = k(kt+n') - k(kt+1) - kn' = -k < 0. \end{aligned}$$

Hence, G is not a fractional (k, n', m)-critical deleted graph.

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