Applied Mathematics E-Notes, 16(2016), 125-132 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Complete Classification Of Bifurcation Diagrams Of A Dirichlet Problem With Cubic Nonlinearity[*][†]

Kuo-Chih Hung[‡]

Received 21 May 2016

Abstract

We study the exact multiplicity of positive solutions and bifurcation diagrams of the Dirichlet boundary value problem

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(u-c) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, & a < b < c, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter. We give a complete classification of totally *seven* qualitatively different bifurcation diagrams. Our results extend those in Hung (J. Differential Equations 255 (2013) 3811–3831).

1 Introduction

In this paper we study the exact multiplicity of positive solutions and bifurcation diagrams of the Dirichlet boundary value problem

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(u-c) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, & a < b < c, \end{cases} \tag{1}$$

where $\lambda > 0$ is a bifurcation parameter. The cubic nonlinearity $f(u) \equiv (u-a)(u-b)(u-c)$ is concave-convex on $(0, \infty)$. We first observe that any positive solution of (1) must be symmetric about the origin. Moreover, the value of $\|u_\lambda\|_\infty = u_\lambda(0) = \rho$ uniquely identifies the solution pair $(\lambda, u_\lambda(x))$ (i.e. there is at most one λ, with at most one solution $u_\lambda(x)$, so that $\|u_\lambda\|_\infty = \rho$). Hence we define the bifurcation diagram of (1)

$$\Sigma = \left\{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1)}\right\}.$$

If $a > 0$, problem (1) becomes

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(u-c) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, & 0 < a < b < c. \end{cases} \tag{2}$$

Hung [1] determined completely the exact multiplicity of positive solutions and bifurcation diagrams of problem (2). See the following Theorem 1.1 and Fig. 1.

---

[*]Mathematics Subject Classifications: 34B18, 74G35.

[†]This work is partially supported by the Ministry of Science and Technology of Taiwan, ROC under the grant number MOST 104-2115-M-167-002.

[‡]Fundamental General Education Center, National Chin-Yi University of Technology, Taichung, Taiwan 411, ROC, kchung@ncut.edu.tw

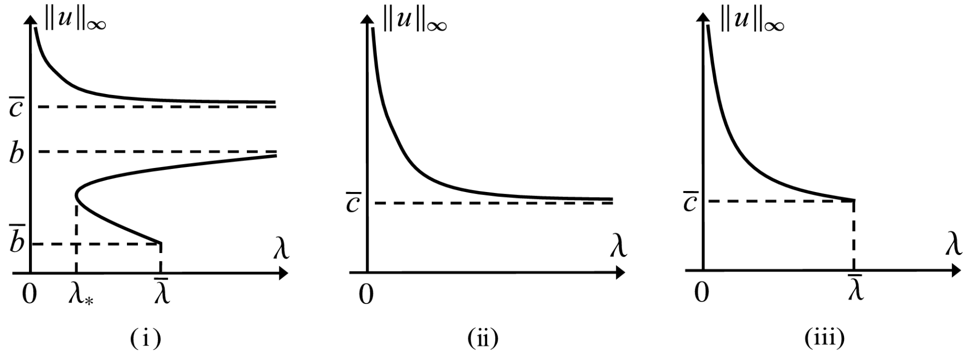

Fig. 1. The bifurcation diagrams of (2). (i) $2b(a+c) > b^2+6ac$. (ii) $2b(a+c) = b^2+6ac$. (iii) $2b(a+c) < b^2+6ac$.

Define $F(u) = \int_0^u f(t)dt$.

THEOREM 1.1. Consider problem (2)

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(u-c) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, & 0 < a < b < c. \end{cases}$$

Then the following assertions (i)–(iii) hold:

(i) (See Fig. 1(i).) If $2b(a+c) > b^2+6ac$, then $F(b) > 0$ and the bifurcation diagram Σ of (2) is broken reversed S-shaped on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exist $\bar{\lambda} > \lambda_* > 0$ such that (2) has exactly three positive solutions u_λ, v_λ, w_λ with $u_\lambda < v_\lambda < w_\lambda$ for all $\lambda_* < \lambda \le \bar{\lambda}$, exactly two positive solutions v_λ, w_λ with $v_\lambda < w_\lambda$ for $\lambda = \lambda_*$ and $\lambda > \bar{\lambda}$, and exactly one positive solution w_λ for $0 < \lambda < \lambda_*$. More precisely,

$$\lim_{\lambda\to\bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \bar{b},\ \lim_{\lambda\to\infty} \|v_\lambda\|_\infty = b,$$

$$\lim_{\lambda\to 0^+} \|w_\lambda\|_\infty = \infty,\ \lim_{\lambda\to\infty} \|w_\lambda\|_\infty = \bar{c},$$

where $\bar{b} \in (a,b)$, $\bar{c} \in (c,\infty)$ satisfying $F(\bar{b}) = 0$ and $F(\bar{c}) = F(b) > 0$.

(ii) (See Fig. 1(ii).) If $2b(a+c) = b^2+6ac$, then $F(b) = 0$ and the bifurcation diagram Σ of (2) is a monotone decreasing curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, (2) has exactly one positive solution u_λ for all $\lambda > 0$. More precisely,

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = \infty \ \text{ and } \ \lim_{\lambda\to\infty} \|u_\lambda\|_\infty = \bar{c},$$

where $\bar{c} \in (c,\infty)$ satisfying $F(\bar{c}) = F(b) = 0$.

(iii) (See Fig. 1(iii).) If $2b(a+c) < b^2+6ac$, then $F(b) < 0$ and the bifurcation diagram Σ of (2) is a monotone decreasing curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exists $\bar{\lambda} > 0$ such that (2) has exactly one positive solution u_λ for $0 < \lambda \le \bar{\lambda}$, and no positive solution for $\lambda > \bar{\lambda}$. More precisely,

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = \infty \ \text{ and } \ \lim_{\lambda\to\bar{\lambda}^-} \|u_\lambda\|_\infty = \|u_{\bar{\lambda}}\|_\infty = \bar{c},$$

where $\bar{c} \in (c,\infty)$ satisfying $F(\bar{c}) = 0$.

Our results in this paper are extensions of those of Hung [1] from $0 < a < b < c$ to $a < b < c$. In Theorem 2.1 stated below, we give a complete classification of totally *seven* qualitatively different bifurcation diagrams of (1).

A dual problem of (1) is

$$\begin{cases} u''(x) + \lambda(u-a)(u-b)(c-u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \ \ a < b < c. \end{cases} \tag{3}$$

The cubic nonlinearity $\tilde{f}(u) \equiv (u-a)(u-b)(c-u)$ is convex-concave on $(0,\infty)$. In a celebrated paper [6, Section 2], Smoller and Wasserman first systematically studied bifurcation diagrams of positive solutions of problem (3). For the case

$$a < b < 0 < c \ \text{ and } \ (a+b+c)/3 > 0,$$

Smoller and Wasserman [6, p. 277, lines 18–19] stated that "This case is rather difficult, and requires some new estimates." They proved partial results and left the remaining part as an open problem. Recently, Hung and Wang [2, Theorem 2.5] proved this open problem completely. For another case

$$0 \leq a < b < c,$$

this case arises from the studies of dynamics of the FitzHugh–Nagumo equation and population biology. Smoller and Wasserman [6, Theorem 2.1] succeeded in solving problem (3) for $a = 0$. For $a > 0$, Wang [7] and Korman, Li and Ouyang [4] independently proved partial results by using the techniques of time-mapping and techniques of bifurcation theory under different conditions, respectively. Further investigations are needed to prove the exact shape of the bifurcation curves without any restriction.

2 Main Result

Our results in this paper are extensions of those of Hung [1] from $0 < a < b < c$ to $a < b < c$. In the following theorem, we give a complete classification of totally *seven* qualitatively different bifurcation diagrams of (1). See Figs. 1 and 2. Note that Fig. 1(ii) is the same as Fig. 2(iii) and Fig. 1(iii) is the same as Fig. 2(iv).

THEOREM 2.1. Consider problem (1). Then the following assertions (i)–(vii) hold:

(i) (See Fig. 1(i)–(iii).) If $0 < a < b < c$, then all the results in Theorem 1.1 hold.

(ii) (See Fig. 2(i).) If $a = 0 < b < c$, then the bifurcation diagram Σ of (1) is broken $\supset$-shaped on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exists $\hat{\lambda} = \frac{\pi^2}{4bc}$ such that (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for all $\lambda > \hat{\lambda}$, and exactly one positive solution v_λ for $0 < \lambda \leq \hat{\lambda}$. More precisely,

$$\lim_{\lambda \to \hat{\lambda}^+} \|u_\lambda\|_\infty = 0, \ \lim_{\lambda \to \infty} \|u_\lambda\|_\infty = b,$$

$$\lim_{\lambda \to 0^+} \|v_\lambda\|_\infty = \infty, \ \lim_{\lambda \to \infty} \|v_\lambda\|_\infty = \bar{c},$$

where $\bar{c} \in (c, \infty)$ satisfying $F(\bar{c}) = F(b) > 0$.

(iii) (See Fig. 2(ii).) If $a < 0 < b < c$, then the bifurcation diagram Σ of (1) is broken $\supset$-shaped on the $(\lambda, ||u||_\infty)$-plane. Moreover, (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for all $\lambda > 0$. More precisely,

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0, \ \lim_{\lambda\to\infty} \|u_\lambda\|_\infty = b,$$
$$\lim_{\lambda\to 0^+} \|v_\lambda\|_\infty = \infty, \ \lim_{\lambda\to\infty} \|v_\lambda\|_\infty = \bar{c},$$

where $\bar{c} \in (c, \infty)$ satisfying $F(\bar{c}) = F(b) > 0$.

(iv) (See Fig. 2(iii) and Fig. 1(ii).) If $a < b = 0 < c$, then all the results in Theorem 1.1(ii) hold.

(v) (See Fig. 2(iv) and Fig. 1(iii).) If $a < b < 0 < c$, then all the results in Theorem 1.1(iii) hold.

(vi) (See Fig. 2(v).) If $a < b < c = 0$, then the bifurcation diagram Σ of (1) is a monotone decreasing curve on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exists $\hat{\lambda} = \frac{\pi^2}{4ab}$ such that (1) has exactly one positive solution u_λ for $0 < \lambda < \hat{\lambda}$, and no positive solution for $\lambda \geq \hat{\lambda}$. More precisely,

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = \infty \ \text{ and } \ \lim_{\lambda\to\hat{\lambda}^-} \|u_\lambda\|_\infty = 0.$$

(vii) (See Fig. 2(vi).) If $a < b < c < 0$, then the bifurcation diagram Σ of (1) is $\supset$-shaped on the $(\lambda, ||u||_\infty)$-plane. Moreover, there exist $\lambda^* > 0$ such that (1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $0 < \lambda < \lambda^*$, exactly one positive solution u_λ for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. More precisely,

$$\lim_{\lambda\to 0^+} \|v_\lambda\|_\infty = \infty \ \text{ and } \ \lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0.$$

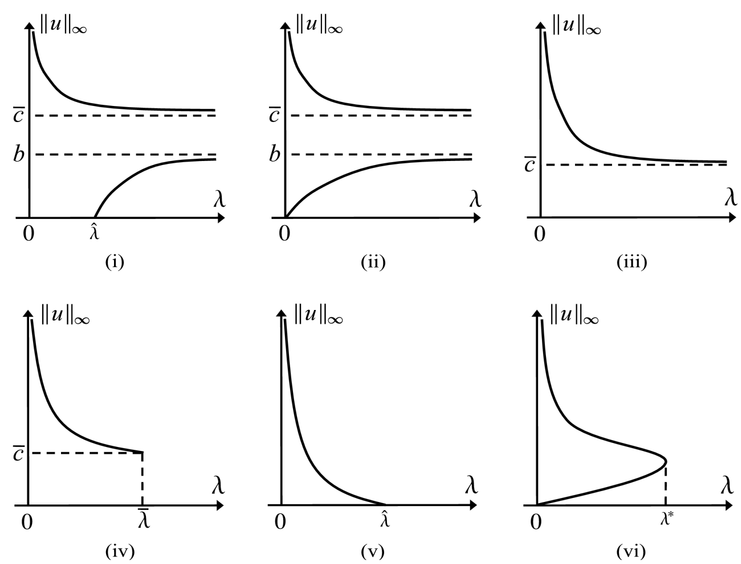

Fig. 2. The bifurcation diagrams of (1). (i) $a = 0 < b < c$. (ii) $a < 0 < b < c$. (iii) $a < b = 0 < c$. (iv) $a < b < 0 < c$. (v) $a < b < c = 0$. (vi) $a < b < c < 0$.

3 Lemmas

First, we give the time map formula as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^{\rho} \frac{1}{\sqrt{F(\rho) - F(u)}} du \equiv G(\rho) \quad \text{for } \rho \in I, \tag{4}$$

where the set $I \subset (0, \infty)$ is the set of $\rho \in (0, \infty)$ such that $G(\rho)$ is well defined; that is,

$$I = \{\rho \in (0, \infty) : F(\rho) \geq F(u) \text{ for all } u \in (0, \rho) \text{ and } G(\rho) \in (0, \infty)\}.$$

It is easy to check

$$I = \begin{cases} (0, b) \cup (\bar{c}, \infty) & \text{if } a \leq 0 < b < c, \\ (\bar{c}, \infty) & \text{if } a < b = 0 < c, \\ [\bar{c}, \infty) & \text{if } a < b < 0 < c, \\ (0, \infty) & \text{if } a < b < c \leq 0. \end{cases}$$

We omit the case $0 < a < b < c$ since it was full discussed in [1]. Note that $G(\rho)$ is a continuous function on I, and positive solutions u_λ of (1) correspond to

$$\|u_\lambda\|_\infty = \rho \in I \quad \text{and} \quad G(\rho) = \sqrt{\lambda}. \tag{5}$$

Thus, studying of the exact number of positive solutions of (1) is equivalent to studying the shape of the time map $G(\rho)$ on I.

To prove Theorem 2.1, we need the following four lemmas.

In the following Lemma 3.1, we determine the limits of $G(\rho)$ on I; that is, we determine the boundary behaviors of the bifurcation diagram Σ of (1). The proofs of Lemma 3.1 are easy but tedious, and hence we omit them. See also Laetsch [5, Theorems 2.5-2.10].

LEMMA 3.1. Consider (4). Then the following assertions (i)–(vi) hold:

(i) $\lim_{\rho\to\infty} G(\rho) = 0$.

(ii) If $f(0) = -abc > 0$, $\lim_{\rho\to 0^+} G(\rho) = 0$.

(iii) If $f(0) = -abc = 0$ and $f'(0) = ab + bc + ca > 0$, $\lim_{\rho\to 0^+} G(\rho) = \frac{\pi}{2\sqrt{ab+bc+ca}} \in (0, \infty)$.

(iv) If $a \leq 0 < c$ and $f(0) = -abc \geq 0$, then $\lim_{\rho\to\bar{c}^+} G(\rho) = \infty$.

(v) If $a \leq 0 < c$ and $f(0) = -abc < 0$, then $\lim_{\rho\to\bar{c}^+} G(\rho) = G(\bar{c}) \in (0, \infty)$ and $\lim_{\rho\to\bar{c}^+} G'(\rho) = -\infty$.

(vi) If $a \leq 0 < b$, then $\lim_{\rho\to b^-} G(\rho) = \infty$.

In the following two lemmas, we give sufficient conditions for the monotonicity of $G(\rho)$.

LEMMA 3.2. Consider (4). Assume that, for some $\beta > 0$, either

$$f(u) - uf'(u) > 0 \text{ on } (0, \beta) \subset I,$$

or the opposite inequality holds. Then $G(\rho)$ is a monotone function on $(0, \beta)$.

PROOF. It is easy to verify by

$$G'(\rho) = \frac{1}{2\sqrt{2}\rho} \int_0^\rho \frac{[f(\rho) - \rho f'(\rho)] - [f(u) - uf'(u)]}{[F(\rho) - F(u)]^{3/2}} du \text{ for } \rho \in (0, \beta).$$

The proof of Lemma 3.2 is complete.

Lemma 3.3 is due to [3, Theorem 2.1] which was proved by applying generalized averages.

LEMMA 3.3. Consider (4). Assume that, for some $\beta > 0$,

$$f(\beta) = 0 \text{ and } f(u),\ f''(u) > 0 \text{ on } (\beta, \infty).$$

Then $G(\rho)$ is a monotone function on the set $J \subset (\beta, \infty)$, where J is the set of $\rho \in (\beta, \infty)$ such that $G(\rho)$ is well defined.

The following lemma mainly follows by applying [5, Theorem 3.2]. We omit the proofs.

LEMMA 3.4. Consider (4). Assume that $f(u)$ is positive and convex on $[0, \infty)$. Then $G(\rho)$ is either monotone increasing on $(0, \infty)$, or $G(\rho)$ is monotone increasing on $(0, \gamma)$ and monotone decreasing on (γ, ∞) for some $\gamma \in (0, \infty)$.

4 Proof of Main Result

In this section, we prove Theorem 2.1. To the end, we consider the next seventh cases.

Case (i): If $0 < a < b < c$, the results was proved by [1] and hence we omit the proofs.

Case (ii): If $a = 0 < b < c$, we obtain the boundary behaviors by Lemma 3.1.

(1) $\lim_{\rho \to 0^+} G(\rho) = \frac{\pi}{2\sqrt{bc}} \in (0, \infty)$, $\lim_{\rho \to b^-} G(\rho) = \infty$, $\lim_{\rho \to \bar{c}^+} G(\rho) = \infty$, and $\lim_{\rho \to \infty} G(\rho) = 0$, where $\bar{c} \in (c, \infty)$ satisfying $F(\bar{c}) = F(b) > 0$.

Let $h(u) = 2F(u) - uf(u)$, then $h'(u) = f(u) - uf'(u)$ and $h''(u) = -uf''(u)$. Since $h(0) = h'(0) = 0$, $h'(b) = -bf'(b) > 0$, and $h(u)$ is convex-concave on $(0, \infty)$ ($f(u)$ is concave-convex on $(0, \infty)$). It can be proved that $h'(u) = f(u) - uf'(u) > 0$ on $(0, b)$. Then by property (1) and Lemma 3.2, we have

(2) $G(\rho)$ is monotone increasing on $(0, b)$.

Since $f(c) = 0$ and $f(u)$, $f''(u) > 0$ on (c, ∞), by property (1) and Lemma 3.3, we have

(3) $G(\rho)$ is monotone decreasing on $(\bar{c}, \infty)$.

Thus by (5) and properties (1)–(3), we immediately obtain the results in case (ii). The bifurcation diagram Σ is depicted in Fig. 2(i).

Case (iii): If $a < 0 < b < c$, we obtain the boundary behaviors by Lemma 3.1.

(1) $\lim_{\rho \to 0^+} G(\rho) = 0$, $\lim_{\rho \to b^-} G(\rho) = \infty$, $\lim_{\rho \to \bar{c}^+} G(\rho) = \infty$, and $\lim_{\rho \to \infty} G(\rho) = 0$, where $\bar{c} \in (c, \infty)$ satisfying $F(\bar{c}) = F(b) > 0$.

By the same analysis in case (ii), it can be proved that $h'(u) = f(u) - uf'(u) > 0$ on $(0, b)$. Then by property (1) and Lemma 3.2, we have

(2) $G(\rho)$ is monotone increasing on $(0, b)$.

Since $f(c) = 0$ and $f(u)$, $f''(u) > 0$ on (c, ∞), by property (1) and Lemma 3.3, we have

(3) $G(\rho)$ is monotone decreasing on $(\bar{c}, \infty)$.

Thus by (5) and properties (1)–(3), we immediately obtain the results in case (iii). The bifurcation diagram Σ is depicted in Fig. 2(ii).

Case (iv): If $a < b = 0 < c$, we obtain the boundary behaviors by Lemma 3.1.

(1) $\lim_{\rho \to \bar{c}^+} G(\rho) = \infty$ and $\lim_{\rho \to \infty} G(\rho) = 0$, where $\bar{c} \in (c, \infty)$ satisfying $F(\bar{c}) = F(b) = 0$.

Since $f(c) = 0$ and $f(u)$, $f''(u) > 0$ on (c, ∞), by property (1) and Lemma 3.3, we have

(2) $G(\rho)$ is monotone decreasing on $(\bar{c}, \infty)$.

Thus by (5) and properties (1) and (2), we immediately obtain the results in case (iv). The bifurcation diagram Σ is depicted in Fig. 2(iii).

Case (v): If $a < b < 0 < c$, we obtain the boundary behaviors by Lemma 3.1.

(1) $\lim_{\rho \to \bar{c}^+} G(\rho) = G(\bar{c}) \in (0, \infty)$, $\lim_{\rho \to \bar{c}^+} G'(\rho) = -\infty$, and $\lim_{\rho \to \infty} G(\rho) = 0$, where $\bar{c} \in (c, \infty)$ satisfying $F(\bar{c}) = 0$.

Since $f(c) = 0$ and $f(u)$, $f''(u) > 0$ on (c, ∞), by property (1) and Lemma 3.3, we have

(2) $G(\rho)$ is monotone decreasing on $[\bar{c}, \infty)$.

Thus by (5) and properties (1) and (2), we immediately obtain the results in case (v). The bifurcation diagram Σ is depicted in Fig. 2(iv).

Case (vi): If $a < b < c = 0$, we obtain the boundary behaviors by Lemma 3.1.

(1) $\lim_{\rho\to 0^+} G(\rho) = \frac{\pi}{2\sqrt{ab}} \in (0,\infty)$ and $\lim_{\rho\to\infty} G(\rho) = 0$.

Since $f(0) = 0$ and $(f(u) - uf'(u))' = -uf''(u) < 0$ on $(0,\infty)$, it can be proved that $f(u) - uf'(u) < 0$ on $(0,\infty)$. Then by property (1) and Lemma 3.2, we have

(2) $G(\rho)$ is monotone decreasing on $(0,\infty)$.

Thus by (5) and properties (1) and (2), we immediately obtain the results in case (vi). The bifurcation diagram Σ is depicted in Fig. 2(v).

Case (vii): If $a < b < c < 0$, we obtain the boundary behaviors by Lemma 3.1.

(1) $\lim_{\rho\to 0^+} G(\rho) = \lim_{\rho\to\infty} G(\rho) = 0$.

Since $f(u)$ is positive and convex on $[0,\infty)$, then by property (1) and Lemma 3.4, we have

(2) $G(\rho)$ is monotone increasing on $(0,\gamma)$ and monotone decreasing on (γ,∞) for some $\gamma \in (0,\infty)$.

Thus by (5) and properties (1) and (2), we immediately obtain the results in case (vii). The bifurcation diagram Σ is depicted in Fig. 2(vi).

The proof of Theorem 2.1 is now complete.

Acknowledgment. The author thanks the anonymous referee for a careful reading of the manuscript and valuable suggestions on the manuscript.

References

[1] K.-C. Hung, Exact multiplicity of positive solutions of a semipositone problem with concave-convex nonlinearity, J. Differential Equations, 255(2013), 3811–3831.

[2] K.-C. Hung and S.-H. Wang, Global bifurcation and exact multiplicity of positive solutions for a positone problem with cubic nonlinearity and their applications, Trans. Amer. Math. Soc., 365(2013), 1933–1956.

[3] P. Korman, Application of generalized averages to uniqueness of solutions, and to a non-local problem, Comm. Appl. Nonlinear Anal., 12(2005), 69–78.

[4] P. Korman, Y. Li and T. Ouyang, Exact multiplicity results for boundary value problems with nonlinearities generalising cubic, Proc. Roy. Soc. Edinburgh Sect. A, 126(1996), 599–616.

[5] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, Indiana Univ. Math. J., 20(1970), 1–13.

[6] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, J. Differential Equations, 39(1981), 269–290.

[7] S.-H. Wang, A correction for a paper by J. Smoller and A. Wasserman, J. Differential Equations, 77(1989), 199–202.