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Strong Domination Numbers Of Vague Graphs With Applications*

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Abstract

A vague graph is a generalized structure of a fuzzy graph that gives more precision, flexibility, and compatibility to a system when compared with systems that are designed using fuzzy graphs. In this paper, we introduce the concept of strong domination numbers of vague graphs. The strong domination numbers of any complete (bipartite) vague graph are determined, and bounds are obtained for the strong numbers of vague graphs. Finally, we present some applications of strong domination numbers.

1 Introduction

In the classical set theory introduced by Cantor, values of elements in a set are either 0 or 1. That is, for any element, there are only two possibilities: the element is either in the set or it is not. Therefore, Cantor set theory cannot handle data with ambiguity and uncertainty. In 1965, Zadeh [16], proposed fuzzy theory and introduced fuzzy set theory. Fuzzy graph theory is defined by Rosenfeld [15] in 1975 as finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful as they aim to reduce the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Since domination of graph measures the influence of each node to the other nodes, then there exists many applications of domination in graphs. Bhutani and Rosenfeld [3] in 2003 introduced the concept of strong arc in fuzzy graphs. In addition Gani and Chandrasekaran [13] in 2006 introduced the concept of domination in fuzzy graphs using strong arcs, and Manjusha and Sunitha [12] in 2015 defined the strong domination number of fuzzy graph using the weights of strong arc. In a fuzzy set, each element is associated with a point value selected from the unit interval $[0, 1]$, which is termed the grade of membership in the set. Instead of using point-based membership as in fuzzy sets, interval based membership is used in a vague set. The interval based membership in vague sets is more flexible in capturing vagueness of data. Gau and

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Buherer [11] in 1993 proposed the concept of vague set, by replacing the value of an element in a set with a subinterval of $[0, 1]$. Namely, a true-membership function $t_A(x)$ and a false membership function $f_A(x)$ are used to describe the boundaries of the membership degree. Ramakrishna [14] in 2009 introduced the concept of vague graph and studied some of their properties. Borzooei and Rashmanlou [4–10] (2015,2016) introduce the concepts of cardinality, dominating set, independent, total dominating number, independent dominating number, regularity, and semi global domination sets in vague graphs. Akram et al. [1, 2] (2013,2014) defined certain types of vague graphs. In this paper, we introduced the new concepts of dominating sets in vague graphs and defined the concepts of strong domination numbers of value graphs using sum of the membership values of strong arcs in dominating sets and we proved interesting results of them.

2 Preliminaries

In this section, we introduced some preliminary notions and definitions which are used in this paper.

DEFINITION 1([14, 11]). A *vague set* A on a non-empty set X is a pair (t_A, f_A) where $t_A\colon X \to [0, 1]$ and $f_A\colon X \to [0, 1]$ are true and false membership functions, respectively such that

$$0 \le t_A(x) + f_A(x) \le 1 \text{ for any } x \in X.$$

Let X and Y be two ordinary non-empty sets. A *vague relation* R of X to Y is a vague set R on $X \times Y$, that is $R = (t_R, f_R)$, where $t_R\colon X \times Y \to [0, 1]$, $f_R\colon X \times Y \to [0, 1]$ which satisfies the condition

$$0 \le t_R(x, y) + f_R(x, y) \le 1 \text{ for all } (x, y) \in X \times Y.$$

Let $G^* = (V, E)$ be a graph. A pair $G = (A, B)$ is called a *vague graph* on G^* or a vague graph where $A = (t_A, f_A)$ is a vague set on V and $B = (t_B, f_B)$ is a vague set on $E \subseteq V \times V$ such that for each $xy \in E$,

$$t_B(xy) \le \min\left(t_A(x), t_A(y)\right) \text{ and } f_B(xy) \ge \max\left(f_A(x), f_A(y)\right).$$

DEFINITION 2([14]). Let $G = (A, B)$ be a vague graph.

(i) An arc uv in G is called a *effective* if

$$t_B(uv) = f_A(u) \wedge f_A(v) \text{ and } f_B(uv) = f_A(u) \vee f_A(v).$$

(ii) G is said to be a *strong vague graph* if for all $uv \in E$, uv is effective arc.

(iii) G is called a *complete vague graph* if for every $u, v \in V$,

$$\big(t_B(uv), f_B(uv)\big) = \big(t_A(u) \wedge t_A(v), f_A(u) \vee f_A(v)\big).$$

A complete vague graph with n nodes is denoted by K_n.

DEFINITION 3 ([5, 14]). Let $G = (A, B)$ be a vague graph and $u, v \in V$.

(i) A *path* ρ in G is a sequence of distinct nodes $v_0, v_1, v_2, ..., v_k$ such that

$$(t_B(v_{i-1}v_i), f_B(v_{i-1}v_i)) > 0, i = 1, ..., k.$$

Here k is called the *length* of the path ρ.

(ii) If u and v are connected by means of a path of length k such as

$$\rho : u = u_0, u_1, ..., u_{k-1}, u_k = v,$$

then $t_B^k(uv)$ and $f_B^k(uv)$ are defined by

$$t_B^k(uv) = \sup \{t_B(u, u_1) \wedge t_B(u_1, u_2) \wedge ... \wedge t_B(u_{k-1}, v)\},$$

$$f_B^k(uv) = \inf \{f_B(u, u_1) \vee f_B(u_1, u_2) \vee ... \vee f_B(u_{k-1}, v)\}.$$

And the *strength of connectedness* between two nodes u and v in G is defined as follows,

$$\big(t_B^\infty(uv), f_B^\infty(uv)\big) = \big(sup_{k\in N}\{t_B^k(uv)\}, inf_{k\in N}\{f_B^k(uv)\}\big).$$

(iii) An arc uv in G is called a *strong* if

$$t_B(uv) \geq t_B^\infty(uv) \text{ and } f_B(uv) \leq f_B^\infty(uv).$$

(iv) For $u, v \in V$ we say that u *dominated* v in G, if there exists a strong arc between them.

(n) A subset D of V is called a *dominating set* in G if for every $v \in V \backslash D$, there exists $u \in D$ such that u dominates v. A dominating set D in G is said to be *minimal dominating set* if no proper subsetof D is a dominating set.

3 Strong (Neighborhood) Domination Number

In this section, we define the strong and the strong neighborhood domination numbers of a vague graph and introduce the concepts of strong size and strong order of a vague graph. Then we prove that in certain types of vague graphs, the strong domination number (or the strong neighborhood domination number) is equal to the strong size (or strong order). Finally, we obtain the upper bounds on strong and strong neighborhood domination numbers in a vague graph.

DEFINITION 4. Let $G = (A, B)$ be a vague graph and $u \in V$. Then $v \in V$ is called a *strong neighbor of* u, if uv is a strong arc. The set of strong neighbors of u is called the *strong neighborhood of* u and is denoted by $N_s(u)$. The *closed strong neighborhood* of u is defined as $N_s[u] = N_s(u) \cup \{u\}$.

DEFINITION 5. Let $G = (A, B)$ be a vague graph and $v \in V$.

(i) The *strong degree* and the *strong neighborhood degree* of v are defined, respectively as

$$d_s(v) = \left(\sum_{u \in N_s(v)} t_B(uv), \sum_{u \in N_s(v)} f_B(uv) \right),$$

$$d_{sN}(v) = \left(\sum_{u \in N_s(v)} t_A(u), \sum_{u \in N_s(v)} f_A(u) \right).$$

(ii) The *strong degree cardinality* and the *strong neighborhood degree cardinality* of v are defined, respectively as

$$|d_s(v)| = \sum_{u \in N_s(v)} \frac{t_B(uv) + \big(1 - f_B(uv)\big)}{2},$$

$$|d_{sN}(v)| = \sum_{u \in N_s(v)} \frac{t_A(u) + \big(1 - f_A(u)\big)}{2}.$$

If $N_s(v) = \{\varnothing\}$, then

$$d_s(v) = (0,1) \longrightarrow |d_s(v)| = 0$$

and also

$$d_{sN}(v) = (0,1) \longrightarrow |d_{sN}(v)| = 0.$$

(iii) The *minimum* and *maximum strong degree* of G are defined, respectively as

$$\delta_s(G) = \wedge\{|d_s(v)| \;\big|\; \forall v \in V\} \text{ and } \Delta_s(G) = \vee\{|d_s(v)| \;\big|\; \forall v \in V\}.$$

(iv) The *minimum* and *maximum strong neighborhood degree* of G are defined, respectively as

$$\delta_{sN}(G) = \wedge\{|d_{sN}(v)| \;\big|\; \forall v \in V\} \text{ and } \Delta_{sN}(G) = \vee\{|d_{sN}(v)| \;\big|\; \forall v \in V\}.$$

EXAMPLE 6. Consider a vague graph $G = (A, B)$ in Figure 1. We see that ab, bc and ad are strong arcs and so we get,

$$d_s(a) = (0.4, 1.1)\ ,\ d_s(b) = (0.5, 1.2)\ ,\ d_s(c) = (0.3, 0.7)\ ,\ d_s(d) = (0.2, 0.6)$$

$$|d_s(a)| = \frac{1.3}{2} = 0.65,\ |d_s(b)| = \frac{1.3}{2} = 0.65,\ |d_s(c)| = \frac{0.6}{2} = 0.3,\ |d_s(d)| = \frac{0.6}{2} = 0.3.$$

Hence

$$\delta_s(G) = 0.3 \text{ and } \Delta_s(G) = 0.65.$$

And also,

$$d_{sN}(a) = (0.6, 0.9),\ d_{sN}(b) = (0.5, 1.1),\ d_{sN}(c) = (0.3, 0.4),\ d_{sN}(d) = (0.2, 0.4)$$

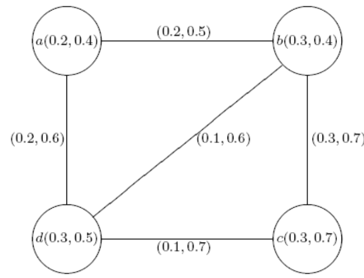

Figure 1: Vague graph G.

$$|d_{sN}(a)| = 0.85,\ \ |d_{sN}(b)| = 0.7,\ \ |d_{sN}(c)| = 0.45,\ \ |d_{sN}(d)| = 0.4.$$

Therefore

$$\delta_{sN}(G) = 0.4 \text{ and } \Delta_{sN}(G) = 0.85.$$

PROPOSITION 7. Let $G = (A, B)$ be a vague graph. Then

$$\delta_s(G) \leq \delta_{sN}(G) \text{ and } \Delta_s(G) \geq \Delta_{sN}(G).$$

REMARK 8. Let G be a vague graph. If t_A and f_A are constant functions, then

$$\delta_s(G) = \delta_{sN}(G) \text{ and } \Delta_s(G) = \Delta_{sN}(G).$$

DEFINITION 9. The *strong size* and the *strong order* of vague graph G are defined respectively as

$$S_s(G) = \left\{ \sum_{uv \in E} \frac{t_B(uv) + \big(1 - f_B(uv)\big)}{2} \ \Big|\ \text{uv is a strong arc} \right\}$$

and

$$O_s(G) = \left\{ \sum_{v \in V} \frac{t_A(v) + \big(1 - f_A(v)\big)}{2} \ \Big|\ \text{v is an end node of a strong arc} \right\}.$$

EXAMPLE 10. Consider the vague graph G in Figure 1. For strong arcs ab, bc and ad in G, we get

$$S_s(G) = \frac{(0.2 + 0.5) + (0.2 + 0.4) + (0.3 + 0.3)}{2} = \frac{1.9}{2} = 0.95,$$

$$O_s(G) = \frac{(0.2 + 0.6) + (0.3 + 0.6) + (0.3 + 0.3) + (0.3 + 0.5)}{2} = \frac{3.1}{2} = 1.55.$$

DEFINITION 11. Let D be a dominating set in a vague graph G. The *arc weight* and the *node weight* of D are defined as follows, respectively

$$W_e(D) = \sum_{u\in D, v\in N_s(u)} \wedge\{t_B(uv)\} + \Big(1 - \vee\Big\{\frac{f_B(uv)}{2}\Big\}\Big)$$

and

$$W_v(D) = \sum_{u\in D, v\in N_s(u)} \wedge\{t_A(v)\} + \Big(1 - \vee\Big\{\frac{f_A(v)}{2}\Big\}\Big).$$

The *strong domination number* and the *strong neighborhood domination number* of G are defined as the minimum arc weight and minimum node weight of dominating sets in G and are denoted by $\gamma_s(G)$ and $\gamma_{sN}(G)$, respectively.

EXAMPLE 12. Consider the vague graph G in Figure 1. The dominating sets in G are $D_1 = \{a, b\}$, $D_2 = \{a, c\}$, $D_3 = \{b, d\}$, $D_4 = \{c, d\}$, $D_5 = \{a, b, c\}$, $D_6 = \{a, b, d\}$, $D_7 = \{b, c, d\}$ and $D_8 = \{c, d, a\}$. Therefore

$$W_e(D_1) = 0.55,\ W_e(D_2) = 0.6,\ W_e(D_3) = 0.55,\ W_e(D_4) = 0.6,$$

$$W_e(D_5) = 0.85,\ W_e(D_6) = 0.85,\ W_e(D_7) = 0.85,\ W_e(D_8) = 0.9.$$

Then $\gamma_s(G) = 0.55$. In addition, we have

$$W_v(D_1) = 0.65,\ W_v(D_2) = 0.85,\ W_v(D_3) = 0.65,\ W_v(D_4) = 0.85,$$

$$W_v(D_5) = 1.1,\ W_v(D_6) = 1.05,\ W_v(D_7) = 1.1,\ W_v(D_8) = 1.25.$$

Then $\gamma_{sN}(G) = 0.65$

THEOREM 13. Let $G = (A, B)$ be a non trivial vague graph. Then

(i) $\gamma_s(G) < S_s(G)$.

(ii) $\gamma_s(G) = S_s(G)$ if and only if each node has at most one strong neighbor.

(iii) $\gamma_{sN}(G) < O_s(G)$.

(iv) $\gamma_{sN}(G) = O_s(G)$ if and only if G has no strong arc.

PROOF. The proofs of (i) and (iii) are trivial. Suppose that, each node has at most one strong neighbor. Since each arc has two end nodes hence for every $u, v \in V$ any dominating set in G is of the form

$$D = \big\{\{u\} \cup \{v\} \big| N_s(u) = \{\varnothing\},\ v \text{ is the only one end node of every strong arc}\big\}.$$

Hence for every dominating set as D in G we get,

$$\gamma_s(G) = \sum_{u\in D, v\in N_s(u)} \frac{t_B(uv) + \big(1 - f_B(uv)\big)}{2} = S_s(G).$$

Conversely, suppose that $\gamma_s(G) = S_s(G)$ and there exists one node as w in G that has exactly two strong neighbors. Hence w is in the dominating sets of G and so

$$\begin{aligned}\gamma_s(G) &= \sum_{v\in D, u\in N_s(v)} \frac{\wedge\{t_B(uv)\} + \big(1-\vee\{f_B(uv)\}\big)}{2} \\ &< |d_s(w)| + \sum_{u,v\neq w, uv\in E} \frac{t_B(uv) + \big(1-f_B(uv)\big)}{2} = S_s(G).\end{aligned}$$

Then $\gamma_s(G) < S_s(G)$, that is a contradiction. So (iii) holds. It is clear that if G hasn't any strong arcs, then $D = G$ is the only dominating set of G therefore $\gamma_{sN}(G) = O_s(G) = 0$. Conversely, suppose that $\gamma_{sN}(G) = O_s(G)$ and there exists one strong arc uv in G. Then

$$D_1 = \{w\big|u \neq w \in V\} \text{ and } D_2 = \{w\big|v \neq w \in V\}$$

are dominating sets of G, so

$$\gamma_{sN}(G) = w_v(D_1) = \frac{t_A(u) + \big(1 - f_A(u)\big)}{2}$$

or

$$\gamma_{sN}(G) = w_v(D_2) = \frac{t_A(v) + \big(1 - f_A(v)\big)}{2}.$$

And also

$$O_s(G) = \frac{t_A(u) + \big(1 - f_A(u)\big)}{2} + \frac{t_A(v) + \big(1 - f_A(v)\big)}{2}.$$

Hence $\gamma_{sN}(G) < O_s(G)$ that is a contradiction. So (iv) holds.

THEOREM 14. Let D be a dominating set in vague graph G. Then

(i) $\gamma_s(G) < \gamma_{sN}(G)$.

(ii) if G is a strong vague graph and for every $u \in D$ and $v \in N_s(u)$, $t_A(u) \geq t_A(v)$ and $f_A(u) \leq f_A(v)$, then $\gamma_s(G) = \gamma_{sN}(G)$.

PROOF. (i) Since

$$t_B(uv) \leq t_A(u) \wedge t_A(v) \text{ and } f_B(uv) \geq f_A(u) \vee f_A(v) \text{ for } u, v \in V,$$

we see that $\gamma_s(G) < \gamma_{sN}(G)$. (ii) Since G is a strong vague graph, then each arc in G is strong. Suppose that D is a dominating set in G and

$$t_A(u) \geq t_A(v), f_A(u) \leq f_A(v) \text{ for } u \in D, v \in N_s(u).$$

So by Definition 13, we get

$$\begin{aligned}W_e(D) &= \sum_{u\in D, v\in N_s(u)} \frac{\wedge\{t_B(uv)\} + \big(1-\vee\{f_B(uv)\}\big)}{2} \\ &= \sum_{u\in D, v\in N_s(u)} \frac{\wedge\{t_A(v)\} + \big(1-\vee\{f_A(v)\}\big)}{2}\end{aligned}$$

and also

$$W_v(D) = \sum_{u\in D, v\in N_s(u)} \frac{\wedge\{t_A(v)\} + \big(1 - \vee\{f_A(v)\}\big)}{2}.$$

Therefore $W_e(D) = W_v(D)$ and $\gamma_s(G) = \gamma_{sN}(G)$.

LEMMA 15. Any effective arc in a vague graph $G = (A, B)$ is a strong arc.

PROOF. Suppose that uv is a effective arc in G. Then

$$\big(t_B(uv), f_B(uv)\big) = \big(t_A(u) \wedge t_A(v), f_A(u) \vee f_A(v)\big).$$

So by Definition 3, we get

$$\begin{aligned} t_B^\infty(uv) &= sup\{t_B^k(uv) \mid k = 1, 2, ...\} \\ &= \Big(t_B(uv)\Big) \bigvee \Big(t_B^2(uv)\Big) \bigvee \Big(t_B^3(uv)\Big) \bigvee ... \\ &= \Big(t_B(uv)\Big) \bigvee \Big(\vee \big(t_B(uu_1) \wedge t_B(u_1v)\big)\Big) \bigvee ... \\ &= \Big(t_A(u) \wedge t_A(v)\Big) \bigvee \Big(t_A(u) \wedge t_A(u_1) \wedge t_A(v)\Big) \bigvee ... \\ &\leq \Big(t_A(u) \wedge t_A(v)\Big) \bigvee \big(t_A(u) \wedge t_A(v)\Big) \bigvee \Big(t_A(u) \wedge t_A(v)\Big) \bigvee ... \\ &= \big(t_A(u) \wedge t_A(v)\big) = t_B(uv). \end{aligned}$$

And similarly we have $f_B(uv) \geq f_B^\infty(uv)$. Therefore

$$t_B(uv) \geq t_B^\infty(uv) \text{ and } f_B(uv) \leq f_B^\infty(uv).$$

By Definition 3, uv is strong arc.

THEOREM 16. If $G = (A, B)$ be a complete vague graph. Then

$$\gamma_s(G) = \frac{\wedge\{t_A(v)\} + \big(1 - \vee\{f_A(v)\}\big)}{2} = \gamma_{sN}(G) \text{ for } v \in V.$$

PROOF. Since G is a complete vague graph, then all arcs in G are effective and so by Lemma 15 are strong. In complete vague graphs every node is adjacent to the other nodes. Hence every dominating set in G contains only one node. Therefore for every $v \in V$

$$\gamma_s(G) = \gamma_{sN}(G) = \frac{\wedge\{t_A(v)\} + \big(1 - \vee\{f_A(v)\}\big)}{2}.$$

DEFINITION 17. A vague graph $G = (A, B)$ is said to be a *complete bipartite vague graph* if the set V can be partitioned into two non-empty sets V_1 and V_2 such that

$$\big(t_B(v_1v_2), f_B(v_1v_2)\big) = (0, 1) \text{ for } v_1, v_2 \in V_1 \text{ or } v_1, v_2 \in V_2.$$

Moreover,

$$\big(t_B(uv), f_B(uv)\big) = \big(t_A(u) \wedge t_A(v), f_A(u) \vee f_A(v)\big) \text{ for } u \in V_1 \text{ and } v \in V_2.$$

REMARK 18. Let $G = (A, B)$ be a complete bipartite vague graph with $k \geq 3$ nods. If V_1 or V_2 has only one node as u, then

$$\gamma_s(G) = \frac{\wedge \{t_A(v) \mid \forall v \in V\} + \big(1 - \vee \{f_A(v) \mid \forall v \in V\}\big)}{2}$$

and

$$\gamma_{sN}(G) = \frac{\wedge \{t_A(v) \mid \forall u \neq v \in V\} + \big(1 - \vee \{f_A(v) \mid \forall u \neq v \in V\}\big)}{2}.$$

4 Strong Perfect Domination Number

In this section, we define the perfect dominating set and strong perfect domination number of a vague graph. Then we prove that under proper conditions, the strong and the strong perfect domination number in a vague graph are equal. We finally, obtain an upper bound for strong perfect domination numbers in vague graphs.

DEFINITION 19. Let $G = (A, B)$ be a vague graph. A subset D of V is called a *perfect dominating set* (or D^p) in G, if for every node $v \in V \backslash D$, there exists only one node $u \in D$ such that u dominates v. A set D^p is said to be *minimal perfect dominating set* if for each $v \in D^p$, $D^p \backslash \{v\}$ is not a perfect dominating set in G.

EXAMPLE 20. Consider the vague graph $G = (A, B)$ in Figure 2. We see that ad,

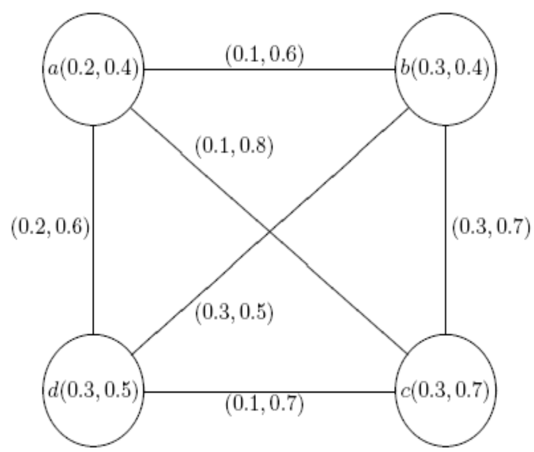

Figure 2: Vague graph G.

bc and bd are strong arcs and so

$$D_1^p = \{a, c\},\ D_2^p = \{b, d\},\ D_3^p = \{a, b, d\} \text{ and } D_4^p = \{b, c, d\}$$

are perfect dominating sets in G where D_1^p and D_2^p are minimal.

PROPOSITION 21. Any perfect dominating set in vague graph G is a dominating set.

REMARK 22. The converse of Proposition 21 is not correct in general. For this consider the vague graph G in Figure 2, we see that $D = \{a, b\}$ is a dominating set in G, but it is not a perfect dominating set. Because d has two strong neighbors in D.

DEFINITION 23. The *strong perfect domination number* of a vague graph G is defined as the minimum arc weight of perfect dominating sets of G, which is denoted by $\gamma_{sp}(G)$.

EXAMPLE 24. Consider the vague graph $G = (A, B)$ in Figure 2. For perfect dominating sets D_1^p, D_2^p, D_3^p and D_4^p in G, we get

$$W_e(D_1^p) = 0.6,\ W_e(D_2^p) = 0.6,\ W_e(D_3^p) = 0.9,\ W_e(D_4^p) = 0.9.$$

Then $\gamma_{sp}(G) = 0.6$.

THEOREM 25. Let $G = (A, B)$ be a vague graph. If each node has at most one strong neighbor, then $\gamma_{sp}(G) = \gamma_s(G)$.

PROOF. Let G be a vague graph. Since each node in G has at most one strong neighbor, therefore for every $u, v \in V$, any dominating set in G is of the form

$$D = \big\{\{u\} \cup \{v\} \big| N_s(u) = \{\varnothing\} \text{ and } v \text{ is the only end node of every strong arc}\big\}.$$

Hence D is a perfect dominating set and so $\gamma_{sp}(G) = \sum_{v \in D} |d_s(v)| = \gamma_s(G)$.

THEOREM 26. Let $G = (A, B)$ be a complete (bipartite) vague graph. Then $\gamma_{sp}(G) = \gamma_s(G)$.

PROOF. Suppose that G is a complete vague graph. Then each dominating set in G such as D contains only one node and so for every node $v \in V \backslash D$ there exists exactly one strong neighbor in D, hence each dominating set in G is a perfect dominating set. Therefore $\gamma_{sp}(G) = \gamma_s(G)$. Now, suppose that G is a complete bipartite vague graph. We consider the following cases:

Case 1: If V_1 (or V_2) has one node as v, then $D = \{v\}$ is the only dominating set in G.

Case 2: If V_1 and V_2 have at most two nodes, then each dominating set in G contains two nodes, V_1 and V_2.

Hence in both cases, each dominating set in G is perfect. Therefore $\gamma_{sp}(G) = \gamma_s(G)$.

COROLLARY 27. Let $G = (A, B)$ be a vague graph where each node has at most one strong neighbor. Then $\gamma_{sp}(G) = S_s(G)$.

REMARK 28. Let $G = (A, B)$ be a vague graph with n nodes and every node in G has exactly one strong neighbor. Then

$$n\big(\delta_s(G)\big) \leq \gamma_{sp}(G) = \gamma_s(G) \leq n\big(\Delta_s(G)\big).$$

THEOREM 29. Let $G = (A, B)$ be a vague graph. Then a perfect dominating set D^p is a minimal perfect dominating set in G if and only if for each node $v \in D^p$, either

(i) $N_s(v) \cap D^p = \{\varnothing\}$ or

(ii) there is a node $u \in V \backslash D^p$ such that $N_s(u) \cap D^p = \{v\}$.

PROOF. Let D^p be a minimal perfect dominating set and $v \in D^p$. Suppose that (i) and (ii) are not established. Then there exists a node $u \in D^p$ such that uv is strong and v has no strong neighbors in $V \backslash D^p$. Therefore $D^p \backslash \{v\}$ is a perfect dominating set in G, which is a contradiction by the minimality of D^p.
Conversely, suppose that (i) or (ii) is established and D^p is not a minimal perfect dominating set in G. Then there exists $v \in D^p$ such that $D^p \backslash \{v\}$ is a perfect dominating set. Hence v has a strong neighbor in D^p and so (i) is not established. Then there is a node $u \in V \backslash D^p$ such that u is a strong neighbor of v and since $D^p \backslash \{v\}$ is a dominating set, then u has a strong neighbor in $D^p \backslash \{v\}$. Therefore $u \in V \backslash D^p$ has two strong neighbors in D^p and so D^p is not a perfect dominating set, that is a contradiction. Then D^p is a minimal perfect dominating set in G.

COROLLARY 30. A dominating set D in a vague graph $G = (A, B)$ is a minimal dominating set if and only if for each node $v \in D$, either

(i) $N_s(v) \cap D = \{\varnothing\}$ or

(ii) there is a node $u \in V \backslash D$ such that $N_s(u) \cap D = \{v\}$.

THEOREM 31. Let G be a vague graph which every its node has at least one strong neighbor. If D^p is a minimal perfect dominating set in G, then $V \backslash D^p$ is a dominating set.

PROOF. Suppose that D^p is a minimal perfect dominating set in vague graph G and $v \in V \backslash (V \backslash D^p) = D^p$. If there is no $u \in V \backslash D^p$ such that $N_s(u) \cap D^p = \{v\}$. Then by Theorem 29, $N_s(v) \cap D^p = \{\varnothing\}$. Therefore there exists a node in G which has no strong neighbors tThat is a contradiction. This implies that $V \backslash D^p$ is a dominating set.

COROLLARY 32. Let $G = (A, B)$ be a vague graph every node of which has at least one strong neighbor. If D is a minimal dominating set in G, then $V\backslash D$ is a dominating set in G.

THEOREM 33. Let $G = (A, B)$ be a vague graph every node of which has exactly one strong neighbor. If D^p is a minimal perfect dominating set in G, then $V\backslash D^p$ is a perfect dominating set in G.

PROOF. Suppose that D^p is a minimal perfect dominating set in the vague graph G. Then by Theorem 31, $V\backslash D^p$ is a dominating set and since every node in G has exactly one strong neighbor, $V\backslash D^p$ is a perfect dominating set in G.

THEOREM 34. Let $G = (A, B)$ be a vague graph where each node in G has exactly one strong neighbor. Then

$$\gamma_{sp}(G) \leq \frac{S_s(G)}{2}.$$

PROOF. Suppose that D is a minimal dominating set in G. Then

$$\gamma_{sp}(G) \leq W_e(D) \leq S_s(G).$$

By Theorem 33, $V\backslash D$ is a perfect dominating set in G. So

$$\gamma_{sp}(G) \leq W_e(V\backslash D) \leq S_s(G)$$

and

$$2\big(\gamma_{sp}(G)\big) \leq S_s(G) \longrightarrow \gamma_{sp}(G) \leq \frac{S_s(G)}{2}.$$

REMARK 35. Let $G = (A, B)$ be a vague graph where each node in G has at least one strong neighbor. Then

$$\gamma_s(G) \leq \frac{S_s(G)}{2}.$$

5 Strong Semi Global Domination Number

In this section, we introduce the concept of semi complementary vague graph and semi global dominating set in vague graphs. Also, we define the strong semi global domination number of a vague graph and obtain some interesting results on these new parameters.

DEFINITION 36. Let $G = (A, B)$ be a vague graph, then *semi complementary vague graph* of G or $G^{sc} = (V^{sc}, E^{sc})$ is defined by

(i) $t_A^{sc}(v) = t_A(v)$ and $f_A^{sc}(v) = f_A(v)$

(ii) $E^{sc} = \{uv \notin E,\ \exists w;\ uw, wv \in E\}$ where for any $uv \in E^{sc}$, $t_B^{sc}(uv) = t_A(u) \wedge t_A(v)$ and $f^{sc}(uv) = f_A(u) \vee f_A(v)$.

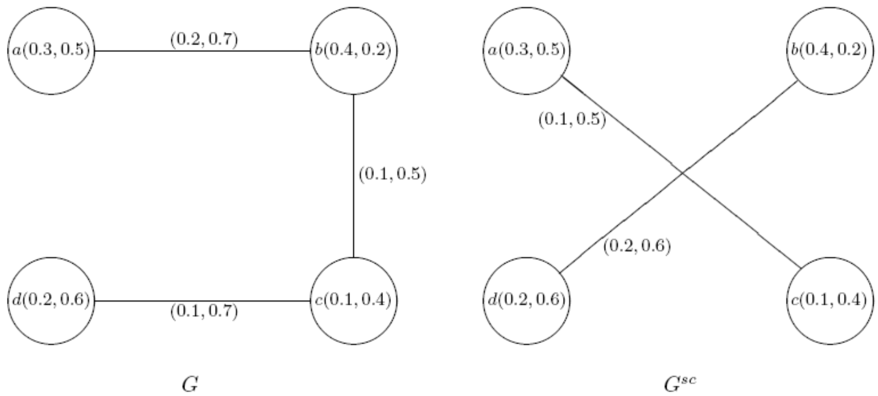

Figure 3: Vague graphs G and G^{sc}.

EXAMPLE 37. Consider vague graphs G and G^{sc} in the Figure 3.

DEFINITION 38. Let $G = (A, B)$ be a vague graph.

(i) The subset $D \subseteq V$ is said to be a *semi global dominating set* (or D^{sg}) in G if D is a dominating set for both G and G^{sc}.

(ii) A semi global dominating set D^{sg} is said to be *minimal semi global dominating set* if for each $v \in D^{sg}$, $D^{sg}\backslash\{v\}$ is not a global dominating set in G.

DEFINITION 39. Let G be a vague graph. The *strong semi global domination number* of a vague graph G is defined as the minimum arc weight of semi global dominating sets in G and it is denoted by $\gamma_{sg}(G)$.

EXAMPLE 40. Consider vague graph G and G^{sc} in the Figure 3, We see that

$$D_1 = \{a, d\} \textit{ and } D_2 = \{b, c\}$$

are minimal semi global dominating sets in G. Then

$$W_e(D_1) = \frac{(0.2 + 0.3)}{2} + \frac{(0.1 + 0.3)}{2} = 0.45,$$

$$W_e(D_2) = \frac{(0.1 + 0.3)}{2} + \frac{(0.1 + 0.3)}{2} = 0.4$$

and $\gamma_{sg}(G) = 0.4$.

THEOREM 41. If G is a complete vague graph with n nodes, then

$$\gamma_{sg}(G) = n\big(\gamma_s(G)\big).$$

PROOF. Since all arcs in G are strong and each node is adjacent to other nodes, we see that G^{sc} has no arc, hence only semi global dominating set in G is $D^{sg} = G$. Then by Theorem 18, we get

$$\begin{aligned}\gamma_{sg}(G) &= \sum_{uv\in E} \frac{\wedge\{t_B(uv)\} + \big(1 - \vee\{f_B(uv)\}\big)}{2} \\ &= n\Big(\frac{\wedge\{t_A(v)\} + (1 - \vee\{f_A(v)\})}{2}\Big) = n\big(\gamma_s(G)\big).\end{aligned}$$

REMARK 42. We observe that $\gamma_{sg}(G) \leq S_s(G)$ for any vague graph $G = (A, B)$.

6 Some Applications of Strong Domination Numbers

A vague set is an extension of Zadeh's fuzzy set theory whose range of membership degree is $[0, 1]$. The vague graph is a generalized structure of a fuzzy graph which gives more precision, flexibility, and compatibility with a system when compared with the fuzzy graphs. These days, graph models are finding many applications in different fields of science and technology. Domination is a rapidly developing area of research in graph theory, and there are many origins to the domination theory. The earliest ideas of dominating sets date back to the origin of game of Chess in India. In society, as well as in administration, the influence of the individual depends on the strength that he derives from his supporters, and these effects may be not effective. Besides,the individual has to depend more on his supporters thanon himself.

Now, we express an application of dominating set. An office consists of 7 employees and elections are being held to determine the new head. We show that a few employees can select person g (who do not have considerable influence on all employees) as the head by using domination set of a vague graph. First we represent the office with a vague graph G as in Figure 4.

In this vague graph, the nodes and the arcs represent employees and friendships between them, respectively. True membership function for each node is considered as the significance of the node in the office, including level of education, work experience, etc., and false membership function for each node is evaluated as lack of compatibility between educational major and occupation, lack of ability, and other cases. In this example, we see that ac, cd, bd, de, cf and cg are strong arcs, and there is a strong relationship between them. Hence, $D_1 = \{c, d\}$, $D_2 = \{c, b, d\}$ and $D_3 = \{c, d, e\}$ are dominating sets in this vague graph and weights are
$W_e(D_1) = (0/2 + 0/5) + (0/5 + 0/6) = 1/8$
$W_e(D_2) = (0/2 + 0/5) + (0/5 + 0/6) + (0/5 + 0/8) = 3/1$
$W_e(D_3) = (0/2 + 0/5) + (0/5 + 0/6) + (0/5 + 0/6) = 2/9$.
And so D_1 is a minimal domination set in this example. Since the nodes in dominating set have the most influence on the other members who are not in D_1, therefore by this

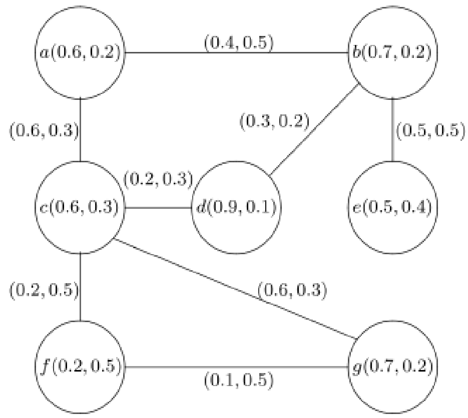

Figure 4: Vague graphs G and G^{sc}.

influence, they can select one of their own as g or anyone else as head of office. (But we see that the crisp graph which is made up of employees and relationships between them, $D = \{c, e\}$ is a minimal domination set, that has less influence on the other members as compared to D_1)

In the following, we have some more applications of strong domination numbers in everyday life.

6.1 Mobile Adhoc Networks

A mobile ad hoc network (MANET) is a continuously self-configuring, infrastructure-less network of mobile devices connected without wires where each device is free to move independently in any direction and will, therefore, change its links to other devices frequently. Each must forward traffic unrelated to its own use and, thus be a router. The primary challenge in building a MANET is equipping each device to continuously maintain the information required to properly route traffic. The dominating sets are useful for the computation of routing in mobile ad hoc networks such that a small dominating set is used as a backbone for communications. The node that is not in this set communicates by passing messages through neighbors in the set.

If the devices and passing of messages between them have value by importance, security, unrelated forward traffic and routing, then a mobile ad hoc network can be represented by a vague graph such that nodes represent devices, and arcs represent pattern of messaging between them and the other network. Since dominating set in a vague graph uses strong arcs, hence the node that is not in dominating set has a strong neighbor in the set and can be faster passing message to its neighbors in the set. So, by the strong domination number the smallest dominating set can be obtained as the smallest and the strongest backbone in this network (compared to the crisp graph). In so doing, we can improve the routing and increase speed of passing of message, and we can also reduce the cost in the mobile ad hoc network.

6.2 Wireless Sensor Network

A wireless sensor network is a group of specialized transducers with a communication infrastructure for monitoring and recording conditions at diverse locations. Commonly monitored parameters are temperature, humidity, pressure, wind direction and speed, illumination intensity, vibration intensity, sound intensity, power-line voltage, chemical concentrations, pollutant levels, and vital body functions. The wireless sensor networks are built of sensor nodes which are widely distributed in the network, and they collect the information of nodes and other networks. Topology control is a fundamental issue in wireless sensor networks. Due to intrinsic characteristic of flatness, hierarchical topology can achieve the scalability and efficiency of a wireless network. To solve this problem, we should represent the network as a vague graph since the place of each node is indefinite and can convey every piece of information, even destructive, to other nodes and, therefore induce intervention in the network. As a result, we can define true (regarding level of importance, necessity, effect, pace of conveyance, etc.) and false (regarding the degree of intervention, vagueness, etc.) membership functions and also valuates each arcs considering importance and necessity of conveyance of information, etc. Accordingly, since each dominating set in a vague graph is gained using strong arcs, we can, in so doing, make the smallest and the most effective minimal backbone set by gaining minimal dominating set (by using minimum arc weight). Virtual backbone is necessary for fault tolerance and routing flexibility.

7 Conclusion

Graph theory has wide applications in computer science and engineering, especially genetic and economics. The importance of this field of mathematics is palpable and undeniable. Since most of the time the aspects of graph problems are uncertain and vague, therefore in these cases, we should respectively make use of fuzzy and vague sets. There are some interesting features for handling vague data that are unique to vague sets. Vague sets allow us to have a better analysis of the relationships among data, defects, and similarity measures by applying a visual-graphic representation of vague data. The notion of vague sets was initially incorporated into relations. Hence, vague graphs may be more important than fuzzy graphs.

The concept of domination in graph is very rich both in theoretical developments and applications. Research in the area of domination theory is interesting due to the diversity of applications and wide variety of domination parameters that can be defined. In this paper, the concept of new type domination number such as, strong neighborhood, strong global and strong perfect domination number are introduced, and some bounds on them are established. Other strong domination parameters can be defined and investigated in future works.

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