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An Integral Representation Of A Symmetrical H_q-Semiclassical Form Of Class One*

Athar Bouanani†, Lotfi Khériji‡

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Abstract

The aim of this paper is to highlight for some values of parameter an integral representation for the form $u(\alpha)$ q-analogue of Bessel kind.

1 Introduction

The monic orthogonal polynomials sequence (MOPS) $\{P_n\}_{n\geq 0}$ satisfying the three-term recurrence relation (see (6) bellow) with

$$\begin{cases} \beta_n = 0 \\ \gamma_{2n+1} = \frac{1-q}{2}\frac{q^{2n+2\alpha}-1}{(q^{4n+2\alpha}-1)(q^{4n+2\alpha+2}-1)}\,q^{2n+2\alpha+2} \\ \gamma_{2n+2} = \frac{q-1}{2}\frac{q^{2n+2}-1}{(q^{4n+2\alpha+2}-1)(q^{4n+2\alpha+4}-1)}\,q^{2n+2\alpha+4} \end{cases}, \quad n\geq 0, \tag{1}$$

is associated with the form $u(\alpha)$ ($\alpha \neq -n$, $n\geq 0$, $q>0$, $q\neq 1$) symmetrical H_q-semiclassical of class one satisfying the q-distributional equation

$$H_q\left(x^3u(\alpha)\right) + \left(\frac{1-q^{-2\alpha-2}}{1-q}\,x^2 - \frac{1}{2}\right)u(\alpha) = 0 \tag{2}$$

were studied in [12] (see also [4]). Moreover, in that work a discrete representation of the form $u(\alpha)$ ($0<q<1$) were established (see [12, (4.9)]). In fact, the form $u(\alpha)$ is the q-analogue of the form $\mathcal{B}[\alpha]$ of Bessel kind which is D-semiclassical of class one for $\alpha\neq -n-1$, $n\geq 0$ satisfying the functional equation [3]

$$D\big(x^3\mathcal{B}[\alpha]\big) - \Big(2(\alpha+1)\,x^2 + \frac{1}{2}\Big)\mathcal{B}[\alpha] = 0,$$

and having the integral representation [3]

$$\langle \mathcal{B}[\alpha], f\rangle = S_\alpha^{-1}\int_{-\infty}^{+\infty}\frac{1}{x^2}\int_{|x|}^{+\infty}\Big(\frac{|x|}{t}\Big)^{2\alpha+1}\exp\Big(\frac{1}{4t^2}-\frac{1}{4x^2}\Big)\,s(t^2)dt\;f(x)\,dx \tag{3}$$

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†Institut Supérieur des Sciences Appliquées et de Technologie de Gabès, Rue Omar Ibn El-Khattab, 6072 Gabès, Tunisia

‡Institut Préparatoire aux Etudes d'Ingénieur El Manar, Campus Universitaire El Manar, B.P. 244, 2092 Tunis, Tunisia

for all polynomial f and $\alpha \geq \frac{1}{2}$ where S_α is the normalization constant, s is the Stieltjes function [13]

$$s(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \exp(-x^{\frac{1}{4}}) \sin x^{\frac{1}{4}} & \text{if } x > 0, \end{cases} \tag{4}$$

and D be the derivative operator. Let us recall the fundamental property [11, 13]

$$\int_0^{+\infty} x^n s(x) dx = 0, \quad n \geq 0. \tag{5}$$

Our aim is to highlight an integral representation corresponding to the form $u(\alpha)$ for $q > 1$ and for some values of the parameter α, according to its H_q-semiclassical character and by solving a suitable q-difference equation.

2 Preliminary

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f \rangle$ the effect of a form $u \in \mathcal{P}'$ (linear functional) on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$ the moments of u. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is called orthogonal (MOPS) if we can associate with it a form u ($(u)_0 = 1$) and a sequence of numbers $\{r_n\}_{n\geq 0}$ ($r_n \neq 0$, $n \geq 0$) such that [1, 10]

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m} \ , \quad n, m \geq 0,$$

and the form u is then said regular. The (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the three-term recurrence relation [1]

$$\begin{cases} P_0(x) = 1, \ \ P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \ \ n \geq 0, \end{cases} \tag{6}$$

where

$$\beta_n = \frac{\langle u, x P_n^2 \rangle}{r_n}, \ \ \gamma_{n+1} = \frac{r_{n+1}}{r_n} \neq 0 \ \text{ for } n \geq 0.$$

The regular form u is positive definite if and only if $\forall n \geq 0$, $\beta_n \in \mathbb{R}$, $\gamma_{n+1} > 0$. Also, its corresponding (MOPS) $\{P_n\}_{n\geq 0}$ is symmetrical if and only if $\beta_n = 0$, $n \geq 0$ or equivalently $(u)_{2n+1} = 0$, $n \geq 0$.

Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any $a \in \mathbb{C} - \{0\}$, any $c \in \mathbb{C}$ and any $q \neq 1$, we let $Du = u'$, $h_a u$, $(x - c)^{-1} u$ and $H_q u$, be the forms defined by duality [8, 10]

$$\langle u', f \rangle := -\langle u, f' \rangle, \ \ \langle h_a u, f \rangle := \langle u, h_a f \rangle, \ \ \langle (x - c)^{-1} u, f \rangle := \langle u, \theta_c f \rangle,$$

and

$$\langle H_q u, f \rangle := -\langle u, H_q f \rangle,$$

for all $f \in \mathcal{P}$ where

$$(h_a f)(x) = f(ax), \quad (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}, \quad (H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

c.f. [5]. We will usually suppose that

$$q \in \widetilde{\mathbb{C}} := \mathbb{C} - \big(\{0\} \bigcup \big(\bigcup_{n \geq 0} \{z \in \mathbb{C},\ z^n = 1\}\big)\big).$$

A form u is called H_q-semiclassical when it is regular and there exist two polynomials Φ and Ψ, Φ monic, $\deg \Phi = t \geq 0$, $\deg \Psi = p \geq 1$ such that

$$H_q(\Phi u) + \Psi u = 0. \tag{7}$$

The corresponding orthogonal sequence $\{P_n\}_{n \geq 0}$ is called H_q-semiclassical [9]. The H_q-semiclassical form u is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if [9]

$$\prod_{c \in \mathcal{Z}_\Phi} \Big\{ \Big| q(h_q \Psi)(c) + (H_q \Phi)(c) \Big| + \Big| \langle u, q(\theta_{cq} \Psi) + (\theta_{cq} \circ \theta_c \Phi) \rangle \Big| \Big\} > 0, \tag{8}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ.

REMARK. When $q \to 1$ in (7)–(8) we meet the D-semiclassical character [10].

Regarding integral representations through true-functions for a H_q-semiclassical form u satisfying (7), we look for a function U such that

$$\langle u, f \rangle = \int_{-\infty}^{+\infty} U(x) f(x) dx, \ f \in \mathcal{P}, \tag{9}$$

where we suppose that U is regular as far as necessary. On account of (7), we get [8]

$$\int_{-\infty}^{+\infty} \Big\{ q^{-1} \big(H_{q^{-1}}(\Phi U)\big)(x) + \Psi(x) U(x) \Big\} f(x) dx = 0, \ f \in \mathcal{P},$$

with the additional condition [8]

$$\lim_{\epsilon \to +0} \int_\epsilon^1 \frac{U(x) - U(-x)}{x} dx \text{ exists or } \ U \text{ is continuous at the origin.} \tag{10}$$

Therefore

$$q^{-1} \big(H_{q^{-1}}(\Phi U)\big)(x) + \Psi(x) U(x) = \lambda g(x), \tag{11}$$

where $\lambda \in \mathbb{C}$ and g is a locally integrable function with rapid decay representing the null form. For instance the function s defined by (4) and satisfying (5) represents the null form.

Lastly, let us recall the following standard expressions needed to the q-calculus in the sequel [2, 6, 7] where q is fixed in $]0,1[$.

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}), \quad n \geq 1, \quad (a;q)_\infty := \prod_{k=0}^{+\infty}(1-aq^k).$$

The q-integral of a function f is defined as

$$\int_0^x f(t)\,d_qt := (1-q)x\sum_{n=0}^{+\infty} f(xq^n)q^n, \quad \int_x^{+\infty} f(t)\,d_qt := (1-q)x\sum_{n=1}^{+\infty} f(xq^{-n})q^{-n},$$

and

$$\int_0^{+\infty} f(t)\,d_qt := (1-q)\sum_{n=-\infty}^{+\infty} f(q^n)q^n,$$

provided the sums converge absolutely.

3 An Integral Representation for $u(\alpha)$

In the sequel, let

$$\alpha \geq \frac{1}{2} \quad \text{and} \quad q > 1. \tag{12}$$

We define the following sequence of numbers

$$|x_k(q)| = \sqrt{\frac{q-1}{2}}q^{\alpha-k}, \quad k \in \mathbb{N}. \tag{13}$$

THEOREM 1. For all $\alpha \geq \frac{1}{2}$, there exists $q > 1$ such that, for all $f \in \mathcal{P}$, the form $u(\alpha)$ has the following integral representation

$$\begin{aligned}\langle u(\alpha), f\rangle &= S_{\alpha,q}^{-1}\int_{\{|x|>|x_0(q)|\}} \frac{\left(x_0^2(q)x^{-2};q^{-2}\right)_\infty}{x^2} \\ &\quad \times \int_{|x|}^{+\infty}\left(\frac{|x|}{t}\right)^{2\alpha+1}\frac{s(t^2-x_0^2(q))}{(q^2x_0^2(q)t^{-2};q^{-2})_\infty}d_{q^{-1}}t\; f(x)dx.\end{aligned}$$

PROOF. Taking into account (12)–(13) we have

$$0 < |x_{k+1}(q)| < |x_k(q)| < |x_0(q)| = \sqrt{\frac{q-1}{2}}q^\alpha \quad \text{and} \quad |x_k(q)| \underset{k\to+\infty}{\longrightarrow} 0. \tag{14}$$

According to (9) and (12), we look for a function U representing $u(\alpha)$ respecting the complementary condition [11]

$$\int_{-\infty}^{+\infty} U(x)\,dx \neq 0. \tag{15}$$

By (9), (12)–(13) and the q-distributional equation in (2), the q-difference equation (11) becomes

$$U(q^{-1}x) - q^{1-2\alpha}\Big(1 - q^2 x_0^2(q)\, x^{-2}\Big) U(x) = \lambda q^3 (1-q) x^{-2} g(x), \tag{16}$$

where $\lambda \neq 0$ and g represents the null form. For instance, let us choose g the following even function [11]

$$g(x) = |x| \; s\big(x^2 - x_0^2(q)\big), \quad x \in \mathbb{R},$$

where s is given by (4). Therefore,

$$g(x) = \begin{cases} |x| \exp\Big(-\sqrt[4]{x^2 - x_0^2(q)}\Big) \sin\Big(\sqrt[4]{x^2 - x_0^2(q)}\Big) & \text{if } |x| > |x_0(q)|, \\ 0 & \text{if } |x| \leq |x_0(q)|. \end{cases} \tag{17}$$

It is easily seen that for all $n \geq 0$

$$\int_{-\infty}^{+\infty} x^{2n+1} g(x)\, dx = 0,$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} x^{2n} g(x) dx &= 2 \int_0^{+\infty} x^{2n} g(x) dx = \int_{x_0(q)}^{+\infty} x^{2n} s\left(x^2 - x_0^2(q)\right) 2x dx \\ &= \int_0^{+\infty} \left[t + x_0^2(q)\right]^n s(t)\, dt = 0, \end{aligned}$$

by (5). Consequently, the even function g in (17) is locally integrable with rapid decay representing the null form.

Let us consider the q-sum $\sum_{n\geq 1} u_n(x)$ and $|x| > |x_0(q)|$ where

$$u_n(x) = \frac{q^{-2n\alpha}\, |x| \exp\Big(-\sqrt[4]{q^{2n}x^2 - x_0^2(q)}\Big) \;\sin\Big(\sqrt[4]{q^{2n}x^2 - x_0^2(q)}\Big)}{\big(q^2 x_n^2(q) x^{-2}; q^{-2}\big)_\infty}. \tag{18}$$

For all $n \geq 1,\ |x| > |x_0(q)|$ we have

$$|u_n(x)| \leq \frac{q^{-2n\alpha}\, |x| \exp\Big(-\sqrt[4]{q^{2n}x^2 - x_0^2(q)}\Big)}{\big(q^2 x_n^2(q) x^{-2}; q^{-2}\big)_\infty} := v_n(x),$$

with

$$\frac{v_{n+1}(x)}{v_n(x)} = q^{-2\alpha}(1 - q^2 x_n^2(q) x^{-2}) \exp\bigg(-q^{\frac{n}{2}}\left(\sqrt[4]{q^2x^2 - q^{-2n}x_0^2(q)} - \sqrt[4]{x^2 - q^{-2n}x_0^2(q)}\right)\bigg).$$

As consequence, the q-sum $\sum_{n\geq 1} u_n(x),\ |x| > |x_0(q)|$ converge absolutely since

$$\frac{v_{n+1}(x)}{v_n(x)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Now, on account of (18) we are able to give a possible solution U of the q-difference equation (16)

$$U(x) = \begin{cases} \lambda q^{2\alpha+2}(1-q)\frac{\left(x_0^2(q)x^{-2};q^{-2}\right)_\infty}{x^2}\sum_{n=1}^{+\infty} u_n(x) & \text{if } |x| > |x_0(q)|, \\ 0 & \text{if } |x| \le |x_0(q)|. \end{cases} \quad (19)$$

For $\alpha \ge \frac{1}{2},\ q > 1$ and $|x| > |x_0(q)|$ and according to (17)–(19) we obtain the definition of the following q^{-1}-integral

$$\int_{|x|}^{+\infty} \left(\frac{|x|}{t}\right)^{2\alpha+1} \frac{s(t^2 - x_0^2(q))}{(q^2x_0^2(q)t^{-2};q^{-2})_\infty} d_{q^{-1}}t := (1-q^{-1})\sum_{n=1}^{+\infty} u_n(x).$$

Consequently, (19) becomes

$$U(x) = \begin{cases} -\lambda q^{2\alpha+2}\frac{\left(x_0^2(q)x^{-2};q^{-2}\right)_\infty}{x^2} \\ \quad \times \int_{|x|}^{+\infty}\left(\frac{|x|}{t}\right)^{2\alpha+1}\frac{s(t^2-x_0^2(q))}{(q^2x_0^2(q)t^{-2};q^{-2})_\infty}d_{q^{-1}}t, & \text{if } |x| > |x_0(q)|, \\ 0 & \text{if } |x| \le |x_0(q)|. \end{cases} \quad (20)$$

Taking into account (13)–(14), (19) and the d'Alembert test an other time, we get for $x \ge a$, $\forall a > |x_0(q)|$, $|U(x)| \le V(x)$, where

$$\begin{aligned} V(x) \quad : \quad &= |\lambda|q^{2\alpha+2}(q-1)\frac{\left(x_0^2(q)x^{-2};q^{-2}\right)_\infty}{\left(x_0^2(q)a^{-2};q^{-2}\right)_\infty}\frac{\exp(-\frac{1}{2}\,(x^2-a^2)^{\frac{1}{4}})}{|x|} \\ &\times \sum_{k=1}^{+\infty} q^{-2k\alpha}\exp\left(-\frac{1}{2}\,q^{\frac{k}{2}}\,(a^2-x_0^2(q))^{\frac{1}{4}}\right) \\ = \quad & o\left(\exp\left(-\frac{1}{2}\,(x^2-a^2)^{\frac{1}{4}}\right)\right), \qquad x \longrightarrow +\infty. \end{aligned}$$

Condition (15) now becomes

$$\int_{\left\{|x|>|x_0(q)|\right\}} U(x)\,dx = -\lambda q^{2\alpha+2}S_{\alpha,q} \neq 0,$$

where

$$S_{\alpha,q} = \int_{\left\{|x|>|x_0(q)|\right\}} \frac{\left(x_0^2(q)x^{-2};q^{-2}\right)_\infty}{x^2}\int_{|x|}^{+\infty}\left(\frac{|x|}{t}\right)^{2\alpha+1}\frac{s(t^2-x_0^2(q))}{(q^2x_0^2(q)t^{-2};q^{-2})_\infty}d_{q^{-1}}t\,dx.$$

Furthermore, $S_{\alpha,q}$ approaches S_α when q tends to 1^+ and $S_\alpha \neq 0$ for $\alpha \ge \frac{1}{2}$, therefore

$$\forall\alpha \ge \frac{1}{2},\ \exists q_\alpha > 1, \quad \forall\ 1 < q < q_\alpha, \qquad S_{\alpha,q} \neq 0.$$

Consequently, for all $\alpha \geq \frac{1}{2},\ 1 < q < q_\alpha$ and $f \in \mathcal{P}$, the form $u(\alpha)$ has the following integral representation (compare with the limiting representation (3))

$$\begin{aligned}\langle u(\alpha), f\rangle &= S_{\alpha,q}^{-1}\int_{\left\{|x|>|x_0(q)|\right\}} \frac{\left(x_0^2(q)x^{-2};q^{-2}\right)_\infty}{x^2}\\ &\quad\times \int_{|x|}^{+\infty}\left(\frac{|x|}{t}\right)^{2\alpha+1}\frac{s(t^2-x_0^2(q))}{(q^2x_0^2(q)t^{-2};q^{-2})_\infty}d_{q^{-1}}tf(x)dx.\end{aligned}$$

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