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Fixed Points For Generalized Geraghty Contractions Of Berinde Type On Partial Metric Spaces*

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Abstract

In the present paper, we consider generalized Geraghty contraction in the sense of Berinde on partial metric spaces and give some fixed point results which extend, generalize and enrich some recent results appearing in the literature. Some examples are provided to illustrate the presented results and that they are proper extensions of the existing ones.

1 Introduction

Fixed point theory plays a major role within as well as outside mathematics, so the attraction of fixed point theory to large numbers of researchers is understandable. The Banach contraction mapping principle is one of the fundamental results of nonlinear functional analysis to prove the existence and uniqueness of fixed points of certain self-maps of metric spaces and provides a constructive method to approximate those fixed points.

In the mathematical field of domain theory, attempts were made in order to equip semantics domain with a notion of distance. In particular, Matthews [21] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. These spaces are generalizations of usual metric spaces where the self-distance for any point need not be equal to zero. At this point it seems interesting to remark the fact that partial metric spaces play an important role in constructing models in the theory of computation.

Matthews [21] obtained, among other results, a partial metric version of the Banach fixed point theorem as follows.

THEOREM 1 ([21]). Let T be a mapping of a complete partial metric space (X, p) into itself such that there is a real number k with $0 \leq k < 1$, satisfying for all $x, y \in X$,

$$p(Tx, Ty) \leq kp(x, y).$$

Then T has a unique fixed point.

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After the appearance of partial metric spaces, some authors started to generalize Banach contraction mapping theorem to partial metric spaces and focus on fixed point theory on partial metric spaces. Neill [22] defined the concept of the dualistic partial metric, which is more general than the partial metric. In [23], Oltra and Valero gave a Banach fixed point theorem on complete dualistic partial metric spaces. Later, Valero [30] generalized the main theorem of [23] using a nonlinear contractive condition instead of a Banach contractive condition. For further works in this direction, we refer the interested reader to [1, 3–7, 13, 19, 24–27] and the references cited therein.

In 1973, Geraghty [17] proved a fixed point result, generalizing Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Recently, Dukić et al. [15] proved the following nice fixed point theorem. Before, we introduce the set $\mathfrak{F}$ of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{as} \quad n \to +\infty \quad \text{implies} \quad t_n \to 0 \quad \text{as} \quad n \to +\infty.$$

THEOREM 2 ([15]). Let (X, p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \mathfrak{F}$ such that

$$p(Tx, Ty) \le \beta\big(p(x, y)\big)p(x, y)$$

holds for all $x, y \in X$. Then T has a unique fixed point $u \in X$ and for each $x \in X$ the Picard sequence $\{T^n x\}$ converges to u when $n \to +\infty$.

The concept of almost contractions was introduced by Berinde [8, 9] on metric spaces. Other results on almost contractions could be found in [10–12]. Recently, Altun and Acar [2] characterized this concept in the setting of partial metric space and proved some fixed point theorems using these concepts. The purpose of this work is to present some fixed point results for self-mappings involving some almost generalized contractions in the setting of partial metric spaces by using functions belonging to $\mathfrak{F}$. Our main results extend, generalize and enrich some existing theorems in the literature. Also, we give some illustrative examples making our results proper.

2 Preliminaries

We begin with some basic concepts and results in partial metric spaces which are needed in this paper.

Following Matthews [21], the notion of a partial metric space is given as follows.

DEFINITION 1 ([21]). A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$, the following conditions hold:

(p_1) $x = y \quad \Leftrightarrow \quad p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \le p(x, y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A pair (X,p) is called a partial metric space where X is a nonempty set and p is a partial metric on X.

EXAMPLE 1 ([21]). Let $X = [0,+\infty)$ and p defined on X by $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. Then (X,p) is a partial metric space.

EXAMPLE 2 ([21]). Let (X,d) and (X,p) be a metric space and partial metric space, respectively. Functions $p_i : X \times X \to [0,+\infty)$ ($i \in \{1,2,3\}$) defined by

$$\begin{aligned} p_1(x,y) &= d(x,y) + p(x,y), \\ p_2(x,y) &= d(x,y) + \max\big\{\omega(x), \omega(y)\big\}, \\ p_3(x,y) &= d(x,y) + a \end{aligned}$$

consider partial metrics on X, where $\omega : X \to [0,+\infty)$ is an arbitrary function and $a \ge 0$.

EXAMPLE 3 ([21]). Let $X = \mathbb{R}$ and $p(x,y) = e^{\max\{x,y\}}$ for all $x,y \in X$. Then (X,p) is a partial metric space.

Other examples of the partial metric spaces which are interesting from a computational point of view may be found in [16, 18, 21].

REMARK 1. It is clear that if $p(x,y) = 0$, then from (p_1) and (p_2) follows $x = y$. On the other hand, if $x = y$, then $p(x,y)$ may not be 0.

Recall that each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) \,:\, x \in X,\, \varepsilon > 0\}$, where

$$B_p(x,\varepsilon) = \big\{y \in X \,:\, p(x,y) < p(x,x) + \varepsilon\big\},$$

for all $x \in X$ and $\varepsilon > 0$.

It is remarkable that if p is a partial metric on X, then the functions $p^s, p^w : X \times X \to [0,+\infty)$, given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

and

$$\begin{aligned} p^w(x,y) &= \max\big\{p(x,y) - p(x,x), p(x,y) - p(y,y)\big\} \\ &= p(x,y) - \min\big\{p(x,x), p(y,y)\big\} \end{aligned}$$

are ordinary metrics on X. It is easy to see that p^s and p^w are equivalent metrics on X.

REMARK 2. Clearly, a limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(\cdot,\cdot)$ need not be continuous in the sense that $x_n \to x$ and $y_n \to y$ implies $p(x_n, y_n) \to p(x, y)$. For example, if $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for $x, y \in X$, then for $\{x_n\} = \{1\}$, $p(x_n, x) = x = p(x, x)$ for each $x \geq 1$ and so, for example, $x_n \to 2$ and $x_n \to 3$ when $n \to +\infty$.

DEFINITION 2 ([21]). Let (X, p) be a partial metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) The sequence $\{x_n\}$ is said to converge to x, with respect to τ_p, if and only if $\lim_{n\to+\infty} p(x_n, x) = p(x, x)$.

(2) The sequence $\{x_n\}$ is said to be Cauchy sequence if $\lim_{n,m\to+\infty} p(x_n, x_m)$ exists and is finite.

(3) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point x such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

The following lemma is crucial in proving our main results.

LEMMA 1 ([21]). Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in (X, p^s).

(2) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\to+\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m).$$

The following Lemma shows that under certain conditions the limit is unique.

LEMMA 2 ([28]). Let $\{x_n\}$ be a convergent sequence in partial metric space X such that $x_n \to x$ and $y_n \to y$ with respect to τ_p. If

$$\lim_{n\to+\infty} p(x_n, x_n) = p(x, x) = p(y, y),$$

then $x = y$.

LEMMA 3 ([20, 28]). Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial metric space X such that

$$\lim_{n\to+\infty} p(x_n, x) = \lim_{n\to+\infty} p(x_n, x_n) = p(x, x)$$

and

$$\lim_{n\to+\infty} p(y_n, y) = \lim_{n\to+\infty} p(y_n, y_n) = p(y, y),$$

then $\lim_{n\to+\infty} p(x_n, y_n) = p(x, y)$. In particular, $\lim_{n\to+\infty} p(x_n, z) = p(x, z)$ for every $z \in X$.

LEMMA 4 ([21]). Let (X, p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to z \in X$, with respect to τ_p, with $p(z, z) = 0$, then $\lim_{n\to+\infty} p(x_n, y) = p(z, y)$ for all $y \in X$.

3 Main Results

The following is the main result of this paper.

THEOREM 3. Let (X, p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that

$$p(Tx, Ty) \leq \beta\big(M(x, y)\big)M(x, y) + LN(x, y) \tag{1}$$

holds for all $x, y \in X$, where

$$M(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}$$

and

$$N(x, y) = \min\big\{p^w(x, Tx), p^w(y, Ty), p^w(x, Ty), p^w(y, Tx)\big\}.$$

Then T has a unique fixed point $u \in X$. Moreover, $p(u, u) = 0$.

PROOF. Let $x_0 \in X$ be an arbitrary point. We construct a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Suppose that $p(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$. So, we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, that is, x_{n_0} is a fixed point of T.

From now on, assume that $p(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. By applying (1), we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq \beta\big(M(x_{n-1}, x_n)\big)M(x_{n-1}, x_n) + LN(x_{n-1}, x_n), \end{aligned} \tag{2}$$

Since

$$\begin{aligned} \frac{p(x_{n-1}, x_{n+1}) + p(x_n, x_n)}{2} &\leq \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2} \\ &\leq \max\big\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\big\}, \end{aligned}$$

we have

$$\begin{aligned}
M(x_{n-1},x_n) &= \max\Big\{p(x_{n-1},x_n), p(x_{n-1},Tx_{n-1}), p(x_n,Tx_n),\\
&\qquad \frac{p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})}{2}\Big\}\\
&= \max\Big\{p(x_{n-1},x_n), p(x_{n-1},x_n), p(x_n,x_{n+1}),\\
&\qquad \frac{p(x_{n-1},x_{n+1})+p(x_n,x_n)}{2}\Big\}\\
&= \max\big\{p(x_{n-1},x_n), p(x_n,x_{n+1})\big\} \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
N(x_{n-1},x_n) &= \min\big\{p^w(x_{n-1},Tx_{n-1}), p^w(x_n,Tx_n), p^w(x_{n-1},Tx_n), p^w(x_n,Tx_{n-1})\big\}\\
&= \min\big\{p^w(x_{n-1},x_n), p^w(x_n,x_{n+1}), p^w(x_{n-1},x_{n+1}), p^w(x_n,x_n)\big\}. \qquad (4)
\end{aligned}$$

As $p^w(x_n,x_n)=0$, it follows that $N(x_{n-1},x_n)=0$. Notice that the case $M(x_{n-1},x_n)=p(x_n,x_{n+1})$ is impossible due to the definition of β. Indeed,

$$\begin{aligned}
p(x_n,x_{n+1}) &\le \beta\big(M(x_{n-1},x_n)\big)M(x_{n-1},x_n)\\
&\le \beta\big(p(x_n,x_{n+1})\big)p(x_n,x_{n+1})\\
&< p(x_n,x_{n+1}).
\end{aligned}$$

Thus, we conclude that $M(x_{n-1},x_n)=p(x_{n-1},x_n)$. Keeping the inequality (2) in the mind, we get $0<p(x_n,x_{n+1})<p(x_{n-1},x_n)$ for all $n\in\mathbb{N}$. Hence, the sequence $\{p(x_n,x_{n+1})\}$ is a nonincreasing sequence of nonnegative numbers which is bounded from below. So, there exists $r\ge 0$ such that $\lim_{n\to+\infty} p(x_n,x_{n+1})=r$. We claim that $r=0$. On the contrary, assume that $r>0$. Then, due to (2), we obtain

$$\frac{p(x_n,x_{n+1})}{p(x_{n-1},x_n)} \le \beta\big(p(x_{n-1},x_n)\big) \le 1$$

for all $n\in\mathbb{N}$ which yields that $\lim_{n\to+\infty}\beta\big(p(x_{n-1},x_n)\big)=1$. Owing to the fact that $\beta\in\mathfrak{F}$, we have $\lim_{n\to+\infty} p(x_n,x_{n+1})=0$, that is, $r=0$, a contradiction. Hence,

$$\lim_{n\to+\infty} p(x_n,x_{n+1})=0. \qquad (5)$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the partial metric space (X,p). By applying Lemma 2, we need to prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s). Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence in the metric space (X,p^s). Then, there exists $\varepsilon>0$ such that for an integer k there exist integers $m(k)>n(k)>k$ such that

$$p^s(x_{n(k)},x_{m(k)})>\varepsilon. \qquad (6)$$

By the definition of p^s, we have $p^s(x,y) \leq 2p(x,y)$ for all $x, y \in X$. Thus, by using (6), we get

$$p(x_{n(k)}, x_{m(k)}) > \frac{\varepsilon}{2}. \tag{7}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (7). Hence,

$$p(x_{n(k)}, x_{m(k)-1}) \leq \frac{\varepsilon}{2}. \tag{8}$$

By applying (7) and (8) and due to (p_4) from Definition 1, we get

$$\begin{aligned}
\frac{\varepsilon}{2} &< p(x_{n(k)}, x_{m(k)}) \\
&\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) - p(x_{m(k)-1}, x_{m(k)-1}) \\
&\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\
&\leq \frac{\varepsilon}{2} + p(x_{m(k)-1}, x_{m(k)}).
\end{aligned}$$

Letting $k \to +\infty$ in the above inequality and using (5), we obtain

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)}) = \frac{\varepsilon}{2}. \tag{9}$$

Further, by using the triangular inequality, we have

$$\left| p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{m(k)}) \right| \leq p(x_{m(k)-1}, x_{m(k)}).$$

Letting again $k \to +\infty$ in the above inequality and using (5) and (9), we obtain

$$\lim_{k\to+\infty} p(x_{n(k)}, x_{m(k)-1}) = \frac{\varepsilon}{2}. \tag{10}$$

By using (p_3) and (p_4) from Definition 1, we have

$$\begin{aligned}
p(x_{n(k)}, x_{m(k)}) &\leq p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)+1}, x_{m(k)}) - p(x_{n(k)+1}, x_{n(k)+1}) \\
&\leq p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)+1}, x_{m(k)}) \\
&\leq p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)+1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\
&\quad - p(x_{m(k)-1}, x_{m(k)-1}) \\
&\leq p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)+1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\
&\leq 2p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\
&\quad - p(x_{n(k)}, x_{n(k)}) \\
&\leq 2p(x_{n(k)}, x_{n(k)+1}) + p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}).
\end{aligned}$$

Letting again $k \to +\infty$ in the above inequalities and using (5),(9) and (10), we obtain

$$\lim_{k\to+\infty} p(x_{n(k)+1}, x_{m(k)}) = \frac{\varepsilon}{2} \tag{11}$$

and

$$\lim_{k\to+\infty} p(x_{n(k)+1}, x_{m(k)-1}) = \frac{\varepsilon}{2}. \tag{12}$$

On the other hand, from

$$\begin{aligned} p^w(x_{n(k)}, x_{n(k)+1}) &= p(x_{n(k)}, x_{n(k)+1}) - \min\left\{p(x_{n(k)}, x_{n(k)}), p(x_{n(k)+1}, x_{n(k)+1})\right\} \\ &\le p(x_{n(k)}, x_{n(k)+1}), \end{aligned}$$

and thanks to (5), we get

$$\lim_{k\to+\infty} p^w(x_{n(k)}, x_{n(k)+1}) = 0. \tag{13}$$

Now, by applying the inequality (1) with $x = x_{n(k)}$ and $y = x_{m(k)-1}$, we have

$$\begin{aligned} p(x_{n(k)+1}, x_{m(k)}) &= p(Tx_{n(k)}, Tx_{m(k)-1}) \\ &\le \beta\big(M(x_{n(k)}, x_{m(k)-1})\big)M(x_{n(k)}, x_{m(k)-1}) \\ &\quad + LN(x_{n(k)}, x_{m(k)-1}), \end{aligned} \tag{14}$$

where

$$\begin{aligned} M(x_{n(k)}, x_{m(k)-1}) &= \max\Bigg\{p(x_{n(k)}, x_{m(k)-1}), p(x_{n(k)}, Tx_{n(k)}), p(x_{m(k)-1}, Tx_{m(k)-1}), \\ &\quad \frac{p(x_{n(k)}, Tx_{m(k)-1}) + p(x_{m(k)-1}, Tx_{n(k)})}{2}\Bigg\} \\ &= \max\Bigg\{p(x_{n(k)}, x_{m(k)-1}), p(x_{n(k)}, x_{n(k)+1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{p(x_{n(k)}, x_{m(k)}) + p(x_{m(k)-1}, x_{n(k)+1})}{2}\Bigg\} \end{aligned} \tag{15}$$

and

$$\begin{aligned} N(x_{n(k)}, x_{m(k)-1}) &= \min\left\{p^w(x_{n(k)}, Tx_{n(k)}), p^w(x_{m(k)-1}, Tx_{m(k)-1})\right. \\ &\quad \left. p^w(x_{n(k)}, Tx_{m(k)-1}), p^w(x_{m(k)-1}, Tx_{n(k)})\right\}, \\ &= \min\left\{p^w(x_{n(k)}, x_{n(k)+1}), p^w(x_{m(k)-1}, x_{m(k)}),\right. \\ &\quad \left. p^w(x_{n(k)}, x_{m(k)}), p^w(x_{m(k)-1}, x_{n(k)+1})\right\}. \end{aligned} \tag{16}$$

Letting again $k \to +\infty$ in (15) and (16) and using (5),(9), (10), (12) and (13), we get

$$\lim_{k\to+\infty} M(x_{n(k)}, x_{m(k)-1}) = \max\left\{\frac{\varepsilon}{2}, 0, 0, \frac{\varepsilon}{2}\right\} = \frac{\varepsilon}{2} \tag{17}$$

and

$$\lim_{k\to+\infty} N(x_{n(k)}, x_{m(k)-1}) = 0. \tag{18}$$

Now, letting again $k \to +\infty$ in (14) and using (11), (17) and (18), we obtain

$$1 \leq \lim_{k\to+\infty} \beta\big(M(x_{n(k)}, x_{m(k)-1})\big),$$

and so $\lim_{k\to+\infty} \beta\big(M(x_{n(k)}, x_{m(k)-1})\big) = 1$. Since $\beta \in \mathfrak{F}$, we have

$$\lim_{k\to+\infty} M(x_{n(k)}, x_{m(k)-1}) = 0.$$

This implies that $\varepsilon = 0$, which is a contradiction. So, $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Since (X, p) is complete, it follows from Lemma 1 that (X, p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $u \in X$ in (X, p^s), that is,

$$\lim_{n\to+\infty} p^s(x_n, u) = 0.$$

Again, from Lemma 1,

$$p(u, u) = \lim_{n\to+\infty} p(x_n, u) = \lim_{n\to+\infty} p(x_n, x_n).$$

On the other hand, thanks to (5) and due to (p_2) from Definition 1,

$$\lim_{n\to+\infty} p(x_n, x_n) = 0, \tag{19}$$

which yields that

$$p(u, u) = \lim_{n\to+\infty} p(x_n, u) = \lim_{n\to+\infty} p(x_n, x_n) = 0. \tag{20}$$

Now, we will prove that $p(u, Tu) = 0$. Assume on the contrary that $p(u, Tu) > 0$. By applying (1), we obtain

$$\begin{aligned} p(x_{n+1}, Tu) &= p(Tx_n, Tu) \\ &\leq \beta\big(M(x_n, u)\big) M(x_n, u) + LN(x_n, u), \end{aligned} \tag{21}$$

where

$$\begin{aligned} M(x_n, u) &= \max\left\{p(x_n, u), p(x_n, Tx_n), p(u, Tu), \frac{p(x_n, Tu) + p(u, Tx_n)}{2}\right\} \\ &= \max\left\{p(x_n, u), p(x_n, x_{n+1}), p(u, Tu), \frac{p(x_n, Tu) + p(u, x_{n+1})}{2}\right\} \end{aligned} \tag{22}$$

and

$$\begin{aligned} N(x_n, u) &= \min\big\{p^w(x_n, Tx_n), p^w(u, Tu), p^w(x_n, Tu), p^w(u, Tx_n)\big\} \\ &= \min\big\{p^w(x_n, x_{n+1}), p^w(u, Tu), p^w(x_n, Tu), p^w(u, x_{n+1})\big\}. \end{aligned} \tag{23}$$

Thanks to (20), it is obvious that $\lim_{n\to+\infty} p(x_n, Tu) = p(u, Tu)$. Hence, by using (5) and again (20), we get

$$\lim_{n\to+\infty} M(x_n, u) = \max\Big\{0, 0, p(u, Tu), \frac{1}{2}p(u, Tu)\Big\} = p(u, Tu). \tag{24}$$

Moreover, from (5) and (19), we conclude that $\lim_{n\to+\infty} p^s(x_n, Tx_n) = 0$. Thus, by using (23), we obtain

$$\lim_{n\to+\infty} N(x_n, u) = 0. \tag{25}$$

Now, letting again $k \to +\infty$ in (21) and using (24) and (25), we have

$$1 \leq \lim_{n\to+\infty} \beta\big(M(x_n, u)\big),$$

which implies that $\lim_{n\to+\infty} M(x_n, u) = 0$, a contradiction. Hence, $p(u, Tu) = 0$, that is, $Tu = u$. Therefore, we conclude that T has a fixed point $u \in X$ and $p(u, u) = 0$.

Finally, if, on the contrary, $v \neq u$ (so $p(u, v) \neq 0$) is another fixed point of T (with $p(v, v) = 0$), then by using (20),

$$\begin{aligned} M(u,v) &= \max\left\{p(u,v), p(u,Tu), p(v,Tv), \frac{p(u,Tv)+p(v,Tu)}{2}\right\} \\ &= \max\left\{p(u,v), p(u,u), p(v,v), \frac{p(u,v)+p(v,u)}{2}\right\} \\ &= p(u,v) \end{aligned} \tag{26}$$

and

$$\begin{aligned} N(u,v) &= \min\left\{p^w(u,Tu), p^w(v,Tv), p^w(u,Tv), p^w(v,Tu)\right\} \\ &= \min\left\{p^w(u,u), p^w(v,v), p^w(u,v), p^w(v,u)\right\} \\ &= 0. \end{aligned} \tag{27}$$

Hence, by applying the inequality (1) and using (26) and (27), we obtain

$$\begin{aligned} p(u,v) &= p(Tu,Tv) \\ &\leq \beta\big(M(u,v)\big)M(u,v) + LN(u,v) \\ &= \beta\big(M(u,v)\big)p(u,v) < p(u,v), \end{aligned}$$

which is a contradiction. Therefore, the fixed point of T is unique. This finishes the proof.

By taking $L = 0$ in Theorem 3, we obtain the following result.

COROLLARY 1. Let (X, p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \mathfrak{F}$ such that

$$p(Tx, Ty) \leq \beta\big(M(x,y)\big)M(x,y)$$

holds for all $x, y \in X$, where

$$M(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty)+p(y,Tx)}{2}\right\}.$$

Then T has a unique fixed point $u \in X$. Moreover, $p(u,u) = 0$.

If in Theorem 3 we take the function $\beta(t) = \lambda$, $\lambda \in [0,1)$, which is in $\mathfrak{F}$, then we have the following corollary.

COROLLARY 2. Let (X,p) be a complete partial metric space and let $T : X \to X$ be a self-mapping. Suppose that there exist $\lambda \in [0,1)$ and $L \geq 0$ such that

$$p(Tx,Ty) \leq \lambda M(x,y) + LN(x,y)$$

holds for all $x, y \in X$, where

$$M(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty) + p(y,Tx)}{2}\right\}$$

and

$$N(x,y) = \min\left\{p^w(x,Tx), p^w(y,Ty), p^w(x,Ty), p^w(y,Tx)\right\}.$$

Then T has a unique fixed point $u \in X$. Moreover, $p(u,u) = 0$.

REMARK 3. Corollary 1 is a generalization of Theorem 3.1 of Dukić et al. [15] which is also noted here as Theorem 2.

REMARK 4. By taking $L = 0$ in Corollary 2, we obtain the Ćirić fixed point theorem [14] in the setting of metric spaces (by considering $p = d$ is a metric).

REMARK 5. Corollary 2 generalizes Theorem 10 (with $f = g = T = S$) of Turkoglu and Özturk [29].

We now present some examples showing that there are situations where our results can be used to conclude about the existence of fixed points, while some other known results cannot be applied.

EXAMPLE 4. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. It is clear that (X,p) is a complete partial metric space. Consider $T : X \to X$ given by $Tx = \frac{x}{6}$. Let the function β be defined by

$$\beta(t) = \begin{cases} \frac{e^{-t}}{t+1}, & \text{if } t > 0, \\ \frac{1}{2}, & \text{if } t = 0. \end{cases}$$

By taking $x, y \in X$ with, for example, $x \geq y$ and $x > 0$, we have

$$p(Tx,Ty) = \max\left\{\frac{x}{6}, \frac{y}{6}\right\} = \frac{x}{6}$$

and

$$\begin{aligned} M(x,y) &= \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty) + p(y,Tx)}{2}\right\} \\ &= \max\left\{x, x, y, \frac{1}{2}\left[x + \max\left\{y, \frac{x}{6}\right\}\right]\right\} = x, \end{aligned}$$

since $\max\{y, \frac{x}{6}\} \leq x$. Hence,

$$\beta\big(M(x,y)\big)M(x,y) = \beta(x)x = \frac{e^{-x}}{x+1}x.$$

Now, from

$$\frac{1}{6} < \frac{1}{2e} \leq \frac{e^{-x}}{x+1}$$

and

$$L\min\big\{p^w(x,Tx), p^w(y,Ty), p^w(x,Ty), p^w(y,Tx)\big\} \geq 0$$

for all $x, y \in X$, we get that (1) holds. Thus, all the hypotheses of Theorem 3 are satisfied. Therefore, T has a unique fixed point in X, which is $u = 0$.

Note that if we use the metric $d(x,y) = 2|x-y|$ for all $x, y \in X$, instead of p, then T does not satisfy the conditions of Geraghty's theorem (see [17]) by considering the above function β in the metric space (X,d). Indeed, by taking $x = 1$ and $y = 0$, we obtain

$$d\big(T1,T0\big) = d\bigg(\frac{1}{6},0\bigg) = 2\bigg|\frac{1}{6} - 0\bigg| = \frac{1}{3}$$

and

$$\beta\big(d(1,0)\big)d(1,0) = 2\beta(2) = 2\frac{e^{-2}}{2+1} = \frac{2e^{-2}}{3} < \frac{1}{3}.$$

Hence, the existence of a fixed point of T cannot be deduced by using Geraghty's theorem.

EXAMPLE 5. Let $X = [0,1]$ and $p(x,y) = e^{\max\{x,y\}} - 1$ for all $x, y \in X$. Then (X,p) is a complete partial metric space. Define $T : X \to X$ by the rule

$$Tx = \begin{cases} 0, & \text{if } x = 1, \\ \frac{x}{2}, & \text{if } x \neq 1. \end{cases}$$

Consider the function β given by $\beta(t) = \frac{1}{2}$. Then, $\beta \in \mathfrak{F}$.

Now we consider the following cases:

Case 1. If $y \leq x < 1$, then

$$\begin{aligned} p(Tx,Ty) &= e^{\max\{Tx,Ty\}} - 1 = e^{\frac{x}{2}} - 1 \\ &\leq \frac{e^x - 1}{2} = \frac{1}{2}p(x,y) \leq \beta\big(M(x,y)\big)M(x,y). \end{aligned}$$

Case 2. If $y < x = 1$, then

$$\begin{aligned} p(T1,Ty) &= e^{\max\{T1,Ty\}} - 1 = e^{\frac{y}{2}} - 1 \\ &\leq \frac{e-1}{2} = \frac{1}{2}p(1,y) \leq \beta\big(M(1,y)\big)M(1,y). \end{aligned}$$

Case 3. If $y = x = 1$, then

$$p(T1, T1) = 0 \le \frac{e-1}{2} = \frac{1}{2}p(1,1) \le \beta\big(M(1,1)\big)M(1,1).$$

Since, for all $x, y \in X$,

$$L \min\left\{p^w(x,Tx), p^w(y,Ty), p^w(x,Ty), p^w(y,Tx)\right\} \ge 0,$$

it follows that (1) is verified. Hence, all conditions of Theorem 3 are satisfied. Therefore, T has a unique fixed point in X, which is $u = 0$.

However, if we consider the standard metric $d(x,y) = |x-y|$ for all $, y \in X$, instead of p, then we cannot find a function $\beta \in \mathfrak{F}$ satisfying the conditions of Geraghty's theorem (see [17]) in the metric space (X,d). Indeed, by taking $x = \frac{3}{4}$ and $y = 1$, we have

$$d\left(T\frac{3}{4}, T1\right) = d\left(\frac{3}{8}, 0\right) = \left|\frac{3}{8} - 0\right| = \frac{3}{8} \qquad \text{and} \qquad d\left(\frac{3}{4}, 1\right) = \left|\frac{3}{4} - 1\right| = \frac{1}{4}.$$

Since $\frac{3}{8} > \beta(\frac{1}{4})\frac{1}{4}$, it follows that Geraghty's theorem cannot be used to prove the existence of a fixed point of T.

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