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Some Generalizations Of Feng Qi Type Integral Inequalities On Time Scales*

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Abstract

In the paper, the authors provide some new generalizations of Feng Qi type integral inequalities on time scales by using elementary analytic methods.

1 Introduction

The following problem was posed by Qi in [16]: under what conditions does the inequality

$$\int_a^b f^p(x)dx \geqslant \left(\int_a^b f(x)dx\right)^{p-1}$$

holds for $p > 1$? Later, this problem attracted great interest of many mathematicians. M. Akkouchi proved the following results in [1, p. 124, Theorem C].

THEOREM. Let $[a, b]$ be a closed interval of $\mathbb{R}$ and $p > 1$. For any continuous function $f(x)$ on $[a, b]$ such that $f(a) \geqslant 0, f'(x) \geqslant p$, we have

$$\int_a^b f^{p+2}(x)dx \geqslant \frac{1}{(b-a)^{p-1}} \left(\int_a^b f(x)dx\right)^{p+1}.$$

Then, the q-analogue of the previous result was obtained in [7, Proposition 3.5] as follows.

THEOREM. Let $p > 1$ be a real number and $f(x)$ a function defined on $[a, b]_q$, such that $f(a) \geqslant 0, D_q f(x) \geqslant p$ for all $x \in (a, b]_q$. Then

$$\int_a^b f^{p+2}(x)d_qx \geqslant \frac{1}{(b-a)^{p-1}} \left(\int_a^b f(qx)d_qx\right)^{p+1}.$$

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Later, V. Krasniqi and A. Sh. Shabani obtained some more sufficient conditions to Qi type h-integral inequalities in [12]. M. R. S. Rahmat got some (q,h)-analogues of integral inequalities on discrete time scales in [17]. Yin et al. obtained some Qi type inequalities on time scales in [21]. For more results, we refer the reader to the papers ([2–6, 8–10, 13, 14, 18–20]). Recently, V. Krasniqi obtained some generalizations of Qi type inequalities in [11]. His main results are following two theorems.

THEOREM 1. If f is a non-negative increasing function on $[a,b]$ and satisfies $f'(x) \geq (t-2)(x-a)^{t-3}$ for $t \geqslant 3$, then

$$\int_a^b f^t(x)dx - \left(\int_a^b f(x)dx\right)^{t-1} \geqslant f^{t-1}(a)\int_a^x f(x)dx.$$

THEOREM 2. Let $p \geqslant 1$. If f is a non-negative increasing function on $[a,b]$ and satisfies $f'(x) \geqslant p\left(\frac{x-a}{b-a}\right)^{p-1}$, then

$$\int_a^b f^{p+2}(x)dx - \frac{1}{(b-a)^{p-1}}\left(\int_a^b f(x)dx\right)^{p+1} \geqslant f^{p+1}(a)\int_a^x f(x)dx.$$

The main aim of this paper is to generalize the above results on time scales.

2 Notations and Lemmas

2.1 Notations

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where the supremum of the empty set is defined to be the infimum of $\mathbb{T}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd (ld)-continuous provided g is continuous at right (left)-dense points and has finite left (right)-sided limits at left (right)-dense points in $\mathbb{T}$. The graininess function μ (ν) for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t(\nu(t) = t - \rho(t))$, and for every function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma (f^\rho)$ means the composition $f \circ \sigma \quad (f \circ \rho)$. We also need below the set $\mathbb{T}^\kappa(\mathbb{T}_\kappa)$ which is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum (right-scattered minimum) m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}(\mathbb{T}_\kappa = \mathbb{T} - \{m\})$.We then define the interval $[a,b]$ in $\mathbb{T}$ by $[a,b]_\mathbb{T} = \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$.

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa(\mathbb{T}_\kappa)$. We define $f^\Delta(f^\nabla)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U ($U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$
$$(|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| < \varepsilon|\rho(t) - s|).$$

If the delta (nabla) derivative $f^\Delta(t)(f^\nabla(t))$ exits for all $t \in \mathbb{T}$, then we say that f is delta (nabla) differentiable on $\mathbb{T}$. We will make use of the following product and rules for the derivatives of the product fg and the quotient f/g (where $gg^\sigma(gg^\rho) \neq 0$) of two delta (nabla) differentiable functions f and g,

$$\begin{aligned}(fg)^\Delta &= f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \\ ((fg)^\nabla &= f^\nabla g + f^\rho g^\nabla = fg^\nabla + f^\nabla g^\rho),\end{aligned} \tag{1}$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma} \quad \left(\left(\frac{f}{g}\right)^\nabla = \frac{f^\nabla g - fg^\nabla}{gg^\rho}\right).$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t, \mu(t) = \nu(t) = 0, f^\Delta(t)\ (f^\nabla(t)) = f'(t)$, and in the case $\mathbb{T} = q\mathbb{Z}$, we have $\sigma(t) = t + q, \rho(t) = t - q, \mu(t) = \nu(t) = q$,

$$f^\Delta(t) = \frac{f(t+q) - f(t)}{q} \text{ and } f^\nabla(t) = \frac{f(t) - f(t-q)}{q}.$$

If $\mathbb{T} = q^\mathbb{Z}, q < 1$, we have $\sigma(t) = qt, \rho(t) = \frac{t}{q}, \mu(t) = (q-1)t$,

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t} \text{ and } f^\nabla(t) = \frac{f(t) - f(t/q)}{t - t/q} \text{ for } t \neq 0.$$

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with D, provided $D \subset \mathbb{T}^\kappa(\mathbb{T}_\kappa)$, $\mathbb{T}^\kappa \backslash D(\mathbb{T}_\kappa \backslash D)$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. Let f be rd (ld)-continuous. Then there exists a function F which is pre-differentiable with region of differentiation D such that $F^\Delta(x) = f(t)$ ($F^\nabla(x) = f(t)$) holds for all $t \in D$. We define the Cauchy integral by

$$\int_b^c f(t)\Delta t = F(c) - F(b) \quad \left(\int_b^c f(t)\nabla t = F(c) - F(b)\right),$$

where F is a pre-antiderivative of f and $b, c \in \mathbb{T}$. The existence theorem [3, p. 27, Theorem 1.74] reads as follows: Every rd (ld)-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by $F(t) = \int_{t_0}^t f(\tau)\Delta\tau$ $\left(F(t) = \int_{t_0}^t f(\tau)\nabla\tau\right)$ is an antiderivative of f.

If f is delta (nabla) differentiable, then f is continuous and rd (ld)-continuous. By using property of rd(ld)-continuous function, one can easily see that

$$\sigma, \rho, f^\sigma(x), (f^\sigma(x))^p, f^\rho(x), (f^\rho(x))^p \qquad p \in \mathbb{N}$$

are rd (ld)-continuous. Thus, all integrals involving main results of this paper are meaningful.

2.2 Lemmas

The following lemmas are useful and some of them can be found in [3].

LEMMA 2.1 ([21, p. 423, Lemma 2.5]). Let $a, b \in \mathbb{T}$ and $p > 1$. Assume $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^\kappa$ and non-negative, increasing function on $[a, b]_\mathbb{T}$. Then

$$pg^{p-1}(x)g^\Delta(x) \leqslant (g^p(x))^\Delta \leqslant p(g^\sigma(x))^{p-1}g^\Delta(x).$$

LEMMA 2.2 ([3, p. 28, Theorem 1.76]). If $f^\Delta(x) \geqslant 0 (f^\nabla(x) \geqslant 0)$, then $f(x)$ is nondecreasing.

LEMMA 2.3 ([3, p. 5, Theorem 1.75]). Assume that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous at $t \in \mathbb{T}^\kappa$. Then

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = f(t)\mu(t).$$

LEMMA 2.4. Let $a, b \in \mathbb{T}$ and $p > 1$. Assume $g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_\kappa$ and non-negative, increasing function on $[a, b]_\mathbb{T}$. Then

$$p(g^\rho(x))^{p-1}g^\nabla(x) \leqslant (g^p(x))^\nabla \leqslant pg^{p-1}(x)g^\nabla(x).$$

PROOF. By (1), we have

$$\left(g^2\right)^\nabla = (g + g^\rho)\, g^\nabla \text{ and } \left(g^3\right)^\nabla = \left((g^\rho)^2 + gg^\rho + g^2\right) g^\nabla.$$

By mathematical induction, we easily obtain

$$(g^p)^\nabla = \left(g^{p-1} + g^\rho g^{p-2} + \cdots + (g^\rho)^{p-1}\right) g^\nabla.$$

Since the function $g(x)$ is an increasing function on $[a, b]_\mathbb{T}$, we get

$$g^\rho(x) \le g(x).$$

So we easily obtain

$$p(g^\rho(x))^{p-1}g^\nabla(x) \leqslant (g^p(x))^\nabla \leqslant pg^{p-1}(x)g^\nabla(x).$$

The proof is completed.

For more discussion on time scales, we refer the reader to [3].

3 Main Results

THEOREM 3.1. Let $a, b \in \mathbb{T}$ and $t \geqslant 3$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^{\kappa}$. If f is a non-negative, increasing function on $[a, b]_{\mathbb{T}}$ and satisfies

$$f^{t-2}(x) f^{\Delta}(x) \geqslant (t-2)(f^{\sigma^2}(x))^{t-2}(\sigma^2(x) - a)^{t-3}\sigma^{\Delta}(x),$$

where $\sigma^2(x) = \sigma(\sigma(x))$, then

$$\int_a^b f^t(x)\Delta x - \left(\int_a^b f(x)\Delta x\right)^{t-1} \geqslant f^{t-2}(a)\left[f(a) - (t-1)\mu^{t-2}(a)\right]\int_a^x f(x)\Delta x.$$

PROOF. Define

$$F(x) = \int_a^x f^t(u)\Delta u - \left(\int_a^x f(u)\Delta u\right)^{t-1}$$

and $g(x) = \int_a^x f(u)\Delta u$. It is easy to see that $g^{\Delta}(x) = f(x)$. By Lemma 2.1, it follows that

$$F^{\Delta}(x) \geqslant f^t(x) - (t-1)(g^{\sigma}(x))^{t-2} g^{\Delta}(x) = f(x)F_1(x),$$

where $F_1(x) = f^{t-1}(x) - (t-1)(g^{\sigma}(x))^{t-2}$. By Lemma 2.2 again, we have

$$F_1^{\Delta}(x) \geqslant (t-1)f^{t-2}(x)f^{\Delta}(x) - (t-1)(t-2)(g^{\sigma^2}(x))^{t-3}f^{\sigma}(x)\sigma^{\Delta}(x).$$

Since f is a non-negative and increasing function, we have

$$g^{\sigma^2}(x) = \int_a^{\sigma^2(x)} f(u)\Delta u \leqslant f^{\sigma^2}(x)(\sigma^2(x) - a). \tag{2}$$

Hence,

$$\begin{aligned} F_1^{\Delta}(x) &\geqslant (t-1)[f^{t-2}(x)f^{\Delta}(x) - (t-2)(f^{\sigma^2}(x))^{t-3}f^{\sigma}(x)(\sigma^2(x)-a)^{t-3}\sigma^{\Delta}(x)] \\ &\geqslant (t-1)[f^{t-2}(x)f^{\Delta}(x) - (t-2)(f^{\sigma^2}(x))^{t-2}(\sigma^2(x)-a)^{t-3}\sigma^{\Delta}(x)] \geqslant 0. \end{aligned}$$

By Lemma 2.2, we conclude that $F_1(x)$ is an increasing function. Hence,

$$F_1(x) \geqslant F_1(a) = f^{t-2}(a)[f(a) - (t-1)\mu^{t-2}(a)],$$

which means that

$$F^{\Delta}(x) \geqslant f^{t-2}(a)[f(a) - (t-1)\mu^{t-2}(a)]f(x)$$

by applying Lemma 2.2. It follows that

$$\left(F(x) - f^{t-2}(a)[f(a) - (t-1)\mu^{t-2}(a)]g(x)\right)^{\Delta} \geqslant 0.$$

Thus, we have

$$F(x)-f^{t-2}(a)[f(a)-(t-1)\mu^{t-2}(a)]g(x) \geqslant F(a)-f^{t-2}(a)[f(a)-(t-1)\mu^{t-2}(a)]g(a) = 0.$$

This finishes the proof.

REMARK 1. If $\mathbb{T} = \mathbb{R}$ and $f(a) \neq 0$ in Theorem 3.1, we deduce Theorem 2.1 in [11].

REMARK 2. If $\mathbb{T} = h\mathbb{Z}$ in Theorem 3.1, then Theorem 3.1 generalizes Theorem 3.2 in [17].

THEOREM 3.2. Let $a, b \in \mathbb{T}$ and $p \geqslant 1$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$. If f is a non-negative, increasing function on $[a, b]_\mathbb{T}$ and satisfies

$$f^p(x)f^\Delta(x) \geqslant \frac{p}{(b-a)^{p-1}}\left(f^{\sigma^2}(x)\right)^p\left(\sigma^2(x)-a\right)^{p-1}\sigma^\Delta(x),$$

then

$$\begin{aligned}&\int_a^b f^{p+2}(x)\Delta x - \frac{1}{(b-a)^{p-1}}\left(\int_a^b f(x)\Delta x\right)^{p+1}\\ \geqslant\ & f^p(a)\left[f(a) - \frac{p+1}{(b-a)^{p-1}}\mu^p(a)\right]\int_a^x f(x)\Delta x.\end{aligned}$$

PROOF. Define

$$G(x) = \int_a^x f^{p+2}(t)\Delta t - \frac{1}{(b-a)^{p-1}}\left(\int_a^x f(t)\Delta t\right)^{p+1}$$

and $g(x) = \int_a^x f(t)\Delta t$. By Lemma 2.1, it follows that

$$\begin{aligned}G^\Delta(x) &= f^{p+2}(x) - \frac{1}{(b-a)^{p-1}}(g^{p+1}(x))^\Delta \geqslant f^{p+2}(x) - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(x))^p g^\Delta(x)\\ &\geqslant f(x)\left[f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(x))^p\right] = f(x)G_1(x),\end{aligned}$$

where $G_1(x) = f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(x))^p$. By Lemma 2.1 and (2), we have

$$\begin{aligned}G_1^\Delta(x) &\geqslant (p+1)f^p(x)f^\Delta(x) - \frac{p(p+1)}{(b-a)^{p-1}}\left(g^{\sigma^2}(x)\right)^{p-1}f^\sigma(x)\sigma^\Delta(x)\\ &\geqslant (p+1)\left[f^p(x)f^\Delta(x) - \frac{p}{(b-a)^{p-1}}\left(f^{\sigma^2}(x)\right)^p\left(\sigma^2(x)-a\right)^{p-1}\sigma^\Delta(x)\right]\\ &\geqslant 0.\end{aligned}$$

Similar to the proof of Theorem 3.1, we have

$$G^{\Delta}(x) \geq f(x)G_1(a) \;\Leftrightarrow\; (G(x) - g(x)G_1(a))^{\Delta} \geqslant 0,$$

which implies

$$G(x) - g(x)G_1(a) \geqslant G(a) - g(a)G_1(a) = 0.$$

The proof is complete.

REMARK 3. If $\mathbb{T} = \mathbb{R}$ and $f(a) \neq 0$ in Theorem 3.2, we deduce Theorem 2.2 in [11].

REMARK 4. If $\mathbb{T} = h\mathbb{Z}$ in Theorem 3.2, then Theorem 3.2 generalizes Theorem 3.3 in [17].

THEOREM 3.3. Let $a, b \in \mathbb{T}$ and $p \geqslant 3$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^{\kappa}$. If f is a non-negative, increasing function on $[a,b]_{\mathbb{T}}$ and satisfies

$$f^{p-3}(x)f^{\Delta}(x) \geqslant (p-2)\Big(f^{\sigma^2}(x)\Big)^{p-3}\Big(\sigma^2(x) - a\Big)^{p-3}\sigma^{\Delta}(x),$$

then

$$\begin{aligned} &\int_a^b f^p(x)\Delta x - \left(\int_a^b f^{\rho}(x)\Delta x\right)^{p-1} \\ \geqslant\; & (f(a))^{p-2}[f(a) - (p-1)\mu^{p-2}(a)]\int_a^x f(\rho(x))\Delta x. \end{aligned}$$

PROOF. Define

$$H(x) = \int_a^x f^p(t)\Delta t - \left(\int_a^x f^{\rho}(t)\Delta t\right)^{p-1}$$

and $g(x) = \int_a^x f^{\rho}(t)\Delta t$. By Lemma 2.1, it follows that

$$H^{\Delta}(x) = f^p(x) - (g^{p-1}(x))^{\Delta} \geqslant f^p(x) - (p-1)(g^{\sigma}(x))^{p-2}g^{\Delta}(x) \geqslant f(\rho(x))H_1(x),$$

where $H_1(x) = f^{p-1}(x) - (p-1)(g^{\sigma}(x))^{p-2}$. By Lemma 2.1 and (2) again, we have

$$\begin{aligned} H_1^{\Delta}(x) &\geqslant (p-1)f^{p-2}(x)f^{\Delta}(x) - (p-1)(p-2)\Big(g^{\sigma^2}(x)\Big)^{p-3} f(x)(\sigma(x))^{\Delta} \\ &\geqslant (p-1)f(x)\Big[f^{p-3}(x)f^{\Delta}(x) - (p-2)\Big(f^{\sigma^2}(x)\Big)^{p-3}\Big(\sigma^2(x) - a\Big)^{p-3}\sigma^{\Delta}(x)\Big] \\ &\geqslant 0. \end{aligned}$$

By Lemma 2.2, we conclude that $H_1(x)$ is an increasing function. Hence,

$$\begin{aligned} H_1(x) &\geqslant H_1(a) = f^{p-1}(a) - (p-1)(g^\sigma(a))^{p-2} \\ &= (f(a))^{p-2}[f(a) - (p-1)\mu^{p-2}(a)] \end{aligned}$$

which means that $(H(x) - g(x)H_1(a))^\Delta \geqslant 0$. The proof is complete.

THEOREM 3.4. Let $a, b \in \mathbb{T}$ and $p \geqslant 1$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$. If f is a non-negative, increasing function on $[a, b]_\mathbb{T}$ and satisfies

$$(f^\sigma(x))^\Delta \geqslant p\sigma^\Delta(x),$$

then

$$\begin{aligned} &\int_a^b (f^\sigma(x))^{p+2}\Delta x - \frac{1}{(b-a)^{p-1}}\left(\int_a^b f^\rho(x)\Delta x\right)^{p+1} \\ \geqslant\ &\left[(f^\sigma(a))^{p+1} - \frac{p+1}{(b-a)^{p-1}}(f^\rho(a)\mu(a))^p\right]\int_a^x f(\rho(x))\Delta x. \end{aligned}$$

PROOF. Define

$$W(x) = \int_a^x (f^\sigma(t))^p \Delta t - \left(\int_a^x f^\rho(t)\Delta t\right)^{p-1}$$

and $g(x) = \int_a^x f^\rho(t)\Delta t$. By Lemma 2.1, it follows that

$$\begin{aligned} W^\Delta(x) &\geqslant (f^\sigma(x))^{p+2} - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(x))^p g^\Delta(x) \\ &\geqslant f^\sigma(x)\left[f^\sigma(x))^{p+1} - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(x))^p\right] \\ &\geqslant f(\rho(x))W_1(x), \end{aligned}$$

where $W_1(x) = (f^\sigma(x))^{p+1} - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(x))^p$. By Lemma 2.1 again, we have

$$W_1^\Delta(x) \geqslant (p+1)\left[\Big(f^\sigma(x)\Big)^p\Big(f^\sigma(x)\Big)^\Delta - \frac{p}{(b-a)^{p-1}}\Big(g^{\sigma^2}(x)\Big)^{p-1} f(x)(\sigma(x))^\Delta\right].$$

Since f is a non-negative and increasing function, then

$$g^{\sigma^2}(x) = \int_a^{\sigma^2(x)} f^\rho(t)\Delta t \leqslant f^{\rho\sigma^2}(x)(\sigma^2(x) - a) \leqslant f^\sigma(x)(b-a). \tag{3}$$

Hence,

$$W_1^\Delta(x) \geqslant (p+1)\Big(f^\sigma(x)\Big)^p\left[\Big(f^\sigma(x)\Big)^\Delta - p(\sigma(x))^\Delta\right].$$

By Lemma 2.2, we conclude that $W_1(x)$ is an increasing function. Hence,

$$\begin{aligned} W_1(x) &\geqslant W_1(a) = (f^\sigma(a))^{p+1} - \frac{p+1}{(b-a)^{p-1}}(g^\sigma(a))^p \\ &= (f^\sigma(a))^{p+1} - \frac{p+1}{(b-a)^{p-1}}(f^\rho(a)\mu(a))^p, \end{aligned}$$

which means that $(W(x) - g(x)W_1(a))^\Delta \geqslant 0$. The proof is complete.

THEOREM 3.5. Let $a, b \in \mathbb{T}$ and $p \geqslant 3$. Assume $f, \sigma : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^\kappa$. If f is a non-negative, increasing function on $[a, b]_\mathbb{T}$ and satisfies

$$\Big(f^\sigma(x)\Big)^\Delta \geqslant (p-2)\Big(\sigma^2(x) - a\Big)^{p-3}\sigma^\Delta(x),$$

then

$$\begin{aligned} &\int_a^b f^\sigma(x))^p\Delta x - \left(\int_a^b f^\rho(x)\Delta x\right)^{p-1} \\ \geqslant\ & (f^\sigma(a))^{p-2}\Big[f^\sigma(a) - (p-1)\mu^{p-2}(a)\Big]\int_a^x f(\rho(x))\Delta x. \end{aligned}$$

PROOF. Define

$$Q(x) = \int_a^x (f^\sigma(t))^p\Delta t - \left(\int_a^x f^\rho(t)\Delta t\right)^{p-1}$$

and $g(x) = \int_a^x f^\rho(t)\Delta t$. By Lemma 2.1, it follows that

$$\begin{aligned} Q^\Delta(x) &= (f^\sigma(x))^p - (g^{p-1}(x))^\Delta \geqslant (f^\sigma(x))^p - (p-1)(g^\sigma(x))^{p-2}g^\Delta(x) \\ &\geqslant f(\rho(x))Q_1(x), \end{aligned}$$

where $Q_1(x) = (f^\sigma(x))^{p-1} - (p-1)(g^\sigma(x))^{p-2}$. By Lemma 2.1 and (3) again, we have

$$\begin{aligned} Q_1^\Delta(x) &\geqslant (p-1)[(f^\sigma(x))^{p-2}(f^\sigma(x))^\Delta - (p-2)(g^{\sigma^2}(x))^{p-3}(g^\sigma(x))^\Delta] \\ &\geqslant (p-1)\Big(f^\sigma(x)\Big)^{p-2}\Big[(f^\sigma(x))^\Delta - (p-2)\Big(\sigma^2(x) - a\Big)^{p-3}\sigma^\Delta(x)\Big] \\ &\geqslant 0. \end{aligned}$$

By Lemma 2.2, we conclude that $Q_1(x)$ is an increasing function. Hence,

$$\begin{aligned} Q_1(x) &\geqslant Q_1(a) = (f^\sigma(a))^{p-1} - (p-1)(g^\sigma(a))^{p-2} \\ &\geqslant (f^\sigma(a))^{p-2}\Big(f^\sigma(a) - (p-1)\mu^{p-2}(a)\Big), \end{aligned}$$

which means that

$$\left(Q(x)-g(x)(f^{\sigma}(a))^{p-2}\left(f^{\sigma}(a)-(p-1)\mu^{p-2}(a)\right)\right)^{\Delta}\geqslant 0.$$

The proof is complete.

Next, we generalized Feng Qi type inequalities related to nabla derivative.

THEOREM 3.6. Let $a,b\in\mathbb{T}$ and $t\geqslant 3$. Assume $f:\mathbb{T}\to\mathbb{R}$ are nabla differentiable at $t\in\mathbb{T}_{\kappa}$. If f is a non-negative, increasing function on $[a,b]_{\mathbb{T}}$ and satisfies

$$\left(f^{\rho}(x)\right)^{t-2}f^{\nabla}(x)\geqslant(t-2)(f(x))^{t-2}(x-a)^{t-3},$$

then

$$\int_a^b f^t(x)\nabla x-\left(\int_a^b f(x)\nabla x\right)^{t-1}\geqslant f^{t-1}(a)\int_a^x f(x)\nabla x.$$

PROOF. Define

$$F(x)=\int_a^x f^t(u)\nabla u-\left(\int_a^x f(u)\nabla u\right)^{t-1}$$

and $g(x)=\int_a^x f(t)\nabla t$. It is easy to see $g^{\nabla}(x)=f(x)$. By Lemma 2.4, it follows that

$$F^{\nabla}(x)\geqslant f^t(x)-(t-1)(g(x))^{t-2}g^{\nabla}(x)=f(x)F_1(x),$$

where $F_1(x)=f^{t-1}(x)-(t-1)(g(x))^{t-2}$. By Lemma 2.2 again, we have

$$F_1^{\nabla}(x)\geqslant(t-1)(f^{\rho}(x))^{t-2}(x)f^{\nabla}(x)-(t-1)(t-2)(g(x))^{t-3}f(x).$$

Since f is a non-negative and increasing function, then

$$g(x)=\int_a^x f(t)\nabla t\leqslant f(x)(x-a). \tag{4}$$

Hence,

$$\begin{aligned}F_1^{\nabla}(x)&\geqslant(t-1)\left[(f^{\rho}(x))^{t-2}f^{\nabla}(x)-(t-2)(f(x))^{t-2}(x-a)^{t-3}\right]\\&\geqslant 0.\end{aligned}$$

By Lemma 2.2, we conclude that $F_1(x)$ is an increasing function. Hence,

$$F_1(x)\geqslant F_1(a),$$

which means that

$$F^{\nabla}(x)\geqslant F_1(a)f(x).$$

It follows that

$$\Big(F(x) - F_1(a)g(x)\Big)^{\nabla} \geqslant 0.$$

Thus, we have

$$F(x) - f^{t-1}(a)g(x) \geqslant F(a) - f^{t-1}(a)g(a) = 0.$$

This finishes the proof.

THEOREM 3.7. Let $a, b \in \mathbb{T}$ and $p \geqslant 1$. Assume $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_\kappa$. If f is a non-negative, increasing function on $[a, b]_{\mathbb{T}}$ and satisfies

$$(f^\rho(x))^p f^\nabla(x) \geqslant \frac{p}{(b-a)^{p-1}}(f^p(x))(x-a)^{p-1},$$

then

$$\int_a^b f^{p+2}(x)\nabla x - \frac{1}{(b-a)^{p-1}}\left(\int_a^b f(x)\nabla x\right)^{p+1} \geqslant f^{p+1}(a)\int_a^x f(x)\nabla x.$$

PROOF. Define

$$G(x) = \int_a^x f^{p+2}(t)\nabla t - \frac{1}{(b-a)^{p-1}}\left(\int_a^x f(t)\nabla t\right)^{p+1}$$

and $g(x) = \int_a^x f(t)\nabla t$. By Lemma 2.4, it follows that

$$\begin{aligned} G^\nabla(x) &= f^{p+2}(x) - \frac{1}{(b-a)^{p-1}}(g^{p+1}(x))^\nabla \\ &\geqslant f^{p+2}(x) - \frac{p+1}{(b-a)^{p-1}}g^p(x)g^\nabla(x) \\ &\geqslant f(x)\left[f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}}g^p(x)\right] \\ &= f(x)G_1(x), \end{aligned}$$

where $G_1(x) = f^{p+1}(x) - \frac{p+1}{(b-a)^{p-1}}g^p(x)$. By Lemma 2.4 again, we have

$$G_1^\nabla(x) \geqslant (p+1)\left[(f^\rho(x))^p f^\nabla(x) - \frac{p}{(b-a)^{p-1}}f^p(x)(x-a)^{p-1}\right] \geqslant 0.$$

Similar to the proof of Theorem 3.6, we have

$$G^\nabla(x) \geqslant f(x)G_1(a) \;\Leftrightarrow\; (G(x) - g(x)G_1(a))^\nabla \geqslant 0,$$

which implies

$$G(x) - g(x)G_1(a) \geqslant G(a) - g(a)G_1(a) = 0.$$

The proof is completed.

REMARK 5. Similar to the deduction of Theorems 3.4, 3.5, and 3.6, we easily obtain other Feng Qi type inequalities related to the nabla derivative. For the sake of simplicity, we omit the details.

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