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The Cone Compression And Expansion Fixed Point Theorem With Convex And α-Homogeneous Boundary Operators*

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Abstract

The purpose of this note is to show that in the Krasnosel'skii fixed point theorem of cone compression and expansion, the norm and more generally the convex functional may be replaced by convex functionals perturbed by α-homogeneous boundary operators. An application to a model boundary value problem is then provided to illustrate the main existence theorem.

1 Introduction

One of the most important tools in fixed point theory is the cone expansion and compression theorem proved by Krasnosel'skii in 1964 (see, e.g., [8] or [9]). It has been proven to be efficient in showing existence of positive solutions to various boundary value problems (BVPs for short).

THEOREM 1. Let E be a Banach space and $P \subseteq E$ a cone. Assume that Ω_1, Ω_2 are two open subsets of E with $\theta \in \Omega_1$ (the zero element) and $\overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow P$ be a completely continuous operator such that either

(i) $\|Au\|_E \leq \|u\|_E$, for $u \in P \cap \partial\Omega_1$ and $\|Au\|_E \geq \|u\|_E$, for $u \in P \cap \partial\Omega_2$, or

(ii) $\|Au\|_E \geq \|u\|_E$, for $u \in P \cap \partial\Omega_1$ and $\|Au\|_E \leq \|u\|_E$, for $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Given a real Banach space E and $P \subset E$ a nonempty closed convex subset, P is called a cone if it satisfies the following two conditions:

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(i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$, and

(ii) $P \cap (-P) = \{\theta\}$.

The proof of Theorem 1 relies upon the following two technical lemmas producing the computation of the fixed point index (see, e.g. [4, 8, 9]).

LEMMA 1. Let E be a Banach space, $P \subset E$ a cone, and $\Omega \subset E$ a bounded open subset with $\theta \in \Omega$. Let $A : P \cap \overline{\Omega} \longrightarrow P$ be a completely continuous operator such that the so-called Leray-Schauder condition is satisfied:

$$Ax \neq \lambda x, \ \forall\, x \in P \cap \partial\Omega \ \text{ and } \ \forall\, \lambda \geq 1. \tag{1}$$

Then the fixed point index $i(A, P \cap \Omega, P) = 1$.

LEMMA 2. Let E be a Banach space, $P \subset E$ a cone, and $\Omega \subset E$ a bounded open subset. Let $A : P \cap \overline{\Omega} \longrightarrow P$ be a completely continuous operator such that

(i) $\inf\limits_{x \in P \cap \partial\Omega} \|Ax\| > 0$, and

(ii) $\|Ax\| \geq \|x\|$ and $Ax \neq x, \forall\, x \in P \cap \partial\Omega$.

Then the fixed point index $i(A, P \cap \Omega, P) = 0$.

In the last couple of years, several authors have generalized Theorem 1. The generalizations have essentially concerned the norm which was replaced by a convex functional in papers [1, 2, 10, 11]) and where some applications to the solvability of some BVPs can be found. As noticed in [7], the compact operator A may be generalized to a k-set contraction mapping. In [10, Theorem 2.4], the norms appearing in conditions (i) and (ii) of Theorem 1 are not necessarily the same. In [13], in one inequality a norm is considered while it is a semi-norm in the other inequality. In [11], Sun et al. considered the case of the sum of convex and concave operators.

In [15], the authors replaced the norm by a convex functional ρ on the cone P, i.e., a mapping that satisfies

$$\rho(tx + (1-t)y) \leq t\rho(x) + (1-t)\rho(y),$$

for all $x, y \in P$ and $t \in [0, 1]$. Their main existence result extends the fixed point index theory as developed in [4, 8] and relies heavily on the proof that the intersection F of a cone with the exterior of a ball defined by a functional is a retract of the whole space; the latter result (see [15, Theorem 2.1]) is itself very interesting. Recall that a subset $F \subset E$ is called a retract of E (see, e.g. [14]) if there exists a continuous mapping $r : E \longrightarrow F$ such that $r(x) = x$ for every $x \in F$. In 1951, Dugundji [6] proved that every closed convex subset of a Banach space, in particular a cone, is a retract. It is important to note that the subset F need not to be convex even in the norm case.

The authors of [12] have established some fixed point theorems of Altman and Rothe types for α-positive-homogeneous operators. Given a real Banach space E and a positive real number α, a mapping $B : E \longrightarrow \mathbb{R}^+$ is called α-positive homogeneous if

$$B(tu) = t^{\alpha} B(u) \ \ \forall\, u \in E, \forall\, t \geq 0.$$

For instance, the functional B defined on a Banach space E by $B(u) = \|u\|_E^\gamma$, for some positive constant γ, is γ-positive homogeneous.

The aim of this note is to show that in [15, Corollary 2.1], we can replace the functional ρ by the sum of a convex functional and a homogeneous operator. In addition, these boundary operators are not necessarily the same in inequalities (i) and (ii) of Theorem 1. This is the content of Section 2. In Section 3, we apply the theoretical result to the solvability of a Dirichlet BVP on $[0, 1]$.

2 Main Results

We start with a result extending Lemma 1. When $B = 0$, we also recover [15, Theorem 2.3].

LEMMA 3. Let Ω be a bounded open set in a Banach space E such that $\theta \in \Omega$ and let $P \subseteq E$ be a cone. Suppose that $A : P \cap \overline{\Omega} \longrightarrow P$ is a completely continuous mapping such that $Au \neq u$ for all $u \in P \cap \partial\Omega$ and that $\rho, B : P \longrightarrow \mathbb{R}^+$ are uniformly continuous functionals satisfying:

(i) B is α-positive homogeneous with $\alpha \geq 1$ and $Bu > 0$, for $u \neq \theta$,

(ii) ρ is convex with $\rho(\theta) = 0$ and $\rho(u) > 0$, for $u \neq \theta$.

Assume further that

$$B(Au) + \rho(Au) \leq B(u) + \rho(u) \text{ for all} \, u \in P \cap \partial\Omega. \tag{2}$$

Then, the fixed point index $i(A, P \cap \Omega, P) = 1$.

PROOF. We prove that Hypothesis (2) implies that the condition of Leray-Schauder (1) is satisfied. Suppose to the contrary that there exist $u_0 \in P \cap \partial\Omega$ and $\lambda_0 \geq 1$ such that $Au_0 = \lambda_0 u_0$. Since operator A has no fixed point on the boundary $\partial\Omega$, then $\lambda_0 > 1$. Moreover we have

$$\begin{aligned} \frac{1}{\lambda_0} B(Au_0) + \rho(u_0) &= \frac{1}{\lambda_0} B(Au_0) + \rho\left(\frac{1}{\lambda_0} Au_0\right) \\ &= \frac{1}{\lambda_0} B(Au_0) + \rho\left(\frac{1}{\lambda_0} Au_0 + \left(1 - \frac{1}{\lambda_0}\right)\theta\right). \end{aligned}$$

Since the functional ρ is convex, and by Hypothesis (2), we obtain

$$\begin{aligned} \frac{1}{\lambda_0} B(\lambda_0 u_0) + \rho(u_0) &\leq \frac{1}{\lambda_0} \left(B(Au_0) + \rho(Au_0)\right) \\ &\leq \frac{1}{\lambda_0} \left(B(u_0) + \rho(u_0)\right) \leq \frac{1}{\lambda_0} B(u_0) + \rho(u_0), \end{aligned}$$

which implies that

$$B(\lambda_0 u_0) \leq B(u_0).$$

Operator B being α-positive homogeneous, we deduce that

$$B(\lambda_0 u_0) = \lambda_0^\alpha B(u_0) \leq B(u_0).$$

Hence $(1-\lambda_0^\alpha)B(u_0) \geq 0$, contradicting the fact that $1-\lambda_0^\alpha < 0$ and $Bu_0 > 0$. Thus, Lemma 1 guarantees that $i(A, P\cap\Omega, P) = 1$.

By replacing the functional ρ by the sum $B+\rho$ in [15, Theorem 2.1] and [15, Theorem 2.2], respectively, and adapting the same proofs, we obtain the following two lemmas. We omit the details.

LEMMA 4. Let P be a cone in a Banach space E. Assume that $B, \rho : P \longrightarrow \mathbb{R}^+$ are uniformly continuous functionals such that B is α-positive homogeneous ($\alpha \geq 1$), sub-additive (i.e., $B(u+v) \leq Bu+Bv$ for $u, v \in P$), and ρ is convex with $(B+\rho)(\theta) = 0$ and $(B+\rho)(u) > 0$ for $u \neq \theta$. Then, for all $R > 0$, the set

$$D_R = \{u \in P : (B+\rho)(u) \geq R\}$$

is a retract of E.

LEMMA 5. Let Ω be a bounded open set in a Banach space E and let $P \subseteq E$ be a cone. Suppose that $A : P\cap\overline{\Omega} \longrightarrow P$ is a completely continuous mapping and $B, \rho : P \longrightarrow \mathbb{R}^+$ are uniformly continuous functionals that satisfy:

(i) B is α-positive homogeneous ($\alpha \geq 1$) and sub-additive,

(ii) ρ is convex with $(B+\rho)(\theta) = 0$ and $(B+\rho)(u) > 0$, for $u \neq \theta$.

Assume further that

(1) $B(Au) + \rho(Au) \geq Bu + \rho(u)$, for $u \in P\cap\partial\Omega$,

(2) $\inf\limits_{u\in P\cap\partial\Omega} \{Bu + \rho(u)\} > 0$ and $Au \neq u$, for $u \in P\cap\partial\Omega$.

Then the fixed point index $i(A, P\cap\Omega, P) = 0$.

As a consequence of Lemma 3 and Lemma 5, we derive a generalization of the Krasnosel'skii fixed point theorem. The proof is immediate.

THEOREM 2. Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $P \subseteq E$ be a cone. Suppose that $A : P\cap(\overline{\Omega}_2\setminus\Omega_1) \longrightarrow P$ is a completely continuous mapping and $B_i, \rho_i : P \longrightarrow \mathbb{R}^+$ are uniformly continuous functionals such that:

(i) For $i = 1, 2$, B_i is α_i-positive homogeneous ($\alpha_i \geq 1$) and B_2 is sub-additive.

(ii) ρ_1 is convex with $\rho_1(\theta) = 0$ and $\rho_1(u) > 0$, for $u \neq \theta$ if $\rho_1 \not\equiv 0$.

(iii) ρ_2 is convex with $\rho_2(\theta) = 0$ and $\rho_2(u) > 0$, for $u \neq \theta$.

Assume further that

(1) $B_1(Au) + \rho_1(Au) \leq B_1(u) + \rho_1(u)$, for $u \in P \cap \partial\Omega_1$,

(2) $B_2(Au) + \rho_2(Au) \geq B_2(u) + \rho_2(u)$, for all $u \in P \cap \partial\Omega_2$ with

$$\inf_{u \in P \cap \partial\Omega_2} \{B_2 u + \rho_2 u\} > 0.$$

Then operator A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Letting $B_1 = B_2 = 0$ and $\rho_1 = \rho_2$, we recapture [15, Corollary 2.1]. Another interesting situation is that of a convex functional combined with an α-positive homogeneous one. The following consequence is derived by letting $\rho_1 \equiv B_2 \equiv 0$.

COROLLARY 1. Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $P \subseteq E$ be a cone. Suppose that $A: P \cap (\overline{\Omega}_2 \backslash \Omega_1) \longrightarrow P$ is a completely continuous operator, $B, \rho : P \longrightarrow \mathbb{R}^+$ are uniformly continuous functionals such that B is an α-positive homogeneous functional with $\alpha \geq 1$, and ρ is convex satisfying $\rho(\theta) = 0, \rho(u) > 0$, for $u \neq \theta$. Assume further that the following two conditions hold:

(i) $B(Au) \leq B(u)$, for $u \in P \cap \partial\Omega_1$,

(ii) $\rho(Au) \geq \rho(u)$, for all $u \in P \cap \partial\Omega_2$ with $\inf\limits_{u \in P \cap \partial\Omega_2} \rho(u) > 0$.

Then operator A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Applications

Consider the Dirichlet boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function. It is clear that if u is a solution of the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) ds,$$

where

$$G(t,s) = \begin{cases} t(1-s) & \text{for } t \leq s, \\ s(1-t) & \text{for } s \leq t \end{cases} \tag{4}$$

is the Green function for $-u''(t) = 0$ and $u(0) = u(1) = 0$, then it is a solution of problem (3). One can show that

$$\begin{cases} G(t,s) \leq G(s,s) \text{ for } (t,s) \in [0,1]^2, \\ G(t,s) \geq \frac{1}{4} G(s,s), \ \forall\, t \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) ds = \frac{3}{32}. \end{cases} \tag{5}$$

Let $E = C([0,1])$ be the Banach space of real continuous functions on $[0,1]$ endowed with the norm $\|u\|_\infty = \max\limits_{t\in[0,1]} |u(t)|$ and let A be the operator defined by

$$Au(t) = \int_0^1 G(t,s) f(s,u(s)) ds.$$

For $u \in E$, Au is solution of the problem

$$\begin{cases} -(Au)''(t) = f(t,u(t)), & t \in [0,1], \\ (Au)(0) = (Au)(1) = 0. \end{cases} \tag{6}$$

Define the cone

$$P = \left\{ u \in E : \, u(t) \geq 0, u(t) \geq \frac{1}{4}\|u\|_\infty, \quad \forall\, t \in \left[\frac{1}{4}, \frac{3}{4}\right] \right\}$$

and let $H_0^1(0,1)$ stand for the Sobolev space of measurable functions u with $u, u' \in L^2(0,1)$ and $u(0) = u(1) = 0$. It is endowed with the norm $\|u\|_{H_0^1} = \|u'\|_{L^2}$. Define the functionals:

$$B(u) = \|u\|_{H_0^1} \text{ and } \rho(u) = \|u\|_\infty.$$

Concerning problem (3), our main result is:

THEOREM 3. Assume that the following conditions hold:

(a)
$$0 < \limsup_{x\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{x} \leq \pi^2,$$

(b)
$$\liminf_{x\to+\infty} \min_{t\in[\frac{1}{4},\frac{3}{4}]} \frac{f(t,x)}{x} \geq \frac{128}{3}.$$

Then the boundary value problem (3) has at least one positive solution $u \in E$ and there exist two positive constants $0 < R_1 < R_2$ such that $R_1 \leq \|u\|_\infty \leq R_2$.

PROOF. Since f is continuous, operator $A : E \longrightarrow E$ is completely continuous. Moreover the functional $\rho : P \longrightarrow \mathbb{R}^+$ defined by $\rho(u) = \|u\|_\infty$ is uniformly continuous and convex. The functional $B : P \longrightarrow \mathbb{R}^+$ defined by $B(u) = \|u\|_{H_0^1}$ is also uniformly continuous and 1-positive homogeneous.

By Hypothesis (a), there exist $0 < \varepsilon < \pi^2$ and $R_1 > 0$ such that $0 \leq f(t,x) \leq \left(\pi^2 - \varepsilon\right) x$, for $0 < x \leq R_1$ and $t \in [0,1]$. Let

$$\Omega_1 = \{u \in E, \; \|u\|_\infty < R_1\}.$$

From Hypothesis (b), there exists $\overline{R}_2 > 0$ such that $f(t,x) \geq \frac{128}{3}x$, for $x \geq \overline{R}_2$ and $t \in [\frac{1}{4}, \frac{3}{4}]$. Let $R_2 = \max\{2R_1, 4\overline{R}_2\}$ and

$$\Omega_2 = \{u \in E : \; \|u\|_\infty < R_2\}.$$

Claim 1: $\|Au\|_{H_0^1} \leq \|u\|_{H_0^1}$, $\forall u \in P \cap \partial\Omega_1$. For all $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned} \|Au\|_{H_0^1} &= \sup_{\|v\|_{H_0^1} \leq 1} \left|(Au, v)_{H_0^1}\right| = \sup_{\|v\|_{H_0^1} \leq 1} \left|\int_0^1 (Au)'(t)v'(t)dt\right| \\ &= \sup_{\|v\|_{H_0^1} \leq 1} \left|\int_0^1 -(Au)''(t)v(t)dt\right| = \sup_{\|v\|_{H_0^1} \leq 1} \int_0^1 f(t, u(t))v(t)dt, \end{aligned}$$

whence the estimates

$$\begin{aligned} \|Au\|_{H_0^1} &\leq \sup_{\|v\|_{H_0^1} \leq 1} \left(\left(\int_0^1 f^2(t, u(t))dt\right)^{\frac{1}{2}} \left(\int_0^1 v^2(t)dt\right)^{\frac{1}{2}}\right) \\ &\leq \sup_{\|v\|_{H_0^1} \leq 1} \left(\left(\int_0^1 (\pi^2 - \varepsilon)^2 u^2(t)dt\right)^{\frac{1}{2}} \|v\|_{L^2}\right) \\ &\leq \pi^2 \sup_{\|v\|_{H_0^1} \leq 1} \left(\left(\int_0^1 u^2(t)dt\right)^{\frac{1}{2}} \|v\|_{L^2}\right) \\ &\leq \pi^2 \sup_{\|v\|_{H_0^1} \leq 1} \left(\frac{1}{\lambda_1}\|u\|_{H_0^1}\|v\|_{H_0^1}\right) \\ &= \pi^2 \frac{1}{\lambda_1}\|u\|_{H_0^1} = \|u\|_{H_0^1}. \end{aligned}$$

Here, we have used Poincaré's inequality (see [3]) with optimal constant, that is, if $\Omega = (a, b)$ is a bounded interval, then

$$\|u\|_{L^2} \leq \frac{1}{\sqrt{\lambda_1}}\|u'\|_{L^2} = \frac{1}{\sqrt{\lambda_1}}\|u\|_{H_0^1}, \quad \forall\, u \in H_0^1(\Omega),$$

where $\lambda_1 = \frac{\pi^2}{(b-a)^2}$ is the first eigenvalue of the Dirichlet problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [a, b], \\ u(a) = u(b) = 0. \end{cases}$$

Claim 2: $\|Au\|_\infty \geq \|u\|_\infty$, $\forall\, u \in P \cap \partial\Omega_2$. For $u \in P \cap \partial\Omega_2$, we have, using (5):

$$\begin{aligned} \|Au\|_\infty &\geq Au\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, s\right) f(s, u(s))ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) f(s, u(s))ds \geq \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) \frac{128}{3}u(s)ds \\ &\geq \frac{128}{3}\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) \frac{1}{4}\|u\|_\infty ds \geq \frac{128}{3}\frac{1}{4}\frac{3}{32}\|u\|_\infty = \|u\|_\infty. \end{aligned}$$

We have then proved that $B(Au) \leq B(u)$, for all $u \in P \cap \partial\Omega_2$ and $\rho(Au) \geq \rho(u)$, for all $u \in P \cap \partial\Omega_2$. Moreover $\inf\limits_{u \in P \cap \partial\Omega_2} \rho(u) = R_2 > 0$.

By Corollary 1, we conclude that problem (3) has at least one positive solution in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

REMARK 1. (a) With $\pi^2 \simeq 9.8596$, the estimate in (a) is optimal. It is better than the weaker one:

$$0 < \limsup_{x \to 0^+} \max_{t \in [0,1]} \frac{f(t,x)}{x} \leq 8.$$

The latter estimate is usually applied with Schauder's fixed point theorem for it provides sub-linear growth condition on the mapping A, i.e., $\|Au\|_\infty \leq \|u\|_\infty, \ \forall \ u \in P \cap \partial\Omega_1$. Indeed, arguing as in the proof of Theorem 3, Claim 1, we rather obtain the estimates:

$$\begin{aligned} \|Au\|_\infty &= \max_{t \in [0,1]} \int_0^1 G(t,s) f(s,u(s)) ds \\ &\leq \max_{t \in [0,1]} \int_0^1 G(t,s)(R-\varepsilon) u(s) ds \\ &\leq \frac{1}{8}(R-\varepsilon)\|u\|_\infty = \frac{R}{8}\|u\|_\infty. \end{aligned}$$

Thus if $R = 8$, then $\|Au\|_\infty \leq \|u\|_\infty$.

(b) The following problem of fractional order can be studied in a similar way; we omit the details:

$$\begin{cases} D^\alpha_{1^-}(D^\alpha_{0^+} u(t) = f(t,u(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{cases}$$

where $0 < \alpha < 1$ and $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous.

(c) The conclusion of Theorem 1 can be extended to the case of a k-set contraction mapping A $(0 < k < 1)$ with respect to some measure of noncompactness (see [4]) and also to a translate of a cone P as developed in [5].

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