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A Note On A Result Due To Ankeny And Rivlin*

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Abstract

Let $p(z) = a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n$ be a polynomial of degree n having no zeros in the unit disk. Then it is well known that for $R \geq 1$, $\max_{|z|=R} |p(z)| \leq \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |p(z)|$. In this paper, we consider polynomials with gaps, having all its zeros on the circle $S(0,K) := \{z : |z| = K\},\ 0 < K \leq 1,$ and estimate the value of $\left(\frac{\max_{|z|=R} |p(z)|}{\max_{|z|=1} |p(z)|}\right)^s$ for any positive integer s.

1 Introduction

Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. We will denote

$$M(p,r) := \max_{|z|=r} |p(z)|, \quad r > 0,$$

$$||p|| := \max_{|z|=1} |p(z)|\,,$$

and

$$D(0,K) := \{z : |z| < K\},\ K > 0.$$

Bernstein observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [8, p. 137]). This inequality is also known as the Bernstein's inequality.

THEOREM 1. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Then for $R \geq 1$,

$$M(p,R) \leq R^n ||p||.$$

Equality holds for $p(z) = \alpha z^n$, α being a complex number.

For polynomial of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [1] proved the following result.

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THEOREM 2 (Ankeny and Rivlin [1]). Let $p(z)=\sum_{j=0}^{n} a_j z^j \neq 0$ in $D(0,1)$. Then for $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)||p||. \tag{1}$$

Here equality holds for $p(z)=\dfrac{\alpha+\beta z^n}{2}$, where $|\alpha|=|\beta|=1$.

In 2005, Gardner, Govil and Musukula [3] proved the following generalization and sharpening of Theorem 2.

THEOREM 3. Let $p(z)=a_0+\sum_{j=t}^{n} a_j z^j,\ 1 \leq t \leq n$, be a polynomial of degree n and $p(z) \neq 0$ in $D(0,K),\ K \geq 1$. Then for $R \geq 1$,

$$\begin{aligned} M(p,R) \quad \leq \quad & \Big(\frac{R^n+s_0}{1+s_0}\Big)||p|| - \Big(\frac{R^n-1}{1+s_0}\Big)m - \frac{n}{1+s_0}\left[\frac{(\|p\|-m)^2-(1+s_0)^2\,|a_n|^2}{(||p||-m)}\right] \\ & \times\left\{\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+s_0)|a_n|} - \ln\left[1+\frac{(R-1)(\|p\|-m)}{(\|p\|-m)+(1+s_0)|a_n|}\right]\right\}, \end{aligned} \tag{2}$$

where $m=\min_{|z|=K}|p(z)|$, and

$$s_0 = K^{t+1}\frac{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t-1}+1}{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t+1}+1}.$$

Several research monographs have been written on this subject of inequalities (see for example Govil and Mohapatra [4], Milovanović, Mitrinović and Rassias [7], Rahman and Schmeisser [9], and recent article of Govil and Nwaeze [5]).

While trying to obtain an inequality analogous to (1) for polynomials not vanishing in $D(0,K), K \leq 1$, Dewan and Ahuja [2] were able to prove this only for polynomials having all the zeros on the circle $S(0,K) := \{z : |z| = K\},\ 0 < K \leq 1$.

THEOREM 4. Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros on $S(0,K),\ K \leq 1$. Then for $R \geq 1$ and for every positive integer s,

$$\{M(p,R)\}^s \leq \left[\frac{K^{n-1}(1+K)+(R^{ns}-1)}{K^{n-1}+K^n}\right]\{M(p,1)\}^s.$$

For $s=1$, Theorem 4 reduces to the following Corollary 5.

COROLLARY 5. Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n having all its zeros on $S(0,K),\ K \leq 1$. Then for $R \geq 1$,

$$M(p,R) \leq \left[\frac{K^{n-1}(1+K)+(R^n-1)}{K^{n-1}+K^n}\right]M(p,1). \tag{3}$$

2 Main Results

THEOREM 6. Let

$$p(z) = z^m \left[a_{n-m} z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j} z^{n-m-j} \right],$$

where $1 \leq \mu \leq n-m$ and $0 \leq m \leq n-1$, be a polynomial of degree n, having $m-fold$ zeros at origin and remaining $n-m$ zeros on $S(0, K)$, $K \leq 1$. Then for $R \geq 1$ and every positive integer s,

$$[M(p,R)]^s \leq L(\mu; K, m, n, s)[M(p,1)]^s,$$

where

$$\begin{aligned} L(\mu; K, m, n, s) \quad &= \quad \frac{1}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})} \Bigg[n(K^{n-m-2\mu+1} + K^{n-m-\mu+1}) \\ &\quad + (R^{ns} - 1)[n + mK^{n-m-2\mu+1} + mK^{n-m-\mu+1} - m] \Bigg]. \end{aligned}$$

For $m = 0$, by Theorem 6, we have

COROLLARY 7. Let $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, be a polynomial of degree n, having all zeros on $|z| = K$, $K \leq 1$. Then for $R \geq 1$ and every positive integer s,

$$[M(p,R)]^s \leq L(\mu; K, n, s)[M(p,1)]^s,$$

where

$$L(\mu; K, n, s) = \frac{K^{n-\mu}(K^{1-\mu} + K) + (R^{ns} - 1)}{K^{n-2\mu+1} + K^{n-\mu+1}}.$$

If we set $\mu = 1$ into Corollary 7, we get the following result of Dewan and Ahuja [2].

COROLLARY 8. Let $p(z) = \sum_{j=0}^{n} a_j z^j$, be a polynomial of degree n, having all zeros on $|z| = K$, $K \leq 1$. Then for $R \geq 1$ and every positive integer s,

$$[M(p,R)]^s \leq L(1; K, n, s)[M(p,1)]^s,$$

where

$$L(1; K, n, s) = \frac{K^{n-1}(1 + K) + (R^{ns} - 1)}{K^{n-1} + K^n}.$$

3 Lemmas

For the proof of Theorem 6, we will need the following lemmas. The first lemma is due to Kumar and Lal [6].

LEMMA 9. Let

$$p(z) = z^m \left[a_{n-m} z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j} z^{n-m-j} \right],$$

where $1 \leq \mu \leq n-m$ and $0 \leq m \leq n-1$, be a polynomial of degree n, having $m-fold$ zeros at origin and remaining $n-m$ zeros on $|z| = K$, $K \leq 1$.

$$\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|.$$

The next lemma is the Bernstein inequality given in Theorem 1.

LEMMA 10. Let $p(z)$ be a polynomial of degree n. Then for $R \geq 1$,

$$M(p, R) \leq R^n M(p, 1).$$

We now turn our attention to proof of the main result.

PROOF OF THEOREM 6. By Lemma 9, we have

$$\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|.$$

Applying Lemma 10 to the polynomial $p'(z)$ which is of degree $n-1$, it follows that for all $R \geq 1$ and $\theta \in [0, 2\pi)$,

$$\begin{aligned} \left|p'(Re^{i\theta})\right| &\leq \max_{|z|=R} |p'(z)| \leq R^{n-1} \max_{|z|=1} |p'(z)| \\ &\leq R^{n-1} \left[\frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \right] \max_{|z|=1} |p(z)|. \end{aligned}$$

So for each $\theta \in [0, 2\pi)$ and $R \geq 1$, we obtain

$$\begin{aligned} \left[p(Re^{i\theta})\right]^s - \left[p(e^{i\theta})\right]^s &= \int_1^R \frac{d\left[p(te^{i\theta})\right]^s}{dt} dt \\ &= \int_1^R s \left[p(te^{i\theta})\right]^{s-1} p'(e^{i\theta}) e^{i\theta} dt. \end{aligned}$$

This implies that

$$\left|p(Re^{i\theta})\right|^s \leq \left|p(e^{i\theta})\right|^s + s \int_1^R \left|p(te^{i\theta})\right|^{s-1} \left|p'(e^{i\theta})\right| dt.$$

Therefore,

$$
\begin{aligned}
&[M(p,R)]^s \\
&\le [M(p,1)]^s + s\int_1^R [t^n M(p,1)]^{s-1}\left|p'(e^{i\theta})\right| dt \\
&\le [M(p,1)]^s + s\int_1^R t^{ns-n}\,[M(p,1)]^{s-1}\, t^{n-1}\frac{n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)}{K^{n-m-2\mu+1}+K^{n-m-\mu+1}} \\
&\times M(p,1)dt \\
&= [M(p,1)]^s + s\left[\frac{n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)}{K^{n-m-2\mu+1}+K^{n-m-\mu+1}}\right][M(p,1)]^s\int_1^R t^{ns-1}dt \\
&= [M(p,1)]^s + [M(p,1)]^s\left[\frac{n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)}{K^{n-m-2\mu+1}+K^{n-m-\mu+1}}\right] s\frac{R^{ns}-1}{ns} \\
&= [M(p,1)]^s\left\{1+\frac{\left[n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)\right](R^{ns}-1)}{n\left(K^{n-m-2\mu+1}+K^{n-m-\mu+1}\right)}\right\}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
[M(p,R)]^s \quad \le \quad & \frac{[M(p,1)]^s}{n\left(K^{n-m-2\mu+1}+K^{n-m-\mu+1}\right)}\Big\{n\left(K^{n-m-2\mu+1}+K^{n-m-\mu+1}\right) \\
&+\Big[n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)\Big](R^{ns}-1)\Big\}.
\end{aligned}
$$

This completes the proof.

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